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Feedback Control of Large-Scale Systems

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Feedback Control of Large-Scale Systems

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Preface

Present-day technologies rely on the cooperation of many different machines, reactors, robots or transportation systems. All their parts are linked by common resources, by material flows or through information networks. If such systems are to be controlled, their analysis and control problems become very complex. That is, these tasks cannot be solved simply by using faster computers with larger memories. They necessitate new ideas for decomposing and dividing the analysis and control problems of the overall system into rather independent subproblems and for dealing with the uncertainties of the model and the lack of information about the system to be controlled.

For a typical large-scale system, there is no complete model available for the overall system, but each model describes only a part of the whole system. The system has to be controlled not by a single unit but by several separate feedback loops, each of which deals with only a subset of the measured signals and operates on a subset of the actuators. All together, these feedback loops represent a decentralized controller. Moreover, the analysis and design tasks have to be solved by different decision makers which can only communicate in a restricted way. Owing to this requirement, decentralized structures of decision making have to be used in the analysis of interconnected systems and the design of decentralized controllers.

The theory of large-scale systems is devoted to the problems that arise from the large size of the system to be controlled, the uncertainties of the models, and the information structure constraints. It is based on several new ideas which utilize the internal structure of interconnected systems, yield to decentralized controllers and reduce the requirements on the scope and accuracy of the model of the plant. The theory answers the fundamental question of how to break down a given control problem into manageable subproblems which are only weakly related to each other and can, therefore, be solved by separate but cooperating decision units. The resulting control strategies and feedback laws can be applied using multicomputer configurations whose separate computing units work independently without coordination or with weak coordination and information exchange.

This book presents the basic methods showing how multivariable feedback theory has to be extended to solve analytical and design tasks for interconnected systems. Emphasis is placed on the derivation of methods which have a decentralized information structure, that is which involve several weakly coupled decision units for analysing the given system or designing the controller. Preference is given to the clear presentation of simple and effective techniques which provide the basis for a large number of specific and sophisticated methods that have been derived only recently. Many of these refinements are outlined or at least mentioned in the bibliographical notes at the ends of the chapters.

The book is aimed at students, researchers and practising engineers. The theoretical background of interconnected feedback systems is presented together with a detailed engineering interpretation of the relevant methods and results. The different approaches, which have led to the large number of available analytical and design methods and many recent results, are presented together with their interrelationships, advantages and drawbacks.

Two different kinds of examples are used. Simple numerical examples give an intuitive understanding of the methodology, illustrate the significance of the results or algorithms, provide counterexamples to conjectures or make trends obvious. Practical application studies demonstrate how control problems for large-scale systems can be solved by means of the various methods. They show that some of the concepts presented here have already been applied to industrial systems such as multiarea power systems, glass furnaces, lines of moving vehicles, or to the water quality control of rivers. Some of the examples are used several times to illustrate different phenomena encountered in interconnected systems.

The presentation follows lectures which I have given regularly since 1976 to undergraduate and graduate students at the Universities of Technology in Dresden and Ilmenau. Readers should be acquainted with the theory of linear multivariable control systems. The basic results of matrix algebra and graph theory are given in the appendices.

Many co-researchers have influenced my thinking on the topic of this book. I am indebted to Professor K. Reinisch who introduced me as a student at the Technische Hochschule Ilmenau to the theory of large-scale systems as early as 1973. It was fascinating to be involved with the development of this theory since those early days. Most of the material of this book has been collected during my research at the Zentralinstitut für Kybernetik und Informationsprozesse in Dresden. My first supervisor, Dr J. Uhlig, channelled my interests towards decentralized control for application to the national electric power system. I have benefited greatly from stimulating discussions, correspondence or

common research work with Professors U. Korn and U. Jumar (Magdeburg), D. D. Šiljak (Santa Clara), K. Reinschke (Cottbus), L. Bakule (Prague), V. Vesely (Bratislava) and many others.

Thanks are due to Mrs B. Mante for her excellent preparation of the figures. Last, but not least, cooperation with Prentice Hall London, particularly with Allison King, is gratefully acknowledged.

Dresden, March 1990 J. Lunze

Symbols

MATHEMATICAL SYMBOLS

In general, matrices and operators are denoted by bold Roman capital letters, vectors by bold lower-case letters, and scalars by italics. In order to follow this notation, the style of a symbol is changed, for example, if vector-valued signals ($\mathbf{x}, \mathbf{u}, \mathbf{x}_i$) are reduced to scalars (x, u, x_i)

$\mathbf{A} = (a_{ij})$	(n, m) matrix $\mathbf{A}$, whose entries are denoted by a_{ij} ($i = 1, \ldots, n$; $j = 1, \ldots, m$)
$\mathbf{A} = (\mathbf{A}_{ij}) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \ldots & \mathbf{A}_{1p} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{q1} & \mathbf{A}_{q2} & \ldots & \mathbf{A}_{qp} \end{pmatrix}$	matrix partitioned into submatrices $\mathbf{A}_{ij}$ ($i = 1, \ldots, q$; $j = 1, \ldots, p$)
diag a_i	diagonal matrix with elements $a_1, a_2, \ldots$ on the main diagonal
diag $\mathbf{A}_i$, diag $\mathbf{A}$	block-diagonal matrix with the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots$ or $\mathbf{A}_1 = \mathbf{A}_2 = \cdots = \mathbf{A}$ on the diagonal
$a^*, \mathbf{A}^*$	complex conjugate of scalar a or matrix $\mathbf{A}$
$\mathbf{A}', \mathbf{A}^{-1}, \mathbf{A}^+$	transpose, inverse, pseudoinverse of $\mathbf{A}$
$\mathbf{I}, \mathbf{0}$	unit matrix, zero matrix
$[\mathbf{A}], [[\mathbf{A}]]$	structure matrices (cf. Section 2.5 and eqn (3.2.3))
$\lambda_i[\mathbf{A}]$	eigenvalue of matrix $\mathbf{A}$
$\sigma[\mathbf{A}] = \{\lambda_1[\mathbf{A}], \ldots, \lambda_n[\mathbf{A}]\}$	spectrum of matrix $\mathbf{A}$
$\Vert \mathbf{x} \Vert, \Vert \mathbf{A} \Vert$	norm of vector $\mathbf{x}$ or matrix $\mathbf{A}$
$\vert \mathbf{x} \vert, \vert \mathbf{A} \vert$	vector or matrix comprising the moduli of the entries of vector $\mathbf{x}$ or matrix $\mathbf{A}$, respectively
$\mathrm{Re}[\lambda], \mathrm{Im}[\lambda]$	real or imaginary part of a complex number or vector
$\mathcal{R}, \mathcal{R}^{m \times r}$	field of real numbers, field of (m, r) matrices

$\mathbf{a} \leqslant \mathbf{b}, \mathbf{A} \leqslant \mathbf{B}$	means $a_i \leqslant b_i$ $(i = 1, \ldots, n)$ for $\mathbf{a} = (a_1 \ldots a_n)'$, $\mathbf{b} = (b_1 \ldots b_n)'$ or $a_{ij} \leqslant b_{ij}$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ for $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$
$\mathscr{S} = \{s_1, s_2, \ldots\}$	set with elements $s_1, s_2, \ldots$
$\emptyset$	empty set

SYSTEM DESCRIPTION

$g(t), \mathbf{G}(t)$	impulse response, impulse response matrix
$g(p), \mathbf{G}(p), \hat{\mathbf{G}}(p)$	transfer function, transfer function matrix
$\hat{\mathbf{A}}, \hat{\mathbf{G}}$	the tilde $\hat{}$ signifies the approximate models
$\bar{\mathbf{A}}, \overline{\mathbf{G}}$	overbar signifies the closed-loop system
$\mathbf{K}_s$	matrix of the static reinforcement
$\delta(t), \sigma(t)$	impulse function, step function
$t \in [0, T], t \in (0, T)$	abbreviation of $0 \leqslant t \leqslant T$, $0 < t < T$ (closed or open intervals, respectively; also used for other parameters)

Introduction

At first sight, feedback control of large-scale systems poses the 'classical' control problem: for a given process with control input $\mathbf{u}(t)$ and control output $\mathbf{y}(t)$ find a controller that ensures closed-loop stability and asymptotic regulation and assigns the loop a suitable input–output (I/O) behaviour. This problem is usually solved in two steps (Figure 1(a)):

1. The design phase: for a given model of the plant and expected classes of disturbances $\mathbf{d}(t)$ and command signals $\mathbf{v}(t)$ a control law
$$\mathbf{u} = \mathbf{K}(\mathbf{y} - \mathbf{v}) \tag{1}$$
is chosen which satisfies the specifications given for the closed-loop system.
2. The execution phase: a controller with the control law (1) is applied to the process, that is at every instant of time t the measurement signal $\mathbf{y}(t)$ and the command $\mathbf{v}(t)$ are combined according to the control law in order to determine the control input $\mathbf{u}(t)$.

The entities which carry out the calculations necessary for both steps are called *decision-making units* or *control agents*.

However, this well-known control problem has been treated by classical and modern control theory under the crucial assumptions that there is a unique plant with a unique controller and that all calculations can be based on the whole information about the plant. That is, the design problem is solved for a model that describes the process as a whole. The controller receives all sensor data available and determines all input signals of the plant. In other words, all information is assumed to be available for a single unit that designs and applies the controller to the plant. This unit can be thought of as a centralized decision maker. Hence, multivariable control theory deals with the *centralized design* of *centralized controllers* (Figure 1(a)).

This assumption can hardly be satisfied if modern technological or societal systems have to be controlled. Present-day technologies rely

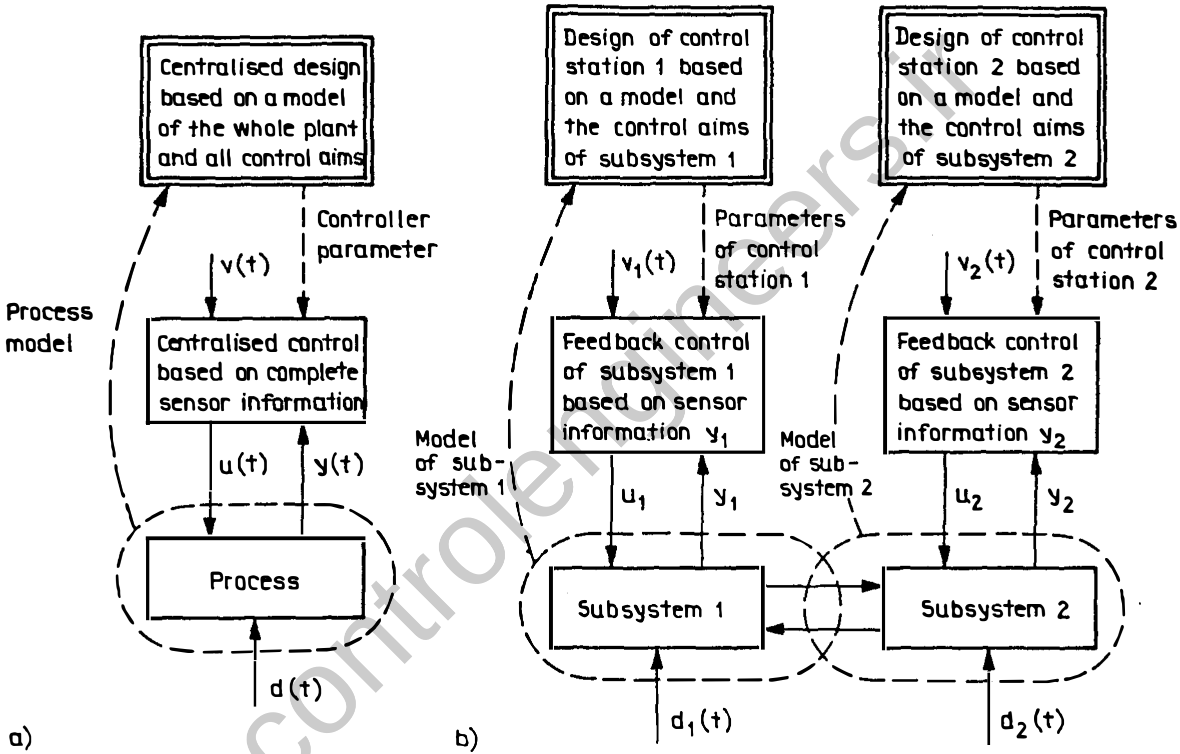


Figure 1 Information structures: (a) centralized control; (b) decentralized control

on the cooperation of many different machines, reactors, robots or transportation systems. All their parts are linked by common resources, by material flows or through information networks. The number N of subsystems may exceed ten or twenty (although Figure 1 is drawn for two subsystems only). Consequently, neither a complete model (*a priori* information) nor a complete set of measurement data (*a posteriori* information) can be made available for a centralized decision maker. Instead, the overall design problem has to be broken down into different, albeit coupled, subproblems. As a result, the overall plant is no longer controlled by a single controller but by several independent controllers, which are called control stations and which all together represent a *decentralized controller*. Moreover, these control stations are no longer designed simultaneously on the basis of a complete knowledge of the plant, but in different design steps by means of models that describe only the relevant parts of the plant. That is, *decentralized design schemes* are used (Figure 1(b)).

This fundamental difference between feedback control of 'small' and 'large' systems is usually described by the idea of information structure. The information structure describes the way in which *a priori* and *a posteriori* information is transferred among decision-making units. The decision-making units in Figure 1 are the design units (doubly framed blocks), which carry out the design step for a given plant model, and the controllers, which determine $\mathbf{u}(t)$ for given $\mathbf{y}(t)$ and $\mathbf{v}(t)$. In centralized control, the whole *a priori* information is transferred to the controller. In contrast to this, in decentralized control the design units and the controllers only have parts of the overall *a priori* or *a posteriori* information at their disposal. This difference is described by the terms *classical information structure* of centralized control or *non-classical information structure* in the case of decentralized control, respectively.

Besides the decentralization of the controller design and the control law, a further kind of decentralization is usually imposed on the implementation of the control law. Different portions of the computational work, which is necessary to apply a given control law, are carried out by different microprocessor units or different software components. Such implementation principles have been elaborated in the field of distributed data processing, which is developing rapidly at the moment. Its results are applied to control engineering problems, for example in large hierarchically structured process control systems. There, the computation necessary to control the whole system is performed within a network of computers of various types and sizes rather than by a single mainframe computer. By choosing an appropriate computer network architecture, a faster response and more flexibility can be obtained with sufficient reliability.

Control theory does not tackle the problem of implementing control algorithms on distributed microprocessor systems or transputers but it does promote the application of such modern hardware facilities. It helps to find the parallelisms inherent in control problems and to decompose the overall control task into subproblems which can be implemented on separate computing entities. The resulting control algorithm is fault tolerant if the subproblems, which are allocated to the individual processors, are sufficiently autonomous. Communication delays, noise, synchronization difficulties or other failures occurring in the processors or processor links merely degrade the overall system performance but operation can continue.

Large-scale systems present control theory with new challenges, as follows:

- The large size of the plant and, thus, the high dimension of the plant model have to be confronted. As a result of restrictions on computer time and memory space the system cannot be analysed and controlled as a whole.
- Every model of a large-scale system has severe uncertainties which have to be considered explicitly in all analysis and design steps. The possible effects of modelling errors cannot be evaluated heuristically but necessitate a systematic treatment.
- The decentralization of the design and of the controller imposes restrictions on the model and on the on-line information links introduced by the controller. This decentralization is necessary because the subsystems are under the control of different authorities, which make their own models and design their own control stations, or because the design problem has to be split into independent subproblems to become manageable.
- The design aims include not only stability or optimality, but also a variety of properties such as reliability, flexibility, robustness against structural perturbations of the plant, or restrictions on the interactions between the subsystems. Such aims have not been dealt with primarily in multivariable control theory.

The ultimate aim is to get a reasonable solution with reasonable effort in modelling, designing and implementing the controller. To achieve this aim, methods for dealing with large-scale systems have to be based on a new methodological background. The structural properties of the plant have to be exploited in order to derive suitable design and control structures. A fundamental question concerns the conditions on the interconnection structure, under which the subsystems are 'weakly coupled' and, thus, can be analysed separately while ignoring their interactions. Decomposition and decentralization methods help to answer this question and lead to new means for testing the stability of interconnected systems, for analysing the I/O behaviour of a certain subsystem under the influence of other subsystems without using a complete model of the overall plant, and for designing decentralized controllers.

Two remarks concerning the interdependencies of the theories of large-scale systems and of 'small' multivariable systems should emphasize this point. First, many problems that can be solved by the methods of multivariable control theory turn out to be subproblems encountered in large-scale systems theory. This is not surprising, for here it is a principal aim to break down complex problems into a combination of easier ones. Simply, it is the way to reduce the complexity that needs novel ideas, which then have to be elaborated.

Second, large-scale systems give rise to problems that are known in multivariable systems theory but have not yet been solved satisfactorily. Rigorous methods are missing for some multivariable control problems, because solutions to these 'small' problems could have been found by engineering common sense rather than by universal algorithms. For example, the Nyquist array method reduces the problem of designing multivariable controllers to several single-loop design tasks. Although there are guidelines for the construction of the decoupling compensator as well as the design of the individual loops, both design steps have to be used in a trial-and-error manner. This is satisfactory for multivariable systems but not for large-scale systems, where several such multivariable design problems occur and have to be solved repeatedly within some iteration loop.

AIMS AND STRUCTURE OF THE BOOK

The purpose of this book is to give a thorough introduction to the feedback control of large-scale systems. Emphasis is placed on the explanation of new phenomena that are encountered in interconnected systems and the new ideas that enable the control engineer to cope with the various problems raised by the need and the aims of decentralization, whereas straightforward extensions of multivariable control strategies to large-scale systems are reflected only insofar as they have proved to be of practical importance for interconnected systems.

Most of the theory of large-scale systems has been developed for linear, stationary, continuous-time deterministic systems. An exception is the field of stability analysis of interconnected systems, where linear and non-linear systems can be dealt with relatively simply within a common framework. Therefore, this text is restricted to linear systems, but extensions to non-linear systems are outlined or mentioned in the bibliographical notes.

The text is essentially divided into four parts. The first part presents the problems encountered in feedback control of large-scale systems and surveys the fundamentals for their solution. Chapter 1 gives a detailed characterization of large-scale control systems and outlines the ideas of decomposition, decentralization, approximation, and robustness analysis, which provide the basis of the analysis and control of large-scale systems in general. The main problems of feedback control are then characterized and a survey of relevant results is given.

Chapter 2 summarizes some results on multivariable feedback systems. Whereas the properties of controllability, observability and stability, and the design of feedback controllers are assumed to be known to readers and are only briefly described here for later reference, structural considerations and robustness analysis are explained in more detail. These methods turn out to be particularly important for the analysis and control of large-scale systems. In Chapter 3, descriptions of the different kinds of composite systems are given. Specific models are presented for systems with hierarchical structure, disjoint subsystems, overlapping subsystems and for temporarily separated subsystems.

The second part of the book describes the extension of the fundamental ideas and methods of centralized to decentralized control. Chapter 4 on decentralized stabilizability is organized in an analogous way to the corresponding part of Chapter 2. This direct comparison highlights the new problems which arise from the structural restrictions inherent in decentralized control. The decentralized servomechanism problem examined in Chapter 5 yields additional requirements for the controller dynamics, which have to be met in order to ensure asymptotic regulation in the closed-loop system.

Chapters 6 and 7 are devoted to two alternative design methods both of which presuppose knowledge of a complete model of the plant. Pole assignment and optimal control procedures are known from multivariable theory. They are extended here in order to satisfy the structural constraints imposed on the feedback law.

The third part presents methods that aim to reduce the knowledge about the plant that is required to solve the analytical and control tasks. The stability tests for interconnected systems given in Chapter 8 use only approximate descriptions of the subsystems and the interaction relations and, thus, can also be applied to incompletely known systems or systems with time-varying properties. They prove the connective stability of composite systems and, thus, ensure that the overall system remains stable even if some subsystems are disconnected. On this basis, design methods are developed in Chapter 9 which likewise do not presuppose complete knowledge of the process and yield controllers that are robust enough to withstand structural perturbations of the plant.

Chapter 10 deals with decentralized design methods, by means of which the control stations can be selected completely independently from each other. These methods are based on specific properties of the interconnected plant.

The final part concerns the decentralized control of specific systems. Chapter 11 describes specific design and on-plant tuning methods for decentralized proportional–integral (PI) controllers. In Chapter 12, large-scale systems are analysed on the basis of symmetry properties. These properties make it possible to derive an explicit relationship between the complexity of overall tasks and the robustness that the subsystem controllers have to possess. Moreover, the system properties can be presented depending upon the number of subsystems coupled together. The final chapter summarizes the results and discusses future advances of the theory of feedback control of large-scale systems.

All results are explained by means of examples, some of which are used repeatedly in different chapters. The relevant literature is summarized at the end of each chapter. These bibliographical notes cannot give a complete review of the literature but indicate the most influential papers in which the results presented here have been described and provide interested readers with adequate entry points to the still-growing literature in this field.

The symbols and conventions used throughout the book are summarized in the List of Symbols. The appendices give concise summaries of the basic results of matrix algebra and graph theory, which are used in the text.

1

Large-scale Control Systems

1.1 MAIN PROBLEMS AND BASIC IDEAS

The notion of large-scale systems came into use when it became obvious that there are practical control problems that cannot be solved efficiently by the principles and methods of multivariable systems and control theory. The reason for this is that the systems to be controlled are too 'large' and the problems to be solved too 'complex', in one sense or another, so that the amount of computation is too large to be manageable and even the basic assumptions of multivariable control are far from being satisfied.

Think, for example, of a multiarea power system with several power-generating units. It distributes electrical power over a wide geographical area and supplies many smaller and larger enterprises or private homes with energy. Voltages and power flows at many different points have to be controlled. Obviously, no precise uniform model can be set up and no unique controller can be implemented for such a system.

As a second example, imagine a system of coupled water reservoirs whose levels have to be controlled so as to ensure sufficient reserves of water and, simultaneously, prevent the reservoirs from overflowing after a period of rain. Here, the difficulties of the control problems result from the complexity of the dynamical interactions between the reservoirs and the uncertainties concerning the amount of outflow to the consumer and inflow from the environment.

Other examples of large-scale systems are traffic systems with complex dynamical behaviour but relatively few measurement data and control inputs, large space structures with many different components, ecological systems with a large number of entities in close interaction, or coupled distillation columns and reactors in the chemical industry, steel-rolling mills, flexible manufacturing systems, or gas distribution networks. These examples illustrate that large-scale systems differ substantially from conventional systems. Their salient features are, independently of their nature, the high dimensionality of the system equations, uncertainties in the information about the system, and constraints on the information flow for modelling, analysing, and controlling the system.

High dimensionality

A system has a large number of inputs and outputs; its components have manifold dynamical interactions; and it is exposed to various kinds of external disturbances. Consequently, the mathematical model of the system itself and of the system environment has a large dynamical order and includes many system parameters. If, for example, a power system consists of twenty power stations of dynamical order 6, then the overall system is of 120th order. The real-time constraints, which are imposed on the implementation of on-line control, make the dimensionality problem more severe. A solution to the control problem has to be found in time. Thus, the number of computational steps must not exceed a certain bound. Therefore, it is uneconomical or even impossible to solve analytical, design or control problems as a whole; these problems have to be simplified or decomposed into smaller ones.

Uncertainties

The properties and behaviour of the system under investigation cannot be completely described by some mathematical model. The reasons for this are manifold. On the one hand, a system with many different subsystems or a wide geographical distribution is not accessible to thorough identification. Even if an exact model is available, it has to be deliberately simplified in order to make analysis and control problems manageable. On the other hand, different phenomena within the system make it impossible to set up a model that is valid over the whole period of operation. Systems with many components cannot be expected to function together for the whole of this period. For example, power stations may be disconnected from the distribution network owing to decreasing energy demands. Every single model can be valid only under certain preconditions or, conversely, a single model can describe the system but only with considerable modelling errors.

These severe uncertainties necessitate an explicit characterization of the model errors and a consideration of these errors when solving analytical and design tasks.

Information structure constraints

There are restrictions on the type and transmission of information about the system. This refers to both *a priori* and *a posteriori* information. It is a typical situation in that the system is controlled not by one but by several control agents, each of which has a complete model and measurement data of its own subsystem but possesses only limited knowledge about the behaviour of the other subsystems. Moreover, the control agents may have contradictory goals. That is, no decision maker knows the system properties and the current system state completely.

Information structure constraints are imposed by the operating conditions of the system or by the control engineer to achieve simplicity of applications or flexibility of on-line control strategies. These constraints produce severe problems that require novel modelling, analysis and design methodologies.

Further peculiarities of large-scale systems result from new *analytical and control aims* as follows:

- The notion of optimality is less important than the aim of satisfying different, partly contradictory, requirements. Flexibility, reliability and robustness of control in the face of changing operating conditions are more important than a temporarily restricted optimality of the system performance. For example, a strategy for controlling interconnected water reservoirs is evaluated not only in terms of the average amount of available water but also with respect to its flexibility under non-typical weather conditions. Generally speaking, for large-scale systems the solution has to be 'reasonable' or 'advantageous' according to different criteria rather than optimal with respect to a single objective, such as the maximum deviation of the system output from a prescribed trajectory or the minimum time or control effort necessary to reach a goal state.
- The solution has to be applicable under structural perturbations of the process to be controlled. As a typical example, a power system with a varying number of power-generating units will be considered later to illustrate this situation.

These peculiarities of the system to be controlled and the scope of the required aims are different signs of conceptual difficulties that are encountered in large-scale systems and usually summarized under the term 'complexity'. Analytical and design problems for large-scale systems are not simply larger in size but more complex. That is, they cannot be solved by using faster computers with larger memories, but they raise new questions which cannot be answered by means of the methods that have been developed for 'small' systems. The complexity of large-scale systems makes the direct application of multivariable control methods unreasonable or even impossible and necessitates new analysis and design philosophies.

In the following, the basic ideas for analysing and controlling large-scale systems are outlined. These ideas are not restricted to feedback control but are also applicable to open-loop control, static optimization or planning and scheduling problems. Their ultimate goal is the reduction of a complex overall problem to conceptually or computationally simplified problems. Their specific form for feedback control will be considered in Section 1.2.

Decomposition

As the amount of computation required to analyse and control a system grows faster than the size of the system, it is beneficial to break down the whole problem into smaller subproblems, to solve these subproblems separately, and then to combine their solutions in order to get a global result for the original task. The subproblems are not independent. Some coordination or modification of the solutions of the subproblems is necessary in order to consider the interrelationships between the subproblems. The effort required to deal with the subproblems and their coordination can be allocated to various processors, which constitute a distributed computing system. Therefore, the concepts and techniques for reformulating a control problem as a set of interdependent subproblems and for solving these subproblems are often referred to as *distributed control.*

The basis for the decomposition of the analytical or control problems is often provided by the internal structure of the process to be controlled. Accordingly, the process is not considered as a single object but as a compound of different interacting subsystems. Decomposition methods utilize the system structure, which can be obtained from the building blocks of the process, or have to impose a structure for mathematical reasons.

A typical result of this decomposition is the *hierarchical structure* depicted in Figure 1.1. The doubly framed blocks indicate decision-making units whereas the single-frame blocks represent the subsystems of the process. In the general discussion here, the term 'decision maker' is used for the elements that do not belong to the process to be controlled but contribute to solving the given analytical or control problems. Later on, when feedback control of large-scale systems is con-

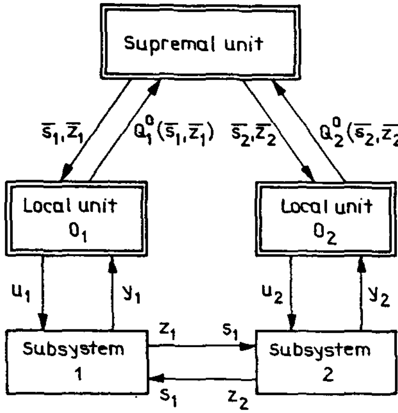


Figure 1.1 Multilevel system

sidered, these blocks will represent off-line design algorithms or feedback controllers.

Multilevel systems

Systems with the multilevel structure as shown in Figure 1.1 have been investigated mainly in connection with optimization problems. A given objective function $Q(\mathbf{y}, \mathbf{u})$, which depends on the process input $\mathbf{u}$ and output $\mathbf{y}$, has to be minimized by choosing an appropriate control input $\mathbf{u}^*$ satisfying the relation

$$\min_{\mathbf{u}} Q(\mathbf{y}, \mathbf{u}) = Q(\mathbf{y}^*, \mathbf{u}^*) = Q^* \tag{1.1.1}$$

for the system

$$\mathbf{y} = \mathbf{g}(\mathbf{u}). \tag{1.1.2}$$

$\mathbf{y}^*$ and Q^* are the optimal output or the optimal value of the objective function, which occur if $\mathbf{u}^*$ is used. Owing to the high dimensionality of the process (1.1.2) the optimal solution $\mathbf{u}^*$ lies within a large search space. Therefore, the objective function (1.1.1) as well as the process (1.1.2) should be decomposed. The overall goal $Q(\mathbf{y}, \mathbf{u})$ is represented as a sum of local goals $Q_i(\mathbf{y}_i, \mathbf{u}_i)$

$$Q(\mathbf{y}, \mathbf{u}) = \sum_{i=1}^{N} Q_i(\mathbf{y}_i, \mathbf{u}_i) \tag{1.1.3}$$

and the process is divided into subsystems

$$\mathbf{y}_i = \mathbf{g}_i(\mathbf{u}_i, \mathbf{s}_i) \qquad \mathbf{z}_i = \mathbf{h}_i(\mathbf{u}_i, \mathbf{s}_i) \tag{1.1.4}$$

which interact according to the relation

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{1.1.5}$$

where the global signals $\mathbf{s}$, $\mathbf{z}$, $\mathbf{y}$ and $\mathbf{u}$ consist of the local signals $\mathbf{s} = (\mathbf{s}_1', \ldots, \mathbf{s}_N')'$, $\mathbf{z} = (\mathbf{z}_1', \ldots, \mathbf{z}_N')'$, $\mathbf{y} = (\mathbf{y}_1', \ldots, \mathbf{y}_N')'$, $\mathbf{u} = (\mathbf{u}_1', \ldots, \mathbf{u}_N')'$. Then the global problem (1.1.1) and (1.1.2) can be replaced by the subproblems

$$\min_{\mathbf{u}_i} Q_i(\mathbf{y}_i, \mathbf{u}_i) \tag{1.1.6}$$

subject to

$$\mathbf{y}_i = \mathbf{g}_i(\mathbf{u}_i, \mathbf{s}_i). \tag{1.1.7}$$

Obviously, the subgoals Q_i depend through the interconnections (1.1.5) on the behaviour of all other subsystems. However, if a supremal unit tentatively prescribes optimal values $\bar{\mathbf{s}}_i$ and $\bar{\mathbf{z}}_i$ for the interaction signals, the subproblems (1.1.6) and (1.1.7) become independent local problems

$$\min_{\mathbf{u}_i} Q_i(\mathbf{y}_i, \mathbf{u}_i) = Q_i(\mathbf{y}_i^o, \mathbf{u}_i^o) = Q_i^o(\bar{\mathbf{s}}_i, \bar{\mathbf{z}}_i) \tag{1.1.8}$$

subject to

$$\mathbf{y}_i = \mathbf{g}_i(\mathbf{u}_i, \bar{\mathbf{s}}_i) \qquad \bar{\mathbf{z}}_i = \mathbf{h}_i(\mathbf{u}_i, \bar{\mathbf{s}}_i) \tag{1.1.9}$$

with fixed $\bar{\mathbf{s}}_i$ and $\bar{\mathbf{z}}_i$. $\mathbf{u}_i^o$ is the solution of the local problem (1.1.8) and (1.1.9), and $\mathbf{y}_i^o$ and Q_i^o are the locally optimal values that result after applying the control $\mathbf{u}^o = (\mathbf{u}_1^{o\prime}, \mathbf{u}_2^{o\prime}, \ldots, \mathbf{u}_N^{o\prime})'$ to the overall system (1.1.2). These problems can be solved in separate local units $\mathbf{O}_i$. The aim of the supremal unit is to determine values $\bar{\mathbf{s}}_i$ for which the global goal is satisfied

$$\min_{\bar{\mathbf{s}}} \sum_{i=1}^{N} Q_i^o(\bar{\mathbf{s}}_i, \bar{\mathbf{z}}_i) = Q^* \quad \text{subject to } \bar{\mathbf{s}} = \mathbf{L}\bar{\mathbf{z}}. \tag{1.1.10}$$

As the interconnection signals serve here as coordination variables, this method is also called ‘direct coordination’.

Instead of solving the high-dimensional overall problem (1.1.1), solutions to the lower-dimensional local problems (1.1.8) and (1.1.9) have to be determined. The price for this simplification is the necessity to solve all these problems more than once. As the solution of the supremal problem (1.1.10) cannot be found in one step, the process of solving the global problem (1.1.1) and (1.1.2) consists of several repetitions of coordination and local optimization steps.

This outline of hierarchical optimization clearly illustrates the main ideas. First, the decomposition of the overall problem (1.1.1) and (1.1.2) yields a set of local problems (1.1.8) and (1.1.9) and a coordination problem (1.1.10). The dependencies of all these subproblems are temporarily neutralized by the prescription of an order in which they have to be solved. First, the supremal unit prescribes interconnection signals $\bar{\mathbf{s}}_i$ and $\bar{\mathbf{z}}_i$. Second, the local units solve their problems (1.1.8) and (1.1.9) for these interactions. Third, the supremal unit makes the next search step for solving the problem (1.1.10) and prescribes new values $\bar{\mathbf{s}}_i, \bar{\mathbf{z}}_i$, etc. The information structure of this multilevel optimization procedure is depicted in Figure 1.1.

The coordination is necessary because the locally optimal solutions $\mathbf{u}_i^o$ do not, in general, coincide with the solution of the given global problem (1.1.1) and (1.1.2). The local problems (1.1.8) and (1.1.9) have to be modified by changing the values $\bar{\mathbf{s}}_i, \bar{\mathbf{z}}_i$ until they lead to globally optimal solutions $\mathbf{u}_i^o = \mathbf{u}_i^*$.

The crucial problem is that decomposition will achieve its aim only if the resulting subproblems are 'weakly coupled' so that the coordination problem (1.1.10) can be solved in a few steps. As the strength of the dependencies of the subproblems is closely related to the strength of the interactions (1.1.5) between the subsystems, the decomposition of the system (1.1.2) is usually carried out with the aim of finding weakly coupled subsystems (1.1.7). The question of what 'weakly coupled' means can be answered in different ways. As will be discussed in Chapter 3 in detail, weak couplings occur, for example, in systems with hierarchical interconnection structure, among subsystems with small-magnitude interactions or with quite different time constants.

In a multilevel system, the overall solution is found only after the subproblems have been solved and information is exchanged several times between the supremal and the local units. Therefore, this scheme is applicable to off-line decision problems such as the design problem of decentralized controllers, but not to on-line feedback control.

Multilayer systems

A second kind of decomposition concerns the division of the functions to be carried. It leads to multilayer systems (Figure 1.2). Unlike multilevel systems where all units contribute to satisfying the same goal, multilayer systems consist of units with different objectives.

In the three-layer system of Figure 1.2(a) the unit on the strategic layer at the top has to attenuate long-term disturbances of possibly high magnitude by prescribing reasonable control aims for the second layer.

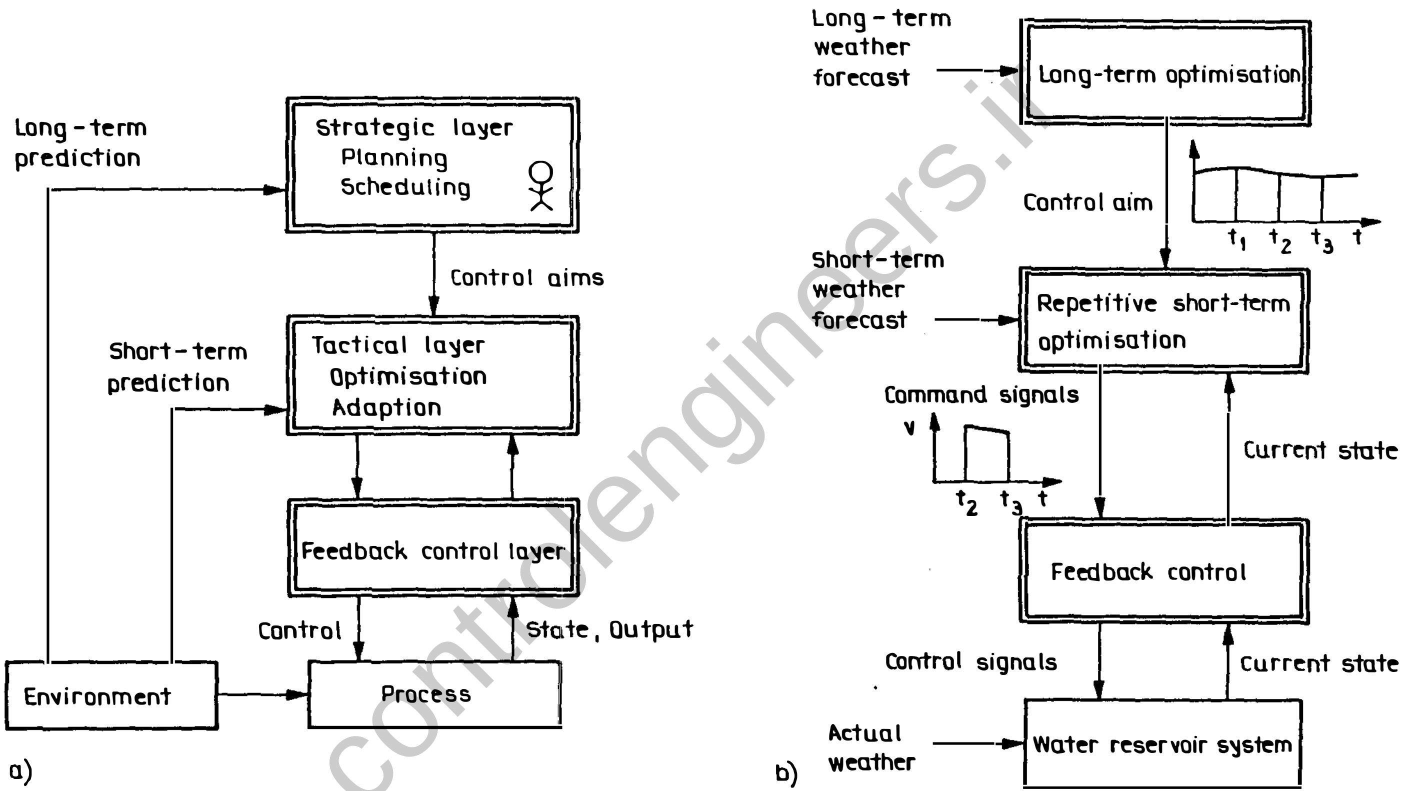


Figure 1.2 Multilayer system: (a) structure; (b) example: management of a water reservoir system

Its task includes planning, scheduling or resource allocation problems. The system serves as a computerized decision support system that proposes solutions, the best of which has to be selected by the human operator. The task on the top layer has to be accomplished usually once in a long time period.

The unit on the tactical layer has to find reasonable set points or to adapt the lower-layer tasks to changing disturbances from the environment. Its tasks have to be repeated often, particularly after unusually strong disturbances have occurred.

The unit on the lowest layer deals with feedback control. It has to ensure set point tracking, to stabilize the process in all operating points, and to attenuate unmeasurable high-frequency disturbances. It works continuously.

The tasks of the units on the different layers can be illustrated for a water management problem (Figure 1.2(b)). The unit on the strategic layer is provided with predictions of the water supply from the environment and the consumer demands for the next year. It has to determine preferable long-term trajectories for the water levels in all the reservoirs. The task of the tactical layer unit is to prescribe actual set points for the feedback controller so as to reach the long-term trajectories during the next month despite current or recent deviations of supply or demand from their long-term predictions. The feedback controller on the lowest layer has to ensure set point tracking from day to day.

Results

Both kinds of decomposition replace the overall problem by a set of subproblems that can be solved by units with limited capability and under time restrictions. Instead of using a unique decision maker for the whole problem, decomposition yields a structure of several decision-making units. Moreover, the relative autonomy of the units improves flexibility and system reliability. After the decomposition of a given problem, information about the control aims and the process is not available in one central unit but distributed over several local and supremal units. For example, the local unit $\mathbf{O}_1$ in the multilevel system of Figure 1.1 needs access only to the model (1.1.9) of the pertinent subsystem and the subgoal Q_1. No decision-making unit receives all the available data.

Large-scale systems are often controlled by a net of decision-making units whose tasks result from mixed multilevel–multilayer decomposition of the overall system. This should be kept in mind when the aims and practical circumstances of the feedback control layer are investigated later. Feedback control represents the lowest of several layers. Furthermore, multilayer and multilevel structures are used in several ways as design or implementation schemes for decentralized controllers.

Decentralization

The units of a hierarchical structure are not completely independent but have to respond to data delivered by other units. Although this communication ensures that the global goals will eventually be satisfied, it prescribes the decision makers certain working regimes.

Decentralization concerns the information structure of the decision-making process. In decentralized decision making the decision units are completely independent or at least almost independent. That is, the network, which describes the information flow among the decision makers, can be divided into completely independent parts. The decision makers belonging to different subnetworks are completely separate from each other. Since such a complete division is possible only for specific problems, the term 'decentralized' will also be used if the decision makers do communicate but this communication is restricted to certain time intervals or to a small part of the available information.

The outline of multilevel systems has shown that some coordination and, thus, communication among the decision-making units is necessary if the overall goal is to be reached. In decentralized structures, such a coordination is impossible or restricted in accordance with the information exchange that is permitted. However, because of the simplifications in the practical implementation of coupled decision makers, which are gained from the absence of information links, decentralized structures are often used but reduce the quality of the solution.

If, for example, one gives up the aim of reaching the best possible solution of the optimization problem (1.1.1) and removes the supremal unit of the multilevel system in Figure 1.1, the local units become completely independent (Figure 1.3). Then the interaction signals $\mathbf{s}_i$ may be fixed at some value $\bar{\mathbf{s}}_i$ once or have to be determined when solving the local problems. In the former case the local problems (1.1.8) and (1.1.9) only have to be solved once. The application of the locally optimal controls $\mathbf{u}_i^o$ yields a suboptimal solution to the global problem (1.1.1) and (1.1.2). In the latter case the local optimization problems are

$$\min_{\mathbf{u}_i, \mathbf{s}_i} Q_i(\mathbf{y}_i, \mathbf{u}_i) = Q_i^1$$

subject to eqn (1.1.4) and yield the locally optimal value Q_i^1. The solutions $(\mathbf{u}_i^o, \mathbf{s}_i^o)$ will, in general, violate the interaction relation $\mathbf{z}^o = \mathbf{L}\mathbf{s}^o$ with $\mathbf{z}_i^o = \mathbf{h}_i(\mathbf{u}_i^o, \mathbf{s}_i^o)$. They are called non-feasible solutions. The process

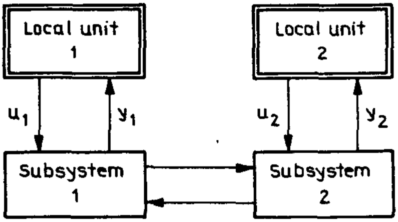


Figure 1.3 Decentralized control structure

under control $\mathbf{u}_i^o$ will lead to interconnection signals $\mathbf{s}_i \neq \mathbf{s}_i^o$ and goal values $Q_i(\mathbf{y}_i, \mathbf{u}_i^o) \neq Q_i^1$.

This example clearly shows the merits and difficulties of decentralized decision making. On the one hand, the solution is obtained by means of completely independent decision makers. Each decision maker needs only a limited amount of information rather than the whole process model and objective function. No synchronization of the activities of the autonomous units has to be implemented. This is advantageous with respect not only to the time and effort required to solve the problems, but also to the flexibility of the control units when perturbations within the plant occur. If the properties of a subsystem change, only the model of the decision maker of this subsystem has to be adapted. If a subsystem is disconnected from the others, its decision-making unit is simultaneously disconnected from the overall system but remains in operation at the subsystem level (cf. Figure 1.3). As far as the implementation of the control algorithm is concerned, decentralization supports the application of parallel computer architectures and fault-tolerant systems.

On the other hand, since the independently received decisions may not be compatible, decentralization turns out to be more involved than decomposition. The subproblems have not only to be temporarily exempted from their interdependencies by a prescription of the order in which they have to work, but also to be made completely independent of each other. This is possible if the interrelations of the subproblems are weak or can be made weak by appropriate modifications.

In feedback control, which can be considered as a two-step decision problem as explained in the Introduction, constraints on the information structure can be imposed on the design phase, the execution phase, or both:

- A decentralized controller is a feedback controller which consists of independent control stations, each of which receives the measurement data $\mathbf{y}_i$ and influences the control input $\mathbf{u}_i$ only of the attached subsystem (cf. Figure 1.3 where the local units represent control stations $\mathbf{u}_i = \mathbf{K}_i\mathbf{y}_i$, or the figure in the Introduction). The control laws, which describe how $\mathbf{u}_i$ has to be chosen for given $\mathbf{y}_i$, are independent of each other. The information flow from the plant through the controller back to the plant is divided into separate flows through the control stations.

- Decentralized design schemes result from a division of the overall design problem into subproblems that can be solved independently (see the figure in the Introduction).

Both kinds of decentralization can only be applied if the overall process can be decomposed into subsystems which are, in one sense or another, weakly coupled.

With this division of the whole control problem into several subproblems, which have to be solved from different information about the same process, new problems occur. The aims followed by the decision makers may be in harmony or contradictory. The amount and kind of information shared by the decision makers influence the solvability of the overall problem. Therefore, the information structure, under which the control problem is to be solved, has to be considered in detail. This is a new aspect of large-scale systems in comparison with 'small' systems, and will become evident in many problems tackled later on.

Approximation and robustness analysis

Owing to the high dimensionality and uncertainties of the models, approximation and robustness analysis are important means of dealing with large-scale systems. Approximation aims at the substitution of the given model or goals by similar but simpler models or goals, respectively. In addition to the uncertainties that appear in the model at hand, approximation brings about further uncertainties. Robustness analysis concerns the evaluation of the dependencies of the solution upon structural or parametric uncertainties in the model used (Figure 1.4). Every solution obtained by means of the approximate model must be robust enough to represent a reasonable solution to the original problem. Robustness analysis is, therefore, one of the main tools in large-scale systems theory.

Approximation and robustness analysis are known from multivariable control theory but have to be extended for large-scale systems. For example, model aggregation has to be applied to the subsystems rather than to the overall process as a whole. Robustness analysis has to exploit the character of uncertainties that are brought about by

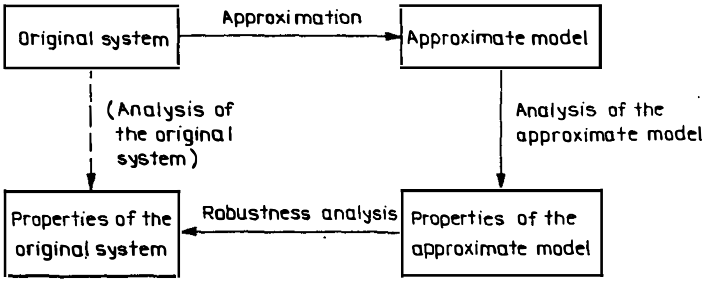


Figure 1.4 Approximation and robustness analysis

structural perturbations within an interconnected system. Such specific methods are the bases of stability analysis of composite systems (Chapter 8) and decentralized design methods (Chapters 9–12).

Summary

These facts clearly show that there is no formal definition of the term 'large-scale system'. Instead, a more pragmatic view has to be adopted: a system is considered large whenever it is necessary to partition the given analysis or design problem in order to come up with manageable subproblems. For feedback control, the use of decentralized controllers is a remarkable feature of large-scale systems.

1.2 DECENTRALIZED CONTROL

Our attention will be focused on the lowest layer of control systems (Figure 1.2(a)). There, the control task can be solved under the following *general assumptions*, which are known from multivariable feedback:

- The process to be controlled can be described by some linear model

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{Md} \qquad \mathbf{y} = \mathbf{Cx} + \mathbf{Du} + \mathbf{Nd}. \tag{1.2.1}$$

- The control problem can be solved by some linear feedback which may have its own dynamics or, in the simplest case, a static control law

$$\mathbf{u} = \mathbf{K}(\mathbf{y} - \mathbf{v}). \tag{1.2.2}$$

- The disturbances $\mathbf{d}(t)$ are, in general, unmeasurable but known to belong to a given class of signals which may be described by some disturbance model

$$\dot{\mathbf{x}}_d = \mathbf{A}_d\mathbf{x}_d \qquad \mathbf{x}_d(0) = \mathbf{x}_{d0} \qquad \mathbf{d} = \mathbf{C}_d\mathbf{x}_d. \tag{1.2.3}$$

The same assumption can be made for the command signal $\mathbf{v}(t)$ generated on the tactical layer (Figure 1.2(a))

$$\dot{\mathbf{x}}_v = \mathbf{A}_v\mathbf{x}_v \qquad \mathbf{x}_v(0) = \mathbf{x}_{v0} \qquad \mathbf{v} = \mathbf{C}_v\mathbf{x}_v. \tag{1.2.4}$$

Typical examples of command or disturbance signals are step functions with unknown step height, sinusoidal functions of given frequencies or ramp functions.

The reasons for using feedback controllers are given by the requirements that the closed-loop system is stable, ensures asymptotic regulation and satisfies restrictions on the dynamical properties as given, for example, by bounds on the system response to given test signals.

Restrictions on the feedback control of large-scale systems

The assumptions and control aims mentioned above are well known from classical and multivariable control theory. Large-scale systems give rise to additional assumptions, requirements or restrictions. For the feedback control layer (Figure 1.2(a)), the typical characteristics of large-scale systems described in Section 1.1 can be given more specifically as follows.

Concerning the feedback law, large-scale systems are characterized by *restrictions on the controller structure* as follows:

- Sensors and actuators are geographically separate. Owing to unreasonable costs or delays in data transmission, centralized on-line control is impossible and a decentralized control structure is therefore required (Figure 1.3).
- A decentralized control structure has to be chosen for reliability reasons. If one decentralized controller fails to operate, only one part rather than the whole feedback layer fails to do its task.
- The process has to be controlled by several feedback loops, but these loops are not loosely coupled. Therefore, the interactions of the controllers through the common plant have to be taken into account in the design phase.

For these reasons, no centralized feedback law can be used. Instead, the controller has to consist of several independent control stations

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{r}i}(t) &= \mathbf{F}_i \mathbf{x}_{\mathrm{r}i}(t) + \mathbf{G}_i \mathbf{y}_i(t) + \mathbf{H}_i \mathbf{v}_i(t) \\ \mathbf{u}_i(t) &= -\mathbf{K}_{xi} \mathbf{x}_{\mathrm{r}i}(t) - \mathbf{K}_{yi} \mathbf{y}_i(t) + \mathbf{K}_{vi} \mathbf{v}_i(t) \end{aligned} \tag{1.2.5}$$

($i = 1, \ldots, N$). Then, the overall controller has the form

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{r}} &= \operatorname{diag} \mathbf{F}_i \mathbf{x} + \operatorname{diag} \mathbf{G}_i \mathbf{y} + \operatorname{diag} \mathbf{H}_i \mathbf{v} \\ \mathbf{u} &= -\operatorname{diag} \mathbf{K}_{xi} \mathbf{x} - \operatorname{diag} \mathbf{K}_{yi} \mathbf{y} + \operatorname{diag} \mathbf{K}_{vi} \mathbf{v} \end{aligned} \tag{1.2.6}$$

where diag $\mathbf{F}_i$ etc. is a block-diagonal matrix with the diagonal blocks $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N$.

Specific control laws are those for the decentralized state feedback

$$\mathbf{u}_i = -\mathbf{K}_i \mathbf{x}_i \tag{1.2.7}$$

for the decentralized output feedback

$$\mathbf{u}_i = -\mathbf{K}_{yi} \mathbf{y}_i \tag{1.2.8}$$

or for the decentralized PI controller

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{r}i} &= \mathbf{y}_i - \mathbf{v}_i \\ \mathbf{u}_i &= -\mathbf{K}_{\mathrm{I}i} \mathbf{x}_{\mathrm{r}i} - \mathbf{K}_{\mathrm{P}i} (\mathbf{y}_i - \mathbf{v}_i) \end{aligned} \tag{1.2.9}$$

with proportional or integral gain matrices $\mathbf{K}_{\mathrm{P}i}$ or $\mathbf{K}_{\mathrm{I}i}$, respectively. The subsystem state $\mathbf{x}_i$ or subsystem output $\mathbf{y}_i$ have only to be locally available. Eqn (1.2.6) shows that the decentralization of the controller imposes constraints on the feedback matrices, which are visible by certain matrix elements fixed at zero. All matrices are block diagonal where the ith diagonal blocks are identical to the corresponding matrices of the ith control station. The set of input and output variables that can be linked by the controller are also referred to as channel $(\mathbf{u}_i, \mathbf{y}_i)$. Since the restrictions on the control law prescribe vanishing elements but do not restrict the range of the non-vanishing controller parameters they are said to be structural restrictions on the control law.

All admissible controller matrices can be described by structure matrices **S**, whose elements are either zero (0) or indeterminate (*) (for an explanation see Section 2.5). The asterisk '*' denotes elements of the feedback matrix that may be chosen freely. The structure matrix $\mathbf{S}_k$ has an indeterminate element in the position (i, j) if an information link may be realized from the jth output to the ith input of the overall system. If [**A**] denotes a matrix that is derived from a given numeric matrix **A** by replacing all non-vanishing elements by *, the admissible matrices **K** of a given structural constraint $\mathbf{S}_k$ belong to the set $\mathscr{K}$

$$\mathbf{K} \in \mathscr{K} = \{\mathbf{K}: \ [\mathbf{K}] = \mathbf{S}_k\}. \tag{1.2.10}$$

For example, the matrix of a decentralized state feedback (1.2.7) of a system with two second-order subsystems and two inputs each has the form

$$\mathbf{K} = \begin{pmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & k_{33} & k_{34} \\ 0 & 0 & k_{43} & k_{44} \end{pmatrix}$$

where eight elements are fixed to zero. With the symbol defined above the set of all possible decentralized feedback matrices is

$$\mathscr{K} = \left\{ \mathbf{K}\colon\ [\mathbf{K}] = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}.$$

Although, in general, the structural constraints need not be restricted to decentralized control but may be given by any structure matrix $\mathbf{S}_f$, the term 'feedback control of large-scale systems' will often be used synonymously with 'decentralized control' as done here.

The following *restrictions on the model* of the overall system are characteristic for large-scale systems:

- The overall system consists of different subsystems controlled by different decision makers, none of which has access to a complete model of the overall plant. Hence, the decentralized control loops have to be designed independently on the basis of partial, uncertain plant models.
- As the subsystems have different aims from a technological point of view, their requirements on the feedback layer are contradictory.
- Structural changes of the plant put requirements on the flexibility and robustness of the feedback controller.

Consequently, no complete model of the overall system is available. Instead, the different control stations have to be designed independently of each other on the basis of partial models of the system. This way of designing systems will be referred to as *decentralized design*.

Obviously, these circumstances of the practical application of feedback control restrict the flow of *a priori* and *a posteriori* information among the control agents which design the controller or the different parts of the controller. Decomposition, decentralization, approximation, and robustness analysis have to be applied to the feedback layer in order to find appropriate analytical and control methods.

Moreover, the requirements on the performance of the closed-loop system have to be extended in comparison with multivariable systems where the following specifications are dealt with:

1. The closed-loop system has to be stable.
2. Asymptotic regulation $\mathbf{y}_i - \mathbf{v}_i \to 0$ for $t \to \infty$ has to occur for a given class of command signals $\mathbf{v}_i$ and disturbances entering the plant at arbitrary points.
3. The dynamical I/O behaviour has to be well suited to give specifications on, for example, the step response (Figure 1.5).
4. The requirements 1–3 above have to be satisfied independently of a given class of model uncertainties.

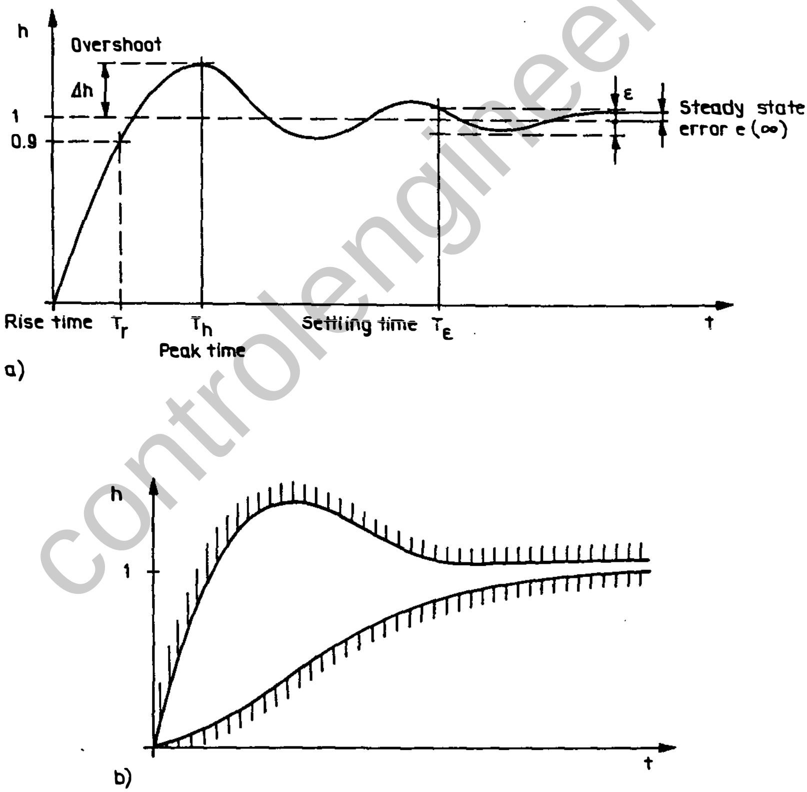


Figure 1.5 Design requirements (3): (a) characteristic values of the step response that have to satisfy given specifications; (b) prescribed band for the command step response

The extensions of these specifications concern the following items:

- The closed-loop system has to remain stable even if some control stations or some subsystems go out of operation. These properties are called integrity (as known from multivariable systems) and connective stability (cf. Section 8.5).
- The autonomy of the subsystems should be preserved or improved by the feedback controller.
- The decentralized controller should be tuned and brought into operation in a sequential way while always satisfying the least requirements such as sequential stability (cf. Section 5.2).

The Main Problems of Feedback Control of Large-scale Systems

The restrictions and requirements posed by large-scale systems include all the basic problems of systems and control theory, that is identification, modelling, model reduction, analysis of the plant concerning controllability, observability, stabilizability, optimization, etc. This book deals mainly with those problems that are related to the design and application of feedback controllers. The following questions outline the scope of this part of control theory.

The fundamental question raised by the feedback control of large-scale systems is the following:

Under what conditions is a decentralization of the design process and control law possible?

This question should be answered not only by developing separation theorems of design problems or existence theorems of decentralized controllers, which describe the possibility of decentralization, but also by restrictions that have to be satisfied if decentralized control should be made practically applicable. The question concerns the kind and amount of locally available *a priori* or *a posteriori* information that has *necessarily* to be transferred to the control agents if a given control aim should be reached. Since the character and strength of the couplings among the subsystems and the possibility of exchanging information among the control agents are important for answering this fundamental question, several important problems have to be solved:

- *In what way does the performance of the overall system depend on the subsystem properties and on the interactions between the subsystems?*

This question requires knowledge of the phenomena that occur within interconnected systems. It has to be investigated by considering the interactions between the subsystems which improve or degrade overall system performance according to global aims. Since this question cannot be answered in general but only for specific control structures or systems (Chapters 11 and 12), answers are given to the more restricted questions of what subsystems can be considered weakly coupled and how the overall system can be decomposed into such subsystems. Methods for detecting a hierarchical order of the subsystems, low-magnitude interactions or subsystems with quite different time constants are described in Chapter 3. The couplings between such subsystems can be neglected during the analysis because the solution obtained for the isolated subsystem is close to the overall system result.

An opposite method is used in the stability analysis of composite systems (Chapter 8). Starting from a given decomposition, upper bounds for the interactions are found so that the stability of the isolated subsystems implies the stability of the overall system.

- *How does a system behave under the control of different independent control stations?*

Since all the individual control stations act on the same plant, they have to be 'compatible' in the sense that together they produce a reasonable system behaviour. The question of which control stations are 'compatible' is rather involved. For example, although the control stations produce nicely stable subsystems, their simultaneous application may lead to an overall system which is unstable.

The existence of stabilizing decentralized controllers can be answered completely in terms of the existence of fixed modes (Chapter 4) which turn out to be generalizations of the unobservable or uncontrollable modes of centralized control. If fixed modes do exist, the stabilization of an unstable system by decentralized controllers is possible only if non-linear or time-varying control laws are used or if the control stations exchange measurement data and, thus, are not completely decentralized.

- *In what way can decentralized feedback controllers be designed?*

New methods for choosing appropriate control laws are necessary. On the one hand, the modern methods of pole assignment or optimal control, which are known from multivariable systems design, can be extended to controllers that satisfy structural constraints. This way leads to centralized design methods which presuppose the availability of a model of the whole plant but lead to decentralized feedback laws

(Chapters 6 and 7). Hierarchical design schemes take advantage of the structure of the plant but belong to centralized design methodologies too.

On the other hand, new methodologies have to be found that take account of information structure constraints on the design process. Such methodologies aim at a reasonable reformulation of the design problem so that the individual control stations of a decentralized controller can be determined independently of each other but are known to satisfy the overall system specifications after all of them have been applied to the plant (Chapters 9 and 10). The main question posed by the decentralization of the design process concerns the distribution of *a priori* information among the design units.

The decentralization of the design process leads to design tasks which refer to some uncertain plant. This is illustrated in Figure 1.6(a). When the ith control station is designed, subsystem i is considered as the system to be controlled. This system has an unknown input $\mathbf{s}_i$, which obviously depends upon the interconnection output $\mathbf{z}_i$ of the ith subsystem. Therefore, the system to be controlled has severe uncertainties which are brought about by the neglected subsystems and the possibly unknown control stations. The unknown signal $\mathbf{s}_i$ cannot be considered as a 'disturbance' input which is produced by some unknown but fixed signal generator independently of $\mathbf{u}(t)$ (Figure 1.6(b)).

Obviously, these questions pose many new problems for feedback control. Decentralization presupposes methods for selecting appropriate design and on-line control structures. That is, the information structure

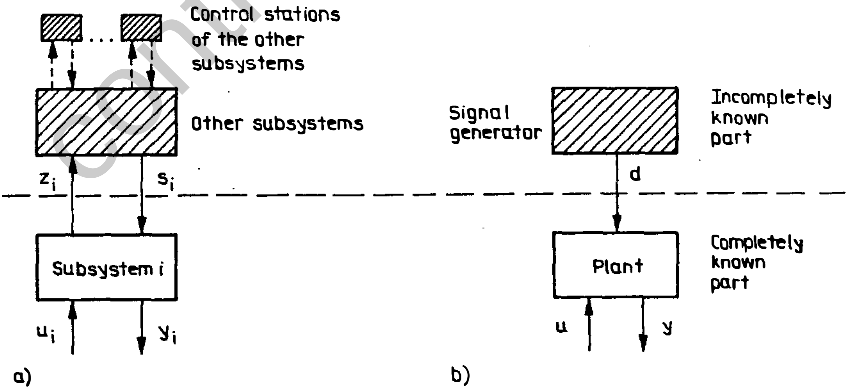


Figure 1.6 Uncertainties of the behaviour of the plant: (a) system with uncertain subsystems and control stations; (b) system subjected to disturbances

in which the control task should be solved is itself an object of investigation. Pertinent questions can be answered only if, in one way or another, the value of information concerning the design or control tasks can be evaluated. This point has never been investigated in the simpler problems of multivariable control, since there complete model and measurement information has been presupposed. It is solved for decentralized control only indirectly by elaborating criteria under which a decentralized solution can be really applied.

BIBLIOGRAPHICAL NOTES

The origin of large-scale systems theory was laid as early as in 1960 by Dantzig and Wolfe who decomposed linear programming problems due to the sparsity of the system matrix. The multilevel approach was proposed by Mesarović and co-workers (Mesarović *et al.* 1970a,b; Cohen 1978). Its application and extension to different kinds of control problems and information structure constraints of practical relevance, in particular to hierarchical closed-loop control and hierarchical structures under disturbances, is reported in the monographs by Singh and Titli (1978, 1979), Findeisen *et al.* (1980), Singh (1980), and Bernussou and Titli (1982). Nearly all publications have been devoted to multilevel structures. Multilayer structures, although often used empirically in practice, are lacking a formal analytical treatment. Restorick (1984) is one of the few authors who tried to elaborate systematic ways for decomposing a problem into a multilayer hierarchy. A readable account of the main ideas of hierarchical control under the consistency or disagreement of the interests of decision makers has been given by Findeisen (1982). The role of model aggregation with particular emphasis on large-scale systems has been described by Aoki (1968, 1971b) and in the monographs by Mahmoud and Singh (1981a) and Jamshidi (1983).

Reinisch (1986) emphasized the problems encountered in non-industrial applications, for example hierarchical schemes for water management problems (Reinisch *et al.* 1987; Reinisch and Hopfgarten 1989), which have been outlined in Section 1.1.

Aoki (1971a) was one of the first who considered decentralized control as a field where intensive research is needed. The field developed rapidly since the late 1970s. Javdan and Richards (1977), Tenney and Sandell (1981a,b), Schmidt (1982), or Šiljak (1983) expressed interesting views on the origin and nature of large-scale control systems. Papers and monographs by Sandell *et al.* (1978), Šiljak (1978), Lunze (1980a), Singh

(1981), Litz (1983), Jamshidi (1983), Voronov (1985), Bakule and Lunze (1988) and Travé *et al.* (1989) surveyed the state of the art. Particular emphasis on stability analysis was made by Michel and Miller (1977) and Grujić *et al.* (1987). Mahmoud and Singh (1981a) summarized methods for modelling large-scale systems.

Applications of decentralized control to river quality control and production management have been described by Tamura (1979) and Hauri and Hung (1979). Electric power systems have been the subject of many application studies. A survey of power systems as large-scale control systems in general was given by Quazza (1976), whereas an overview of decentralized control problems was published by Jamshidi and Etezadi (1982). Joshi (1989) described the feedback control of large flexible space structures. Iftar and Davison (1990) used robust decentralized controllers for dynamic routeing in communication networks.

2

Results on Multivariable Feedback Systems

2.1 MODELS OF MULTIVARIABLE SYSTEMS

Models of feedback systems have to describe the performance of the plant or the closed-loop system in the vicinity of a given operating point. This section surveys standard forms of the system description in the time domain.

State space description

Non-linear dynamical systems are described by a set of n first-order differential equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t), t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t), t) \end{aligned} \tag{2.1.1}$$

where $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{d}$ are the vectors of the system states, inputs, outputs, or disturbances of dimensions n, m, r or q, respectively (Figure 2.1); $\mathbf{f}$ and $\mathbf{g}$ are non-linear vector functions which, for every instant of time t, unambiguously determine vectors $\dot{\mathbf{x}}(t)$ and $\mathbf{y}(t)$ in terms of $\mathbf{x}(t)$, $\mathbf{u}(t)$ and $\mathbf{d}(t)$. The model (2.1.1) is used for $t > 0$ under the assumption

$$\mathbf{u}(t) = 0 \qquad \mathbf{d}(t) = 0 \quad \text{for } t < 0 \tag{2.1.2}$$

because the effect of the inputs for $t < 0$ are described by the initial state x_0.

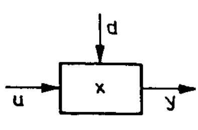


Figure 2.1 System described by eqn (2.1.1)

For linear systems, eqn (2.1.1) has the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{M}(t)\mathbf{d}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) + \mathbf{N}(t)\mathbf{d}(t)\end{aligned} \tag{2.1.3}$$

where **A**, **B**, **C**, **D**, **M** and **N** are matrices of appropriate dimensions. If the system parameters do not depend on time, a linear time-invariant model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{M}\mathbf{d}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{N}\mathbf{d}(t)\end{aligned} \tag{2.1.4}$$

with constant matrices can be used.

I/O description

In the time domain, the I/O description involves the convolution integrals

$$\mathbf{y}(t) = \mathbf{G}_0(t)\mathbf{x}_0 + \mathbf{G}_{yu}(t) * \mathbf{u}(t) + \mathbf{G}_{yd}(t) * \mathbf{d}(t) \tag{2.1.5}$$

where $\mathbf{G}_{yu}$ and $\mathbf{G}_{yd}$ are the impulse response matrices of the system with respect to input **u** or disturbance **d**, respectively. The asterisk symbolises the convolution operation

$$\mathbf{G}(t) * \mathbf{u}(t) = \int_0^\infty \mathbf{G}(t-\tau)\mathbf{u}(\tau)\,\mathrm{d}\tau.$$

If the model (2.1.5) is used for the linear time-invariant system (2.1.4), then

$$\begin{aligned}\mathbf{G}_0(t) &= \begin{cases}\mathbf{C}\exp(\mathbf{A}t) & \text{for } t \geqslant 0 \\ \mathbf{0} & \text{for } t < 0\end{cases} \\ \mathbf{G}_{yu}(t) &= \begin{cases}\mathbf{D}\delta(t) + \mathbf{C}\exp(\mathbf{A}t)\mathbf{B} & \text{for } t \geqslant 0 \\ \mathbf{0} & \text{for } t < 0\end{cases} \\ \mathbf{G}_{yd}(t) &= \begin{cases}\mathbf{N}\delta(t) + \mathbf{C}\exp(\mathbf{A}t)\mathbf{M} & \text{for } t \geqslant 0 \\ \mathbf{0} & \text{for } t < 0\end{cases}\end{aligned} \tag{2.1.6}$$

hold where $\delta(t)$ denotes the Dirac impulse.

The time domain I/O model of feedback systems (Figure 2.2) is

$$\mathbf{y}(t) = \overline{\mathbf{G}}(t) * \mathbf{u}(t)$$

with

$$\overline{\mathbf{G}}(t) = \mathbf{G}_1(t) - \mathbf{G}_1(t) * \mathbf{G}_2(t) * \overline{\mathbf{G}}(t). \tag{2.1.7}$$

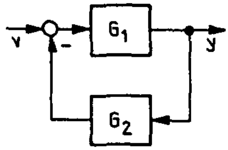


Figure 2.2 Feedback system

Since the inverse convolution operation has no customarily used symbol, $\overline{\mathbf{G}}(t)$ is described by eqn (2.1.7) implicitly. Eqn (2.1.7) can be solved recursively and yields $\overline{\mathbf{G}}(kT)$ for $k=1,2,\ldots$ for a given time step T.

If an initially quiescent system (eqn (2.1.4) or (2.1.5) for $\mathbf{x}_0=\mathbf{0}$) is excited by a step input $\mathbf{u}=\bar{\mathbf{u}}\sigma(t)$ and $\mathbf{d}=\mathbf{0}$, where $\sigma(t)$ denotes the unit step function, the output is given by

$$\mathbf{y}(t)=\int_0^t \mathbf{G}_{yu}(\tau)\,\mathrm{d}\tau\bar{\mathbf{u}}.$$

The matrix

$$\mathbf{H}(t)=\int_0^t \mathbf{G}_{yu}(\tau)\,\mathrm{d}\tau \tag{2.1.8}$$

denotes the step response matrix of the system (2.1.5). For stable systems (2.1.5) the matrix

$$\mathbf{K}_s=\lim_{t\to\infty}\mathbf{H}(t)=\mathbf{H}(\infty)=\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} \tag{2.1.9}$$

represents the static transmission matrix (matrix of static reinforcement) of the system (2.1.4) or (2.1.5), respectively. The static I/O behaviour of the system (2.1.4) or (2.1.5) is described by

$$\mathbf{y}=\mathbf{K}_s\mathbf{u}. \tag{2.1.10}$$

Model of the controller

The same kind of models as described so far for the system to be controlled can be used for the controller. In a rather general form the linear control law is written as

$$\begin{aligned}\dot{\mathbf{x}}_r(t)&=\mathbf{F}\mathbf{x}_r(t)+\mathbf{G}\mathbf{y}(t)+\mathbf{H}\mathbf{v}(t) \qquad \mathbf{x}_r(0)=\mathbf{x}_{r0}\\ \mathbf{u}(t)&=-\mathbf{K}_x\mathbf{x}_r(t)-\mathbf{K}_y\mathbf{y}(t)+\mathbf{K}_v\mathbf{v}(t)\end{aligned} \tag{2.1.11}$$

where $\mathbf{x}_r$ and $\mathbf{v}$ represent the n_r-dimensional vector of controller states or the r-dimensional vector of command signals, respectively. The controller (2.1.11) with a dynamical part is often referred to as a compensator. Particular forms of (2.1.11) are used for PI controllers

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= \mathbf{y}(t) - \mathbf{v}(t) \qquad \mathbf{x}_r(0) = \mathbf{x}_{r0} \\ \mathbf{u}(t) &= -\mathbf{K}_P(\mathbf{y}(t) - \mathbf{v}(t)) - \mathbf{K}_I\mathbf{x}_r(t) \end{aligned} \tag{2.1.12}$$

proportional output feedback controllers

$$\mathbf{u}(t) = -\mathbf{K}_y\mathbf{y}(t) \tag{2.1.13}$$

and proportional state feedback controllers

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t). \tag{2.1.14}$$

Eigenvalues and zeros

The eigenvalues λ of the system matrix $\mathbf{A}$ occurring in eqn (2.1.4) are defined by

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0. \tag{2.1.15}$$

They are also called system eigenvalues or system poles (if no cancellations occur in the corresponding transfer function matrix). The invariant zeros λ_0 of the system (2.1.4) are defined by

$$\operatorname{rank}\begin{pmatrix} \lambda_0\mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} < n + \min(r, m). \tag{2.1.16}$$

2.2 CONTROLLABILITY AND OBSERVABILITY

The following definitions describe the stability, controllability and observability of a linear time-invariant system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x}. \end{aligned} \tag{2.2.1}$$

Definition 2.1 (*Lyapunov stability*)

The equilibrium $\mathbf{x}_e = \mathbf{0}$ of the autonomous system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is said to be *asymptotically stable* if for all $\varepsilon > 0$ there exists some δ such that $\| \mathbf{x}_0 \| < \varepsilon$ implies $\| \mathbf{x}(t) \| < \delta$ for all $t \geqslant 0$ and if

$$\lim_{t \to \infty} \| \mathbf{x}(t) - \mathbf{x}_e \| = 0$$

holds.

Definition 2.2

A system (2.2.1) is said to be *completely controllable* if for every initial state $\mathbf{x}_0$ there exists a finite time T and a control $\mathbf{u}(t)$, $t \in [0, T]$, such that $\mathbf{x}(T) = \mathbf{0}$ holds. The system (2.2.1) is said to be *completely observable* if for $\mathbf{u}(t) = \mathbf{0}$ every initial state $\mathbf{x}(0)$ can be reconstructed from the measurements of $\mathbf{y}(t)$, $t \in [0, T]$ for some T.

Both properties are dual, that is the following considerations on controllability hold for observability if $\mathbf{A}$ and $\mathbf{B}$ are replaced by $\mathbf{A}'$ and $\mathbf{C}'$.

Theorem 2.1

The system (2.2.1) is completely controllable if and only if one of the following conditions is satisfied:

(i) The test matrix $\mathbf{S}_c$ has full rank

$$\operatorname{rank} \mathbf{S}_c = \operatorname{rank}(\mathbf{B} \;\; \mathbf{AB} \;\; \ldots \;\; \mathbf{A}^{n-1}\mathbf{B}) = n. \tag{2.2.2}$$

(ii) The condition

$$\operatorname{rank}(\mathbf{A} - \lambda_i \mathbf{I} \;\; \mathbf{B}) = n \quad (i = 1, 2, \ldots, n) \tag{2.2.3}$$

is satisfied where λ_i denotes the ith eigenvalue of $\mathbf{A}$.

It is also said that the pair $(\mathbf{A}, \mathbf{B})$ is controllable. Controllability and observability are often considered in connection with the system modes. The transformation

$$\tilde{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x} \tag{2.2.4}$$

is applied to the model (2.2.1) with $\mathbf{V}$ being the modal matrix of the matrix $\mathbf{A}$, that is

$$\mathbf{V} = (\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \ldots \;\; \mathbf{v}_n) \tag{2.2.5}$$

where $\mathbf{v}_i$ denotes the right eigenvector of the ith eigenvalue of $\mathbf{A}$

$$\mathbf{A}\mathbf{v}_i = \lambda_i[\mathbf{A}]\, \mathbf{v}_i$$

(cf. Appendix 1). Then, eqn (2.2.1) has the form

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \qquad \tilde{\mathbf{x}}(0) = \mathbf{V}^{-1}\mathbf{x}_0 \\ \mathbf{y} &= \tilde{\mathbf{C}}\tilde{\mathbf{x}} \end{aligned} \tag{2.2.6}$$

with

$$\tilde{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \text{diag } \lambda_i[\mathbf{A}] \qquad \tilde{\mathbf{B}} = \mathbf{V}^{-1}\mathbf{B} \qquad \tilde{\mathbf{C}} = \mathbf{C}\mathbf{V}. \tag{2.2.7}$$

The solution of the differential equation (2.2.6) is

$$\mathbf{y}(t) = \mathbf{C} \sum_{i=1}^{n} \mathbf{v}_i \exp(\lambda_i t)\tilde{\mathbf{x}}_{0i} + \mathbf{C} \sum_{i=1}^{n} \mathbf{v}_i \exp(\lambda_i t)\tilde{\mathbf{b}}_i' * \mathbf{u} \tag{2.2.8}$$

where $\tilde{\mathbf{B}}$ has been decomposed according to

$$\tilde{\mathbf{B}} = \begin{pmatrix} \tilde{\mathbf{b}}_1' \\ \tilde{\mathbf{b}}_2' \\ \vdots \\ \tilde{\mathbf{b}}_n' \end{pmatrix}.$$

Definition 2.2 can be extended to the controllability or observability of the system modes $\mathbf{v}_i \exp(\lambda_i t)$. Complete controllability of the system (2.2.1) means that all modes may be influenced by the input $\mathbf{u}(t)$. Hence, no $\tilde{\mathbf{b}}_i'$ may be equal to the zero row. If some zero rows occur, the corresponding modes are said to be not controllable. In practice the term controllability is also used with the eigenvalues $\lambda_i[\mathbf{A}]$ occurring in the corresponding modes.

Theorem 2.2

The mode $\mathbf{v}_i \exp(\lambda_i t)$ and the eigenvalue $\lambda_i[\mathbf{A}]$ are controllable if and only if one of the following equivalent conditions is satisfied:

(i) $\text{rank}(\mathbf{A} - \lambda_i\mathbf{I} \;\; \mathbf{B}) = n$ (2.2.9)

(ii) $\tilde{\mathbf{b}}_i \neq 0.$ (2.2.10)

Fixed eigenvalues

If the linear output feedback

$$\mathbf{u} = -\mathbf{K}_y\mathbf{y} \tag{2.2.11}$$

is applied to the system (2.2.1) the closed-loop system is described by

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x}$$

with

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C}. \tag{2.2.12}$$

A comparison of the open-loop spectrum

$$\sigma[\mathbf{A}] = \{\lambda_i[\mathbf{A}], i = 1, \ldots, n\}$$

with the closed-loop spectrum

$$\sigma[\bar{\mathbf{A}}] = \{\lambda_i[\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C}], i = 1, \ldots, n\}$$

for arbitrary $\mathbf{K}_y$ yields those eigenvalues of $\mathbf{A}$ that appear unchanged as eigenvalues of $\bar{\mathbf{A}}$ for all feedbacks (2.2.11).

Definition 2.3

The elements of the set

$$\Lambda_f = \bigcap_{\mathbf{K}_y \in \mathscr{R}^{m \times r}} \sigma[\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C}] \tag{2.2.13}$$

are called *centralized fixed modes* (or *centralized fixed eigenvalues*).

The symbol $\cap$ denotes intersection of sets. Here again the terms 'mode' and 'eigenvalue' came into use as synonyms although the elements of the set Λ_f are eigenvalues rather than modes. For centralized control (2.2.11) the fixed eigenvalues can be easily determined.

Theorem 2.3

The set of centralized fixed eigenvalues is identical to the set of uncontrollable or unobservable eigenvalues of the plant (2.2.1)

$$\Lambda_f = \{\lambda_i[\mathbf{A}]: \text{rank}(\mathbf{A} - \lambda_i\mathbf{I} \;\; \mathbf{B}) < n \text{ or } \text{rank}(\mathbf{A}' - \lambda_i\mathbf{I} \;\; \mathbf{C}') < n\}. \tag{2.2.14}$$

The importance of the notion of centralized fixed modes lies in the fact that these modes cannot be changed even by any dynamic output feedback (2.1.11).

Theorem 2.4

The spectrum σ_0 of the closed-loop system (2.2.1) and (2.1.11) includes the centralized fixed eigenvalues

$$\Lambda_f \subseteq \sigma_0 \tag{2.2.15}$$

for arbitrary controller order dim $\mathbf{x}_r$ and controller matrices $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}_x$, $\mathbf{K}_y$ and $\mathbf{K}_v$.

Stabilizability and pole assignability

The question of under what conditions there exists a dynamic output feedback (2.1.11) that stabilizes a given unstable system (2.2.1) can be answered in terms of the fixed modes.

Theorem 2.5

The following assertions hold for every system (2.2.1) with the set Λ_f of centralized fixed eigenvalues.

(i) There exists a centralized feedback controller (2.1.11) such that the closed-loop system (2.2.1) and (2.1.11) is stable if and only if all centralized fixed eigenvalues have negative real parts

$$\mathrm{Re}\,[\lambda] < 0 \quad \text{for all } \lambda \in \Lambda_f. \tag{2.2.16}$$

(ii) There exists a centralized feedback controller (2.1.11) such that the closed-loop system (2.2.1) and (2.1.11) has an arbitrarily prescribed set σ_0 of eigenvalues if and only if the system (2.2.1) has no centralized fixed eigenvalues

$$\Lambda_f = \emptyset. \tag{2.2.17}$$

Here and in what follows it is taken for granted that each prescribed set σ_0 of eigenvalues is symmetric in the sense that if $\lambda \in \sigma_0$ is complex then the complex conjugate value λ^* also belongs to σ_0. Theorem 2.5 says that the changeability of the non-fixed eigenvalues by a static output feedback (2.2.11) implies the possibility of assigning these eigenvalues arbitrary values by means of a dynamic output feedback (2.1.11). If all the state variables of the plant are measurable ($\mathbf{y} = \mathbf{x}$ in eqn (2.2.1)) then stabilization and pole assignment can be performed by means of a static feedback

$$\mathbf{u} = -\mathbf{K}\mathbf{x}. \tag{2.2.18}$$

Theorem 2.6

Under conditions (2.2.16) or (2.2.17) of Theorem 2.5 there exists a static state feedback (2.2.18) such that the closed-loop system (2.2.1) and (2.2.18) is stable or has an arbitrarily prescribed set of eigenvalues, respectively.

Observers

Many design procedures for centralized controllers aim at constructing a state feedback (2.2.18). Since under practical circumstances the state vector $\mathbf{x}$ cannot be expected to be completely measurable, the problem of reconstructing the state from the known input $\mathbf{u}$ and output $\mathbf{y}$ has to be tackled. The idea is to create a model with the available data $\mathbf{u}$ and $\mathbf{y}$ as inputs

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_b\hat{\mathbf{x}} + \mathbf{B}_b\mathbf{u} + \mathbf{E}_b\mathbf{y} \qquad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \tag{2.2.19}$$

which yields an asymptotically correct state estimate $\hat{\mathbf{x}}$

$$\mathbf{x}(t) - \hat{\mathbf{x}}(t) \xrightarrow[t \to \infty]{} 0. \tag{2.2.20}$$

Theorem 2.7

If the system (2.2.1) is completely observable then eqn (2.2.20) holds for the system (2.2.19) if the conditions

$$\mathbf{A}_b = \mathbf{A} - \mathbf{E}_b\mathbf{C} \qquad \mathbf{B}_b = \mathbf{B} \tag{2.2.21}$$

are satisfied and the matrix $\mathbf{E}_b$ is chosen so that all eigenvalues of $\mathbf{A}_b$ have negative real parts.

Note that the observer consists of the extended plant model

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{E}_b\mathbf{u}_b \qquad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0$$

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}$$

which is excited by the plant input $\mathbf{u}$ and the observation error

$$\mathbf{u}_b = \mathbf{y} - \hat{\mathbf{y}}$$

(cf. Figure 2.3).

If instead of the actual state $\mathbf{x}$ the reconstructed state $\hat{\mathbf{x}}$ is available, the state feedback (2.2.18) can be applied as

$$\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}}. \tag{2.2.22}$$

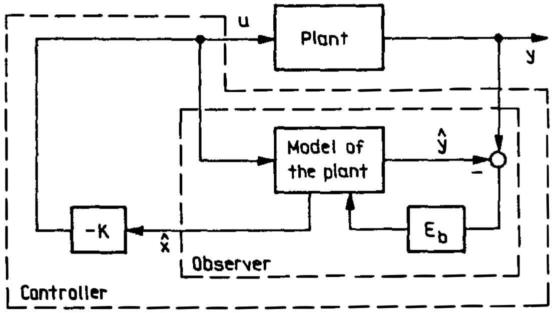


Figure 2.3 Feedback system with observer

Then the controller consists of the dynamical part (2.2.19) and the static part (2.2.22). Both parts together form a compensator (2.1.11) which stabilizes the plant (2.2.1) and has the form

$$\begin{aligned} \dot{\mathbf{x}}_r &= (\mathbf{A}_b - \mathbf{B}_b\mathbf{K})\mathbf{x}_r + \mathbf{E}_b\mathbf{y} \\ \mathbf{u} &= -\mathbf{K}\mathbf{x}_r. \end{aligned} \tag{2.2.23}$$

For the closed-loop system the following separation theorem is important.

Theorem 2.8 (*Separation theorem*)

The spectrum σ_0 of the closed-loop system (2.2.1), (2.2.19) and (2.2.22) consists of the spectrum of the observer (2.2.19) and the spectrum of the state feedback system (2.2.1) and (2.2.18)

$$\sigma_0 = \sigma[\mathbf{A}_b] \cup \sigma[\mathbf{A} - \mathbf{B}\ \mathbf{K}]. \tag{2.2.24}$$

The symbol '∪' denotes the union of the two sets. Hence, for given closed-loop system eigenvalues the feedback matrix **K** can be designed independently of the observer under the assumption that the state vector is measurable.

2.3 STABILITY ANALYSIS OF FEEDBACK SYSTEMS

Stability of multivariable systems

The stability of linear systems is defined either for the autonomous system as in Definition 2.1 or, alternatively, for I/O considerations as follows.

Definition 2.4 (*Input–output (I/O) stability*)

The system

$$\mathbf{y}(t) = \mathbf{G}(t) * \mathbf{u}(t) \tag{2.3.1}$$

is said to be $\mathscr{L}_2$-stable (bounded-input bounded-output stable) if

$$\int_0^\infty \|\mathbf{G}(t)\| \, \mathrm{d}t < \infty$$

holds.

$\|.\|$ denotes a matrix norm (cf. Appendix 1) which is applied here to the matrix $\mathbf{G}(t)$ at a fixed time t. For linear systems the relation between Lyapunov stability and I/O stability is very simple.

Theorem 2.9

If the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.3.2}$$

is completely controllable and observable then it is asymptotically stable if and only if it is $\mathscr{L}$-stable.

The following theorems summarize the tests for proving asymptotic stability.

Theorem 2.10

A linear system (2.3.1) is asymptotically stable if and only if all the uncontrollable and unobservable modes are stable and there exists a finite matrix $\mathbf{P}$ such that

$$\int_0^\infty \|\mathbf{G}(t)\| \, \mathrm{d}t < \mathbf{P} \tag{2.3.3}$$

holds.

Theorem 2.11

A linear system (2.3.2) is asymptotically stable if and only if all the eigenvalues of the matrix $\mathbf{A}$ have negative real parts.

Lyapunov's Direct Method for Stability Analysis

Lyapunov's direct method is a method of stability analysis which is general enough to be simultaneously applicable to autonomous linear or non-linear systems. It will be summarized here for systems

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.3.4}$$

with the equilibrium state $\mathbf{x}_e = \mathbf{0}$. The Lyapunov stability according to Definition 2.1 is considered with (2.3.4) instead of (2.1.1). Lyapunov's direct method is based on a scalar function $v(\mathbf{x}, t)$ whose value may be interpreted as the internal energy of the system (2.3.4) because $v(\mathbf{x}, t)$ is non-negative and vanishes only if the system has reached its equilibrium state $\mathbf{x}_e$:

$$\begin{aligned} &v(\mathbf{x}, t) \geqslant 0 \quad \text{for all } \mathbf{x} \\ &v(\mathbf{x}, t) = \mathbf{0} \quad \text{if and only if } \mathbf{x} = \mathbf{0}. \end{aligned} \tag{2.3.5}$$

If this function is monotonically decreasing along the trajectory of the system (2.3.4), that is if

$$\dot{v}(\mathbf{x}, t) = \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}v'}{\mathrm{d}\mathbf{x}} \quad \dot{\mathbf{x}} < 0 \quad \text{for } \mathbf{x} \neq \mathbf{0} \tag{2.3.6}$$

holds, then the energy decreases and, consequently, the system (2.3.4) eventually reaches its equilibrium state.

Theorem 2.12

If for the system (2.3.4) there is a function $v(\mathbf{x}, t)$ with the property (2.3.5), for which eqn (2.3.6) holds for all initial states $\mathbf{x}_0$, then the equilibrium $\mathbf{x}_e = \mathbf{0}$ of the system (2.3.4) is asymptotically stable.

A function $v(\mathbf{x}, t)$ with the properties (2.3.5) and (2.3.6) is called a Lyapunov function. This stability test poses the problem of finding functions with the property (2.3.5) (tentative Lyapunov functions) and checking the condition (2.3.6). For linear time-invariant systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.3.7}$$

tentative Lyapunov functions are given by

$$v(\mathbf{x}) = \mathbf{x}'\mathbf{P}\mathbf{x} \tag{2.3.8}$$

or

$$v(\mathbf{x}) = \sqrt{\mathbf{x}'\mathbf{P}\mathbf{x}} \tag{2.3.9}$$

where $\mathbf{P}$ is some symmetric positive definite matrix. The derivative of $v(\mathbf{x})$ in (2.3.8) along the trajectory of (2.3.7) is given by

$$\dot{v}(\mathbf{x}) = \mathbf{x}'(\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}.$$

It is negative definite if and only if the sum $\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A}$ is negative definite, that is if

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{2.3.10}$$

holds for some symmetric positive definite matrix $\mathbf{Q}$.

Theorem 2.13

A linear time-invariant autonomous system (2.3.7) is stable if and only if there exists a symmetric positive definite matrix $\mathbf{Q}$ such that the Lyapunov equation (2.3.10) has a positive definite solution $\mathbf{P}$.

Note that for the stability of linear systems the existence of some Lyapunov function is not only sufficient but also necessary. Moreover, if for some matrix $\mathbf{Q}$ there exists a positive definite solution to eqn (2.3.10) then its solution is positive definite for an arbitrary positive definite matrix $\mathbf{Q}$. Hence, for linear systems Lyapunov functions (2.3.8) or (2.3.9) can be found if and only if the system is stable. Algorithm A1.2 in Appendix 1 can be used to solve eqn (2.3.10).

2.4 SOME DESIGN METHODS FOR CENTRALIZED CONTROLLERS

This section summarizes some centralized feedback design methods for later use in the design of decentralized controllers. The methods have been selected according to their convenience for the development of decentralized control but may be replaced by more sophisticated or efficient ones.

Dyadic State Feedback

For the completely controllable plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{2.4.1}$$

a state feedback controller

$$\mathbf{u} = -\mathbf{Kx} \tag{2.4.2}$$

is to be designed such that the closed-loop system (2.4.1) and (2.4.2) has a prescribed set σ_0 of eigenvalues

$$\sigma[\mathbf{A} - \mathbf{B}\ \mathbf{K}] = \sigma_0. \tag{2.4.3}$$

The controller matrix $\mathbf{K}$ is decomposed into the dyadic product

$$\mathbf{K} = \mathbf{fm}' \tag{2.4.4}$$

where $\mathbf{f}$ and $\mathbf{m}$ are m- or n-dimensional vectors, respectively. The vector $\mathbf{f}$ is chosen so as to make the auxiliary single-input plant

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bf}\bar{u}$$

completely controllable via $\bar{u}$. This is possible whenever the original plant (2.4.1) is controllable and the matrix $\mathbf{A}$ is cyclic (i.e. its characteristic polynomial equals its minimal polynomial). The vector $\mathbf{m}$ is determined from the eigenvalues λ_i, the corresponding reciprocal eigenvectors $\mathbf{w}_i'$, which satisfy the equation

$$\mathbf{w}_i'\mathbf{A} = \lambda_i[\mathbf{A}]\mathbf{w}_i' \tag{2.4.5}$$

and the prescribed closed-loop eigenvalues $\bar{\lambda}_i \in \sigma_0$ according to

$$\mathbf{m} = \sum_{i=1}^{n} k_i \mathbf{w}_i \tag{2.4.6}$$

with

$$k_i = \prod_{i=1}^{n} (\lambda_i - \bar{\lambda}_i) \Big/ \mathbf{w}_i'\mathbf{Bf} \prod_{\substack{j=1 \\ j \neq i}}^{n} (\lambda_i - \lambda_j).$$

If the system state is not completely measurable, the feedback (2.4.2) can be applied by using an observer (2.2.19). Then the set of the closed-loop system eigenvalues comprises the set σ_0 prescribed for the design of the state feedback (2.4.2) as well as the set of observer eigenvalues (cf. Theorem 2.8).

LQ Regulation

For the controllable plant (2.4.1) a state feedback (2.4.2) should be chosen so as to minimize the quadratic performance index

$$I = \frac{1}{2} \int_0^\infty (\mathbf{x}'\mathbf{Qx} + \mathbf{u}'\mathbf{Ru})\, \mathrm{d}t \to \min_{\mathbf{K}} \tag{2.4.7}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are symmetric non-negative definite or positive definite matrices, respectively. The pair $(\mathbf{A}, \mathbf{L})$ is assumed to be observable where $\mathbf{L}$ satisfies the relation $\mathbf{L}'\mathbf{L} = \mathbf{Q}$.

The solution to this design problem is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}'\mathbf{P} \tag{2.4.8}$$

where $\mathbf{P}$ denotes the symmetric positive definite solution of the algebraic matrix Riccati equation

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P} + \mathbf{Q} = \mathbf{0}. \tag{2.4.9}$$

Under the assumptions made above a unique solution exists. The closed-loop system (2.4.1), (2.4.2) and (2.4.8)

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P})\mathbf{x} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{2.4.10}$$

is stable. The functions (2.3.8) and (2.3.9) with $\mathbf{P}$ from eqn (2.4.9) are Lyapunov functions of the closed-loop system (2.4.10).

If the performance index (2.4.7) includes a third term

$$I = \frac{1}{2}\int_0^\infty (\mathbf{x}'\mathbf{Q}\mathbf{x} + 2\mathbf{u}'\mathbf{S}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u})\, dt \to \min_{\mathbf{K}}$$

then the transformation $\bar{\mathbf{u}} = \mathbf{u} + \mathbf{R}^{-1}\mathbf{S}\mathbf{x}$ reduces the new design problem to the standard form (2.4.7)

$$I = \frac{1}{2}\int_0^\infty [\mathbf{x}'(\mathbf{Q} - \mathbf{S}'\mathbf{R}^{-1}\mathbf{S})\mathbf{x} + \bar{\mathbf{u}}'\mathbf{R}\bar{\mathbf{u}}]\ dt \to \min_{\mathbf{K}}$$

subject to

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{S})\mathbf{x} + \mathbf{B}\bar{\mathbf{u}}.$$

The solution of this reformulated problem is given by

$$\bar{\mathbf{u}} = -\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}}\mathbf{x}$$

and leads to

$$\mathbf{u} = -\mathbf{R}^{-1}(\mathbf{B}'\bar{\mathbf{P}} + \mathbf{S})\mathbf{x} \tag{2.4.11}$$

as the solution of the original **LQ** problem. $\bar{\mathbf{P}}$ is the symmetric positive definite solution of the algebraic Riccati equation

$$\begin{aligned} (\mathbf{A}' - \mathbf{B}\mathbf{R}^{-1}\mathbf{S})'\bar{\mathbf{P}} + \bar{\mathbf{P}}(\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{S}) \\ - \bar{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\bar{\mathbf{P}} + \mathbf{Q} - \mathbf{S}'\mathbf{R}^{-1}\mathbf{S} = \mathbf{0}. \end{aligned} \tag{2.4.12}$$

If the performance index (2.4.7) is extended to

$$I = \frac{1}{2}\int_0^\infty \exp(2dt)(\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u})\, dt \to \min_{\mathbf{K}} \tag{2.4.13}$$

with some scalar $d > 0$, then the optimal state feedback (2.4.2) is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}'\tilde{\mathbf{P}} \tag{2.4.14}$$

with $\tilde{\mathbf{P}}$ as the symmetric positive definite solution of

$$(\mathbf{A}' - d\mathbf{I})\tilde{\mathbf{P}} + \tilde{\mathbf{P}}(\mathbf{A} - d\mathbf{I}) - \tilde{\mathbf{P}}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\tilde{\mathbf{P}} + \mathbf{Q} = \mathbf{0}. \tag{2.4.15}$$

This feedback ensures a stability degree of the closed-loop system (2.4.1), (2.4.2) and (2.4.14) of at least d, that is

$$\max_i \operatorname{Re}\{\lambda_i[\mathbf{A} - \mathbf{B}\ \mathbf{K}]\} < -d \tag{2.4.16}$$

holds.

Design of PI Controllers

If the plant is exposed to stepwise disturbances or command signals, PI controllers

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= \mathbf{y}(t) - \mathbf{v}(t) \qquad \mathbf{x}_r(0) = \mathbf{x}_{r0} \\ \mathbf{u}(t) &= -\mathbf{K}_P(\mathbf{y}(t) - \mathbf{v}(t)) - \mathbf{K}_I\mathbf{x}_r(t) \end{aligned} \tag{2.4.17}$$

ensure asymptotic regulation whenever the closed-loop system (2.4.1) and (2.4.17) is stable. A necessary condition for the closed-loop stability and a class of controller parameters can be given in terms of the static reinforcement $\mathbf{K}_s$ (cf. eqn (2.1.9)) of the plant (2.4.1).

Theorem 2.14

For a stable plant (2.4.1) under PI control (2.4.17) the following assertions hold.

(i) A necessary condition for the stability of the closed-loop system (2.4.1) and (2.4.17) is given by

$$\det(\mathbf{K}_s\mathbf{K}_I) > 0. \tag{2.4.18}$$

(ii) Consider a PI controller where the controller matrices have been decomposed as follows

$$\mathbf{K}_I = a\hat{\mathbf{K}}_I \qquad \mathbf{K}_P = b\hat{\mathbf{K}}_P. \tag{2.4.19}$$

There exist positive scalars $\bar{a}$ and $\bar{b}$ such that the closed-loop system (2.4.1), (2.4.17) and (2.4.19) is stable for all $a \in (0, \bar{a})$ and $b \in [0, \bar{b})$ if and only if $\hat{\mathbf{K}}_I$ satisfies the condition

$$\operatorname{Re}\{\lambda_i[\mathbf{K}_s\hat{\mathbf{K}}_I]\} > 0. \tag{2.4.20}$$

Eqn (2.4.18) can be referred to as the negative feedback condition for multivariable PI control systems. It claims that the controller (2.4.17) should counteract the control error. For single-input single-output systems it has the form $k_s k_I > 0$. As a consequence, the plant (2.4.1) must satisfy the condition

$$\operatorname{rank} \mathbf{K}_s = \min(m, r) \tag{2.4.21}$$

if a PI controller (2.4.17) should exist such that the closed-loop system is stable.

The second part of the theorem concerns the following low-gain feedback strategy. Consider the closed-loop system (2.4.1) and (2.4.17), where the loop has been opened at the plant input $\mathbf{u}(t)$. Since the plant is assumed to be stable, the only unstable eigenvalues of the resulting open-loop system are brought about by the integrators within the controller. Closed-loop stability can be ensured by choosing small controller parameters in such a way that the unstable eigenvalues are moved into the left-half complex plane while not destabilizing the plant eigenvalues (Figure 2.4). The matrix $\hat{\mathbf{K}}_I$ is chosen so that the root loci of the integrator eigenvalues run into the open left-half complex plane at least for small positive values of the scalars a and b. If $\bar{a}$ and $\bar{b}$ are small enough, the root loci beginning in the plant eigenvalues cannot cross the imaginary axis for $a \in (0, \bar{a})$ and $b \in [0, \bar{b})$.

Since Theorem 2.14 refers only to the static reinforcement $\mathbf{K}_s$ of the plant (2.4.1), its results can be used even if only step response measurement data rather than a dynamic model (2.4.1) are available. The controller can be tuned on-plant by means of experiments with the plant to receive $\mathbf{K}_s$ and with the closed-loop system to choose reasonable tuning factors a and b.

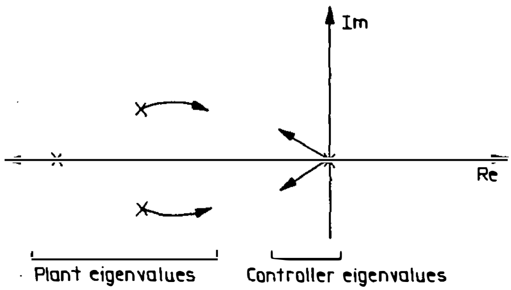


Figure 2.4 Root loci for low-gain feedback (small scalar a)

2.5 STRUCTURAL PROPERTIES OF CONTROL SYSTEMS

In many practical applications the plant cannot be described by some precise linear model. Uncertainties may result from incomplete knowledge of the plant, from non-linear or time-varying properties which cannot be reflected by some linear model, or from approximations made to reduce the model size. The possible effects of the model errors on the analytical or design results have to be estimated. The following discussion points to the fact that certain system properties exist as a result of the system structure and, thus, can be found without knowing the precise system parameters. Section 2.6 summarizes the results concerning the quantitative assessment of the robustness of linear controllers.

The Structure of Dynamical Systems

In general, structural properties are properties that are not strongly dependent upon certain parameter values but hold for a large variety of parameter values. In the following, the system structure is defined on the basis of a classification of the entries of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ of the model

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{2.5.1}$$

into vanishing or non-vanishing elements. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are transformed into structure matrices $[\mathbf{A}]$, $[\mathbf{B}]$, and $[\mathbf{C}]$ by replacing all non-vanishing elements by asterisks, which indicate indeterminate entries. That is, the structure of the system (2.5.1) is described by the places of the entries of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ that are known to be zero under all operating conditions.

On the other hand, given structure matrices $\mathbf{S}_a$, $\mathbf{S}_b$ and $\mathbf{S}_c$ define a class of structurally equivalent systems. To explain this it has to be stressed first that a structure matrix is a matrix whose elements are either 0 or $*$. A given structure matrix $\mathbf{S}_a$ defines the set of all admissible numerical matrices $\mathbf{A}$, which comprises all $\mathbf{A}$ for which $[\mathbf{A}] = \mathbf{S}_a$ holds. This relation between a numerical matrix $\mathbf{A}$ and a structure matrix $\mathbf{S}_a$ is sometimes abbreviated as $\mathbf{A} \in \mathbf{S}_a$, although this relation has to be read as

$$\mathbf{A} \in \mathscr{S}(\mathbf{S}_a) = \{\mathbf{A}\colon\ [\mathbf{A}] = \mathbf{S}_a\}.$$

Two systems that are described by the model (2.5.1) with $\mathbf{A} = \mathbf{A}^1$, $\mathbf{B} = \mathbf{B}^1$, $\mathbf{C} = \mathbf{C}^1$ or $\mathbf{A} = \mathbf{A}^2$, $\mathbf{B} = \mathbf{B}^2$, $\mathbf{C} = \mathbf{C}^2$, respectively, are structurally equivalent if they have identical structure matrices $[\mathbf{A}^1] = [\mathbf{A}^2] = \mathbf{S}_a$,

$[\mathbf{B}^1] = [\mathbf{B}^2] = \mathbf{S}_b$, $[\mathbf{C}^1] = [\mathbf{C}^2] = \mathbf{S}_c$. That is, their matrices have vanishing entries at least in the positions described by $\mathbf{S}_a$, $\mathbf{S}_b$ and $\mathbf{S}_c$. The class of structurally equivalent systems is described by

$$\mathscr{S}(\mathbf{S}_a, \mathbf{S}_b, \mathbf{S}_c) = \{(\mathbf{A}, \mathbf{B}, \mathbf{C}): [\mathbf{A}] = \mathbf{S}_a, [\mathbf{B}] = \mathbf{S}_b, [\mathbf{C}] = \mathbf{S}_c\} \tag{2.5.2}$$

where $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is the abbreviation of a system (2.5.1). The results obtained by means of $\mathbf{S}_a$, $\mathbf{S}_b$ and $\mathbf{S}_c$ are structural in the sense that they hold for all systems of the set $\mathscr{S}(\mathbf{S}_a, \mathbf{S}_b, \mathbf{S}_c)$.

Example 2.1

The structure matrices of the system

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x} \end{aligned} \tag{2.5.3}$$

are

$$[\mathbf{A}] = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \qquad [\mathbf{B}] = \begin{pmatrix} * & 0 \\ 0 & 0 \\ 0 & * \end{pmatrix} \qquad [\mathbf{C}] = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}. \tag{2.5.4}$$

If the structure of the given system (2.5.3) should be investigated, the structure matrices (2.5.4) are used instead of the model (2.5.3). This move from the numeric matrices to the structure matrices is accompanied by the extension of the system with matrices **A, B, C** to the class of systems $\mathscr{S}([\mathbf{A}], [\mathbf{B}], [\mathbf{C}])$. □

A square structure matrix $[\mathbf{Q}]$ of order n can be interpreted by means of a directed graph $G([\mathbf{Q}])$ (cf. Appendix 2) with n vertices in which there exists an edge from $v_i \in \mathscr{V}$ towards $v_j \in \mathscr{V}$ if and only if the element q_{ji} of the matrix $\mathbf{Q}$ does not vanish. This definition can be used for non-square (n, m) matrices if the matrix is extended by $n - m$ zero columns (for $n > m$) or $m - n$ zero rows (for $m > n$) to a square matrix of order $\max(n, m)$.

The graph $G(\mathbf{Q}_0)$ is associated with the dynamical system (2.5.1) where the $(n + r + m, n + r + m)$ structure matrix $\mathbf{Q}_0$ is given by

$$\mathbf{Q}_0 = \begin{array}{c} \begin{array}{ccc} \mathscr{X} & \mathscr{U} & \mathscr{Y} \end{array} \\ \begin{pmatrix} \mathbf{S}_a & \mathbf{S}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{S}_c & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \begin{array}{c} \\ \mathscr{X} \\ \mathscr{U}. \\ \mathscr{Y} \end{array} \tag{2.5.5}$$

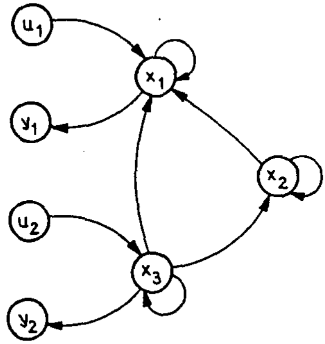


Figure 2.5 Structure of the system (2.5.3)

The vertices can be labelled by x_i, u_i and y_i in such a way that the edges between vertices x_i and x_j, u_i and x_j, or x_i and y_j exist if and only if an asterisk is positioned in the jth row and ith column of the matrix $\mathbf{S}_a$, $\mathbf{S}_b$ or $\mathbf{S}_c$, respectively. To make this relation obvious, the rows and columns of $\mathbf{Q}_0$ in eqn (2.5.5) have been labelled accordingly. In correspondence with this decomposition, the vertices are called 'state vertices', 'input vertices' or 'output vertices', respectively. For the example (2.5.3) the graph $G(\mathbf{Q}_0)$ with $\mathbf{S}_a = [\mathbf{A}]$, $\mathbf{S}_b = [\mathbf{B}]$, $\mathbf{S}_c = [\mathbf{C}]$ is shown in Figure 2.5. It illustrates the direct influence of the signals upon each other for all systems of the class $\mathcal{S}$, that is for all systems that have the same structure as the system (2.5.3).

Properties of structure matrices

Structural investigations of the class $\mathcal{S}$ of systems are based on the properties of the structural matrices. Those which will be used later are summarized below.

Definition 2.5

For a structure matrix $\mathbf{S}_a$ a set of *independent entries* is defined as a set of indeterminate entries '$*$', no two of which lie on the same row or column. The *structural rank* (s rank) of $\mathbf{S}_a$ is defined to be the maximum number of elements contained in a set of independent entries.

The s-rank of a structure matrix $\mathbf{S}_a$ is equal to the maximum rank of all admissible matrices $\mathbf{A} \in \mathbf{S}_a$:

$$s\text{-rank } \mathbf{S}_a = \max_{\mathbf{A} \in \mathbf{S}_a} \text{rank } \mathbf{A}. \tag{2.5.6}$$

Eqn (2.5.6) points to a typical relation between structural and numerical investigations. Almost all matrices $\mathbf{A} \in \mathbf{S}_a$ have a numerical rank that is equal to the structural rank of $\mathbf{S}_a$. In other words, rank $A < s$-rank $\mathbf{S}_a$ is valid only for exceptional matrices $\mathbf{A}$. It can be shown that for (m, n) matrices these exceptional values of the entries of $\mathbf{A}$ lie on a hypersurface in the $(m \times n)$-dimensional parameter space and, thus, are relatively 'infrequent'.

Structural Controllability and Structural Observability

The structural counterparts of complete controllability and observability are defined as follows.

Definition 2.6

A class $\mathscr{S}$ of systems is said to be *structurally controllable* (s-controllable) or *structurally observable* (s-observable) if there exists at least one admissible realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathscr{S}$ which is completely controllable or completely observable, respectively. A class of systems is called *structurally complete* (s-complete) if it is both s-controllable and s-observable.

Although this definition is based on the existence of only one admissible system, which is controllable and observable, s-controllability and s-observability are really structural properties. If one admissible system $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathscr{S}$ is controllable and observable then almost all systems of $\mathscr{S}$ turn out to have these properties.

Corollary 2.1

Structural controllability and structural observability of the class $\mathscr{S}([\mathbf{A}], [\mathbf{B}], [\mathbf{C}])$ are necessary for the complete controllability and observability of a system $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathscr{S}$.

Prerequisites for s-completeness are the following properties.

Definition 2.7

A class of systems is said to be *input-connectable* (or *input-reachable*) if in the graph $G(\mathbf{Q}_0)$ there is for every state vertex v a path from at least one of the input vertices to v. It is said to be *output-connectable* (or *output-reachable*) if there exists for every state vertex v a path from v to at least one output vertex.

As seen in Figure 2.5, the example is input-connectable and output-connectable. These properties ensure that the rank of the matrices

$$(\mathbf{A} - \lambda\mathbf{I} \;\; \mathbf{B}) \quad \text{and} \quad \begin{pmatrix} \mathbf{A} - \lambda\mathbf{I} \\ \mathbf{C} \end{pmatrix}$$

equals n for all scalars $\lambda \neq 0$ and almost all admissible systems. Consequently, the structural counterpart of part (ii) of Theorem 2.1 has to be checked only for $\lambda = 0$ (cf. part (ii) below).

Theorem 2.15

A class $\mathcal{S}$ of systems is s-controllable if and only if

(i) it is input-connectable and
(ii) the relation

$$s\text{-rank}(\mathbf{S}_a \;\; \mathbf{S}_b) = n \tag{2.5.7}$$

holds.

It is s-observable if and only if

(iii) it is output-connectable and
(iv) the relation

$$s\text{-rank}\begin{pmatrix} \mathbf{S}_a \\ \mathbf{S}_c \end{pmatrix} = n \tag{2.5.8}$$

holds.

Structurally Fixed Modes

Theorem 2.15 can be used to distinguish between centralized fixed modes that occur for structural reasons and fixed modes that are caused by an exceptional combination of parameters.

Definition 2.8

A class $\mathcal{S}$ of systems has *structurally fixed modes* if all admissible systems have centralized fixed modes according to Definition 2.3.

Corollary 2.2

An actual system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is known to possess centralized fixed modes if there exist structurally fixed modes in a class $\mathscr{S}([\mathbf{A}], [\mathbf{B}], [\mathbf{C}])$.

Note that the implications which relate the structural and the numerical properties have different directions in Corollaries 2.1 and 2.2.

According to Definition 2.8, a class of systems has *no* structurally fixed modes if there is at least one $\mathbf{S} \in \mathscr{S}$ that is completely controllable and completely observable. Hence, the existence of structurally fixed modes can be tested by means of the conditions of Theorem 2.15.

Lemma 2.1

A class $\mathscr{S}$ of systems has structurally fixed modes if and only if at least one of the conditions (i)–(iv) of Theorem 2.15 is not satisfied.

Although these conditions refer to the plant (2.5.1), a good graphical interpretation of this result can be made if the structure of the system (2.5.1) under output feedback

$$\mathbf{u} = \mathbf{K}_y \mathbf{y} \tag{2.5.9}$$

is investigated. The graph $G(\mathbf{Q}_E)$ of the structure matrix

$$\mathbf{Q}_E = \begin{pmatrix} \mathbf{S}_a & \mathbf{S}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{E} \\ \mathbf{S}_c & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{2.5.10}$$

with the (r, m) matrix

$$\mathbf{E} = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{pmatrix}$$

differs from $G(\mathbf{Q}_0)$ in the edges from all output vertices y_i to all input vertices u_i, which visualize the information flow through the feedback (2.5.9) (compare Figure 2.5 with Figure 2.6). Since the modes of the plant can be changed only if they lie within the closed loop of the plant and the controller, fixed modes can occur for structural reasons if there are state vertices in the graph $G(\mathbf{Q}_E)$ that do not lie within such loops. Since all output and input vertices are connected to each other by the controller, the only reason for this is that some state vertex is not connected to some input vertex (in the opposite direction of the edges) and

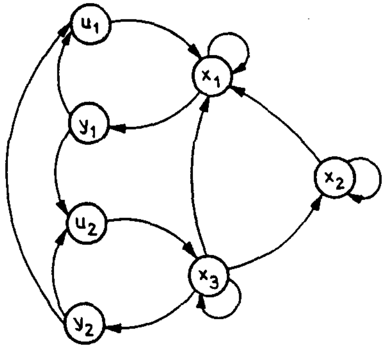


Figure 2.6 Structure of the system (2.5.2) under centralized output feedback

some output vertex and, thus, cannot be included in a loop that consists of plant and controller edges. Structurally fixed modes that occur because of the violation of the reachability conditions (i) and (iii) in Theorem 2.15 are called *structurally fixed modes of type I*.

The reachability test is a powerful means of investigating the controllability and observability since reachability alone ensures that the condition (2.2.3) is satisfied for all $\lambda \neq 0$ and almost all $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathcal{S}$. A second condition has only to check whether eqn (2.2.3) is valid for $\lambda = 0$. In graph-theoretic terms this condition can be formulated in terms of the cycle families. A cycle family is a set of cycles that have no common vertex (for details see Appendix 2). For example, the cycles $u_1 \rightarrow x_1 \rightarrow y_1 \rightarrow u_1$ and $x_3 \rightarrow x_3$ represent a cycle family of the graph in Figure 2.6.

The width of the cycle family is defined as the number of state vertices included in the cycle family. Systems which are input- and output-connectable are structurally complete only if there exists a cycle family of width n. Otherwise *structurally fixed modes of type II* occur.

Theorem 2.16

For a class $\mathcal{S}$ of nth-order systems there exist structurally fixed modes if and only if at least one of the following conditions is satisfied for the graph $G(\mathbf{Q}_\mathbf{E})$.

(i) $\mathcal{S}$ is neither input-connectable nor output-connectable.
(ii) There does not exist a cycle family of width n.

Instead of a proof of the equivalence of part (ii) of Theorem 2.16 and parts (ii) and (iv) of Theorem 2.15 only a short explanation of the correspondence between them will be given here. The cycle family of width n contains n edges whose final vertices are the n state vertices of $G(\mathbf{Q}_\mathbf{E})$. These edges correspond to indeterminate entries of S_a in n different rows. Consequently, the s-rank of $\mathbf{S}_a$ is n (cf. Definition 2.5) and, thus, parts (ii) and (iv) of Theorem 2.15 are satisfied.

Example 2.1 (cont.)

The class of systems depicted in Figure 2.6 has no structurally fixed modes because all state vertices are input-connectable and output-connectable and there is a cycle family of width 3, for example the loop $x_1 \to y_1 \to u_2 \to x_3 \to x_2 \to x_1$. However, if the system input u_2 was deleted the state x_3 would not be input-connectable and the system would have a structurally fixed mode of type I. □

Example 2.2

It is well known that two integrators in parallel are not completely controllable. That this fact is based on structural properties can be shown by means of Theorem 2.16. Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} b \\ 1 \end{pmatrix} u \qquad (2.5.11)$$

$$y = (c \;\; 1)\mathbf{x}$$

for $a = 0$. The graph $G(\mathbf{Q}_\mathbf{E})$ with

$$[\mathbf{A}] = \mathbf{0} \qquad [\mathbf{B}] = \begin{pmatrix} * \\ * \end{pmatrix} \qquad [\mathbf{C}] = (* \;\; *)$$

is shown in Figure 2.7. Although the reachability condition is satisfied the system is not structurally controllable because there is no cycle family of width 2 within the graph. This means that there is no sufficient freedom in the choice of controller parameters so as to influence both system modes independently of each other. The structurally fixed modes are of type II. The observation that these fixed modes are located at zero is not specific for this example but generally true. If fixed modes of type II occur then all systems of the given class have centralized fixed eigenvalues of value 0.

Structurally fixed modes of type II can only be avoided if the structure of the system is changed. If the parameter in (2.5.11) is changed

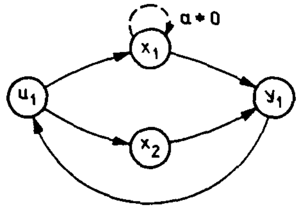


Figure 2.7 Structure of a closed-loop system with two integrators in parallel

($a \neq 0$) then a new edge occurs in Figure 2.6 and a cycle family of width 2 can be found, which consists of the cycles $u_1 \to x_2 \to y_1 \to u_1$ and $x_1 \to x_1$. The class of systems no longer has structurally fixed modes.□

2.6 ROBUSTNESS OF FEEDBACK SYSTEMS

Robustness expresses the ability of a dynamical system to retain a certain property (stability, I/O behaviour) in spite of a set of parametrical or structural perturbations within the plant. The following review summarizes those results which will be used later.

An uncertain system can be described by a model with the structure shown in Figure 2.8. The completely known part is represented by a state space model of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{f} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{E}\mathbf{f} \\ \mathbf{d} &= \mathbf{C}_d\mathbf{x} + \mathbf{D}_d\mathbf{u} + \mathbf{F}_d\mathbf{f}.\end{aligned} \tag{2.6.1}$$

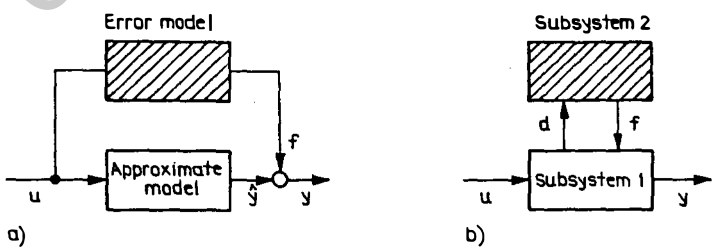


Figure 2.8 Structure of the model of uncertain systems: (a) parallel connection; (b) general model structure

The incompletely known part can, in principle, be considered as a system with input $\mathbf{d}$ and output $\mathbf{f}$

$$\mathbf{f} = \mathbf{G}_2 * \mathbf{d} \tag{2.6.2}$$

where the asterisk denotes the convolution operation. However, since eqn (2.6.2) represents the uncertainties of the process under consideration, the impulse response matrix $\mathbf{G}_2(t)$ is not known and an evaluation

$$|\mathbf{f}(t)| \leqslant \mathbf{V}_2 * |\mathbf{d}| \tag{2.6.3}$$

is used. The lines $|\,.\,|$ signify that in the vector or matrix all elements are replaced by their absolute values. $\mathbf{V}_2$ represents a known impulse response matrix. It can be shown that eqn (2.6.3) holds for $\mathbf{f}(t)$ from eqn (2.6.2) if and only if $\mathbf{V}_2(t) \geqslant |\mathbf{G}_2(t)|$ holds for all time t. Eqns (2.6.1) and (2.6.2) together represent a set $\mathscr{S}$ of systems with input $\mathbf{u}$ and output $\mathbf{y}$, whose dynamics can be represented by eqn (2.6.1) and a system (2.6.2) that satisfies the inequality (2.6.3).

The stability of any system $\mathbf{S} \in \mathscr{S}$ can be checked as follows. The model (2.6.1) leads to

$$\begin{aligned} |\mathbf{y}(t)| &\leqslant \mathbf{V}_{yu} * |\mathbf{u}| + \mathbf{V}_{yf} * |\mathbf{f}| \\ |\mathbf{d}(t)| &\leqslant \mathbf{V}_{du} * |\mathbf{u}| + \mathbf{V}_{df} * |\mathbf{f}| \end{aligned} \tag{2.6.4}$$

with

$$\begin{aligned} \mathbf{V}_{yu}(t) &= |\mathbf{D}|\,\delta(t) + |\mathbf{C}\exp(\mathbf{A}t)\mathbf{B}| \\ \mathbf{V}_{yf}(t) &= |\mathbf{F}|\,\delta(t) + |\mathbf{C}\exp(\mathbf{A}t)\mathbf{E}| \\ \mathbf{V}_{du}(t) &= |\mathbf{D}_d|\,\delta(t) + |\mathbf{C}_d\exp(\mathbf{A}t)\mathbf{B}| \\ \mathbf{V}_{df}(t) &= |\mathbf{F}_d|\,\delta(t) + |\mathbf{C}_d\exp(\mathbf{A}t)\mathbf{E}|. \end{aligned}$$

The stability can be proved by using $\mathbf{V}_2$ and $\mathbf{V}_{df}$.

Theorem 2.17

The system (2.6.1) and (2.6.3) is I/O-stable if the system (2.6.1) is stable and if

$$\lambda_{\mathrm{p}}[\mathbf{M}_2\ \mathbf{M}_{df}] < 1 \tag{2.6.5}$$

holds with

$$\mathbf{M}_{df} = \int_0^\infty \mathbf{V}_{df}(t)\,\mathrm{d}t \quad \text{and} \quad \mathbf{M}_2 = \int_0^\infty \mathbf{V}_2(t)\,\mathrm{d}t.$$

Note that condition (2.6.5) can be checked if $\mathbf{V}_2$ is known as the upper bound of the incompletely known part (2.6.2) of the original system.

The I/O performance of the system (2.6.1) and (2.6.2) is approximately described by eqn (2.6.1) for $\mathbf{f}=0$

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}\\ \hat{\mathbf{y}} &= \mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{u}.\end{aligned} \tag{2.6.6}$$

The model error $\mathbf{y}-\hat{\mathbf{y}}$ can be evaluated by means of all the impulse response matrices introduced in eqns (2.6.3) and (2.6.4) by

$$|\mathbf{y}(t)-\hat{\mathbf{y}}(t)| \leqslant \mathbf{V}_{yf} * \overline{\mathbf{V}} * \mathbf{V}_{du} * |\mathbf{u}| \tag{2.6.7}$$

with

$$\overline{\mathbf{V}}(t) = \mathbf{V}_2 + \mathbf{V}_2 * \overline{\mathbf{V}}. \tag{2.6.8}$$

Theorem 2.18

If the stability of the system (2.6.1) and (2.6.3) can be guaranteed by Theorem 2.17 then the I/O behaviour of the system is approximately described by eqn (2.6.6) where the model errors are bound by (2.6.7).

Condition (2.6.7) describes the maximum deviation between the original system output $\mathbf{y}$ and the approximate model output $\hat{\mathbf{y}}$. Within this interval, which can be determined for all time instances, lie the output of all systems that belong to the set $\mathcal{S}$.

BIBLIOGRAPHICAL NOTES

The results summarized here are thoroughly described in standard textbooks on multivariable control, for example those by Reinisch (1979), Patel and Munro (1982), Korn and Wilfert (1982), or Tolle (1983). Structural investigations of dynamical systems have been summarized by Reinschke (1988). Algorithms for searching directed graphs, as they are necessary for the application of structural investigations, can be found in the books by Even (1979) or Walther and Nägler (1987). A survey of the large number of methods for the robustness analysis of feedback systems has been given by Lunze (1988) and Morari and Zafiriou (1989).

3

Models and Structure of Interconnected Systems

3.1 SUBSYSTEM AND OVERALL SYSTEM MODELS

In this section, the models of interconnected systems are summarized for later use. They are distinguished by the degree to which they reflect the internal structure of the overall system.

Unstructured model

From a global point of view, the plant is a dynamical system with m-dimensional input vector $\mathbf{u}$ and r-dimensional output vector $\mathbf{y}$ (Figure 3.1(a)). Its state space representation has the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{3.1.1}$$

where $\mathbf{x}$ denotes the n-dimensional state vector of the overall system. Since time-invariant systems will be considered, the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have constant elements and are of appropriate dimensions. The model (3.1.1) is well known from multivariable system theory, but is of minor importance for large-scale systems because it says nothing about the subsystems of the overall system.

I/O-oriented model

For decentralized control, the sensors and actuators are grouped to m_i- or r_i-dimensional vectors $\mathbf{u}_i$ and $\mathbf{y}_i$ ($i = 1, \ldots, N$), where the ith control station has access to $\mathbf{y}_i$ and determines $\mathbf{u}_i$ (Figure 3.1(b)). That is, the overall system input and output is decomposed into subvectors $\mathbf{u} = (\mathbf{u}_1' \ \mathbf{u}_2' \ \ldots \ \mathbf{u}_N')'$ and $\mathbf{y} = (\mathbf{y}_1' \ \mathbf{y}_2' \ \ldots \ \mathbf{y}_N')'$. Instead of eqn (3.1.1) the

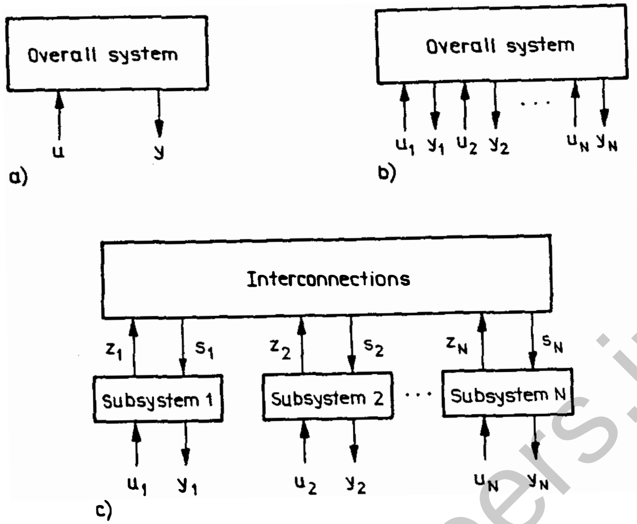


Figure 3.1 Structure of the models of interconnected systems: (a) unstructured model; (b) I/O oriented model; (c) interaction-oriented model

model

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\
\mathbf{y}_i(t) &= \mathbf{C}_{si}\mathbf{x}(t) + \sum_{j=1}^{N} \mathbf{D}_{ij}\mathbf{u}_j(t) \qquad (i = 1, \ldots, N)
\end{aligned}
\tag{3.1.2}
$$

is used which makes the structural constraints of the decentralized control perceptible. The matrices of eqn (3.1.2) can be obtained from (3.1.1) by decomposing **B**, **C** and **D** into submatrices, the dimensions of which are compatible with the dimensions of the vectors $\mathbf{u}_i$ and $\mathbf{y}_i$:

$$
\mathbf{B} = (\mathbf{B}_{s1} \;\; \mathbf{B}_{s2} \;\; \ldots \;\; \mathbf{B}_{sN})
$$

$$
\mathbf{C} = \begin{pmatrix} \mathbf{C}_{s1} \\ \mathbf{C}_{s2} \\ \vdots \\ \mathbf{C}_{sN} \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \ldots & \mathbf{D}_{1N} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \ldots & \mathbf{D}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{D}_{N1} & \mathbf{D}_{N2} & \ldots & \mathbf{D}_{NN} \end{pmatrix}. \tag{3.1.3}
$$

The model (3.1.2) exhibits the structure of the inputs and outputs but does not show how the overall system dynamics depends on the subsystems as the next form of the model will do.

Interaction-oriented model

Many large-scale systems emerge as a result of interactions between different subsystems. These couplings can have the nature of energy, material, or information flows. They are represented by signals $\mathbf{s}_i$ and $\mathbf{z}_i$ through which the ith subsystem interacts with other subsystems (Figure 3.1(c)). These additional input and output signals of the subsystems are internal signals of the overall system.

Since every subsystem represents a dynamical system of its own, it can be described by a state space model

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t) + \mathbf{E}_i\mathbf{s}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i(t) &= \mathbf{C}_i\mathbf{x}_i(t) + \mathbf{D}_i\mathbf{u}_i(t) + \mathbf{F}_i\mathbf{s}_i(t) \\ \mathbf{z}_i(t) &= \mathbf{C}_{zi}\mathbf{x}_i(t) + \mathbf{D}_{zi}\mathbf{u}_i(t) + \mathbf{F}_{zi}\mathbf{s}_i(t) \end{aligned} \tag{3.1.4}$$

where $\mathbf{x}_i$ is the n_i-dimensional state vector of the ith subsystem. Eqn (3.1.4) will be referred to as the ith subsystem. If the interactions between the subsystems are neglected ($\mathbf{s}_i(t) = \mathbf{0}$), eqn (3.1.4) yields the model of the isolated subsystem

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i(t) &= \mathbf{C}_i\mathbf{x}_i(t) + \mathbf{D}_i\mathbf{u}_i(t). \end{aligned} \tag{3.1.5}$$

The interconnections of the subsystems (3.1.4) are described by

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{3.1.6}$$

where the vectors $\mathbf{s}$ and $\mathbf{z}$ of dimension m_s or r_z, respectively, consist of the interconnection inputs $\mathbf{s}_i$ and outputs $\mathbf{z}_i$ of the subsystems with dimensions m_{si} and r_{zi}: $\mathbf{s} = (\mathbf{s}_1' \ \mathbf{s}_2' \ \dots \ \mathbf{s}_N')'$, $\mathbf{z} = (\mathbf{z}_1' \ \dots \ \mathbf{z}_N')'$. The interconnection relation can be represented by the algebraic equation (3.1.6) if all the dynamical elements of the system are considered as part of some subsystem. The model (3.1.4) and (3.1.6) makes clear which subsystems comprise the whole system and which interactions exist among these subsystems.

Relation between the Unstructured Model and the Interaction-oriented Model

A representation of the overall system matrices **A**, **B**, **C** and **D** in terms of the subsystem matrices $\mathbf{A}_i$, $\mathbf{B}_i$, ... and the interconnection matrix **L** can be formulated as follows. Writing the subsystem equations (3.1.4)

$(i = 1, \ldots, N)$ one below the other leads to

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \text{diag } \mathbf{A}_i\mathbf{x}(t) + \text{diag } \mathbf{B}_i\mathbf{u}(t) + \text{diag } \mathbf{E}_i\mathbf{s}(t)\\ \mathbf{y}(t) &= \text{diag } \mathbf{C}_i\mathbf{x}(t) + \text{diag } \mathbf{D}_i\mathbf{u}(t) + \text{diag } \mathbf{F}_i\mathbf{s}(t)\\ \mathbf{z}(t) &= \text{diag } \mathbf{C}_{zi}\mathbf{x}(t) + \text{diag } \mathbf{D}_{zi}\mathbf{u}(t) + \text{diag } \mathbf{F}_{zi}\mathbf{s}(t)\end{aligned} \tag{3.1.7}$$

and $\mathbf{x}(0) = \mathbf{x}_0$ where

$$\mathbf{x} = (\mathbf{x}_1' \;\; \mathbf{x}_2' \;\; \ldots \;\; \mathbf{x}_N')' \tag{3.1.8}$$

and $\mathbf{u} = (\mathbf{u}_1' \;\; \ldots \;\; \mathbf{u}_N')'$ hold; diag $\mathbf{A}_i$ stands for a block-diagonal matrix with the diagonal blocks $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$. Eqns (3.1.6) and (3.1.7) yield

$$\begin{pmatrix}\mathbf{I} & \mathbf{0} & -\text{diag } \mathbf{E}_i\mathbf{L}\\ \mathbf{0} & \mathbf{I} & -\text{diag } \mathbf{F}_i\mathbf{L}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L}\end{pmatrix}\begin{pmatrix}\dot{\mathbf{x}}\\ \mathbf{y}\\ \mathbf{z}\end{pmatrix} = \begin{pmatrix}\text{diag } \mathbf{A}_i\\ \text{diag } \mathbf{C}_i\\ \text{diag } \mathbf{C}_{zi}\end{pmatrix}\mathbf{x} + \begin{pmatrix}\text{diag } \mathbf{B}_i\\ \text{diag } \mathbf{D}_i\\ \text{diag } \mathbf{D}_{zi}\end{pmatrix}\mathbf{u}. \tag{3.1.9}$$

The matrix on the left-hand side of eqn (3.1.9) is invertible if and only if

$$\det(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L}) \neq 0 \tag{3.1.10}$$

holds. If so, a model of the form (3.1.1) can be derived from eqn (3.1.9) where

$$\begin{aligned}\mathbf{A} &= \text{diag } \mathbf{A}_i + \text{diag } \mathbf{E}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{C}_{zi}\\ \mathbf{B} &= \text{diag } \mathbf{B}_i + \text{diag } \mathbf{E}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{D}_{zi}\\ \mathbf{C} &= \text{diag } \mathbf{C}_i + \text{diag } \mathbf{F}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{C}_{zi}\\ \mathbf{D} &= \text{diag } \mathbf{D}_i + \text{diag } \mathbf{F}_i\mathbf{L}(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})^{-1} \text{ diag } \mathbf{D}_{zi}\end{aligned} \tag{3.1.11}$$

hold. Eqn (3.1.11) shows how the subsystem and interconnection parameters combine with the overall system parameters. These relations are easier to understand under the reasonable assumption that the subsystem models (3.1.4) have no direct throughput of $\mathbf{u}_i$ and $\mathbf{s}_i$ towards $\mathbf{z}_i$ and $\mathbf{y}_i$, that is

$$\mathbf{D}_i = \mathbf{0} \qquad \mathbf{F}_i = \mathbf{0} \qquad \mathbf{D}_{zi} = \mathbf{0} \qquad \mathbf{F}_{zi} = \mathbf{0} \tag{3.1.12}$$

$(i = 1, \ldots, N)$ hold. Then, after partitioning the interconnection matrix $\mathbf{L}$ in (3.1.5) according to the structure of $\mathbf{s}$ and $\mathbf{z}$

$$\mathbf{L} = \begin{pmatrix}\mathbf{L}_{11} & \mathbf{L}_{12} & \ldots & \mathbf{L}_{1N}\\ \mathbf{L}_{21} & \mathbf{L}_{22} & \ldots & \mathbf{L}_{2N}\\ \vdots & \vdots & & \vdots\\ \mathbf{L}_{N1} & \mathbf{L}_{N2} & \ldots & \mathbf{L}_{NN}\end{pmatrix} \tag{3.1.13}$$

eqn (3.1.11) has the simpler form

$$\begin{aligned}
&\mathbf{A} = (\mathbf{A}_{ij}) \quad \text{with} \quad \mathbf{A}_{ii} = \mathbf{A}_i + \mathbf{E}_i\mathbf{L}_{ii}\mathbf{C}_{zi} \\
&\qquad\qquad\qquad\qquad\; \mathbf{A}_{ij} = \mathbf{E}_i\mathbf{L}_{ij}\mathbf{C}_{zj} \quad \text{for } i \neq j \\
&\mathbf{B} = \operatorname{diag} \mathbf{B}_i \\
&\mathbf{C} = \operatorname{diag} \mathbf{C}_i \\
&\mathbf{D} = \mathbf{0}.
\end{aligned} \tag{3.1.14}$$

Obviously, the subsystem matrices $\mathbf{A}_i$ occur as diagonal blocks of $\mathbf{A}$ whereas the interactions as described by $\mathbf{E}_i$, $\mathbf{C}_{zi}$ and $\mathbf{L}$ are parts of the non-diagonal blocks $\mathbf{A}_{ij}$ $(i \neq j)$. In particular, if, as often happens, the diagonal blocks of $\mathbf{L}$ vanish ($\mathbf{L}_{ii} = \mathbf{0}$ in eqn (3.1.13)) because the interconnection input $\mathbf{s}_i$ does not directly depend on the interconnection output $\mathbf{z}_i$ of the same subsystem, the diagonal blocks of $\mathbf{A}$ equal the subsystem matrices $\mathbf{A}_i$ ($\mathbf{A}_{ii} = \mathbf{A}_i$). $\mathbf{B}$ and $\mathbf{C}$ are block diagonal. If the subsystems have no direct throughput the same holds for the overall system ($\mathbf{D} = \mathbf{0}$).

Equation (3.1.14) says that under the assumption (3.1.12) the matrices $\mathbf{B}_{si}$ and $\mathbf{C}_{si}$ of the I/O-oriented model (3.1.2) can be written as

$$\mathbf{B}_{si} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{B}_i \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \qquad \mathbf{C}_{si} = (\mathbf{0} \;\ldots\; \mathbf{0} \;\; \mathbf{C}_i \;\; \mathbf{0} \;\ldots\; \mathbf{0}) \tag{3.1.15}$$

where only the ith block is non-vanishing. By using eqns (3.1.14) and (3.1.15) a further form of the overall system model is obtained

$$\begin{aligned}
&\dot{\mathbf{x}}_i(t) = \mathbf{A}_{ii}\mathbf{x}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\
&\mathbf{y}_i(t) = \mathbf{C}_i\mathbf{x}_i(t) \qquad (i = 1, \ldots, N).
\end{aligned} \tag{3.1.16}$$

This model is said to have an *input–output decentralized form* (cf. Section 3.3). It will be used if the dependencies between the subsystem states $\mathbf{x}_i$ are investigated. In eqn (3.1.16) these dependencies are described by the matrices $\mathbf{A}_{ij}$.

In this context, the overall system matrix $\mathbf{A}$ is sometimes decomposed into

$$\mathbf{A}_{\mathrm{D}} = \operatorname{diag} \mathbf{A}_{ii} \tag{3.1.17}$$

and

$$\mathbf{A}_\mathrm{C} = \mathbf{A} - \mathbf{A}_\mathrm{D} = \begin{pmatrix} \mathbf{0} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1N} \\ \mathbf{A}_{21} & \mathbf{0} & \dots & \mathbf{A}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \dots & \mathbf{0} \end{pmatrix} \tag{3.1.18}$$

which represents the interaction relation.

Two further remarks have to be made concerning the relation of the models (3.1.1) and (3.1.4) and (3.1.6). First, (3.1.10) represents not merely a condition under which the overall system model (3.1.1) and (3.1.11) can be derived from (3.1.4) and (3.1.6), but it also ensures the existence of some model of the form (3.1.1) due to the uniqueness of the solution of (3.1.4) and (3.1.6).

Theorem 3.1

The equations (3.1.4) and (3.1.6) have a unique solution and can be represented in the form (3.1.1) if and only if the condition (3.1.10) is satisfied.

Proof

The sufficiency has been proved by constructing the model (3.1.1) and (3.1.11) from (3.1.4) and (3.1.6). In order to prove the necessity consider the last row

$$(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})\mathbf{z} = \text{diag } \mathbf{C}_{zi}\mathbf{x} + \text{diag } \mathbf{D}_{zi}\mathbf{u} \tag{3.1.19}$$

of eqn (3.1.9). If the matrix $(\mathbf{I} - \text{diag } \mathbf{F}_{zi}\mathbf{L})$ is singular, a zero row can be made to appear in this matrix by elementary row operations. Then, eqn (3.1.19) has the form

$$\begin{pmatrix} * \\ \mathbf{0} \end{pmatrix} \mathbf{z}(t) = \begin{pmatrix} * \\ \mathbf{a}' \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} * \\ \mathbf{b}' \end{pmatrix} \mathbf{u}(t)$$

where the asterisks denote arbitrary blocks and $\mathbf{a}'$ and $\mathbf{b}'$ row vectors. For $\mathbf{a}' \neq \mathbf{0}$ and $\mathbf{b}' \neq \mathbf{0}$ the last line

$$\mathbf{a}'\mathbf{x}(t) + \mathbf{b}'\mathbf{u}(t) = 0$$

states a linear dependence between $\mathbf{x}(t)$ and $\mathbf{u}(t)$. Otherwise, $\mathbf{a}' = \mathbf{0}$ or $\mathbf{b}' = \mathbf{0}$ implies a restriction on $\mathbf{x}$ or $\mathbf{u}$, respectively. Both implications contradict the assumptions that the input $\mathbf{u}(t)$ can be chosen arbitrarily. If both $\mathbf{a}' = \mathbf{0}$ and $\mathbf{b}' = \mathbf{0}$ hold, $\mathbf{z}(t)$ and, thus, $\mathbf{x}(t)$ cannot be uniquely determined from eqn (3.1.9). Hence, no overall system model (3.1.1) exists. □

The second remark concerns the order of the overall system. The model (3.1.1) and (3.1.11) has been derived under the assumption (3.1.8). That is, the subsystem state spaces $\mathbf{X}_i$ are assumed to be disjoint so that the overall system state $\mathbf{x}$ is simply the collection (3.1.8) of all subsystem states. Equivalently,

$$\mathbf{X} = \mathbf{X}_1 \oplus \mathbf{X}_2 \oplus \cdots \oplus \mathbf{X}_N \tag{3.1.20}$$

holds, where $\oplus$ denotes the direct sum of the vector spaces $\mathbf{X}_i$.

Although a model with this system state $\mathbf{x}$ exists under condition (3.1.9), this model need not be a minimal realization. Several state variables may coincide or some linear combination of them may be replaced by a single state variable. Problems with such overlapping subsystem states will be considered in connection with symmetric systems (Chapter 12), where the overlapping occurs due to the system structure, and in a generalized decomposition method (Section 3.4), where the overlapping is deliberately introduced by an expansion of the overall system state space.

3.2 HIERARCHICALLY STRUCTURED SYSTEMS

Most of the difficulties of analytical and control problems are raised by the complete interdependence of the subsystems. That is, there are links between arbitrary pairs of subsystems. Such a link from the ith to the jth subsystem need not be direct but may be mediated by one or more other subsystems.

Indirect couplings are typical of systems with sparse interconnections. They render more difficult the question of which subsystems are really coupled. The *sparsity* of interconnection means that the number of direct couplings among the subsystems is small in relation to the maximum number N^2. Sparsity must not be confused with the *weakness* of interconnections, which refers to the fact that the existing links do not severely influence the overall system performance, so that the subsystems behave similarly when coupled together or when isolated from each other.

Conceptual simplifications of analytical and control problems can be obtained if some subsystems have only a one-way effect on some others. The way in which this situation can be recognized will be investigated now.

The Interconnection Structure

The interactions among the subsystems (3.1.4) are described by the relation (3.1.6)

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{3.2.1}$$

where $\mathbf{s} = (\mathbf{s}_1' \ \mathbf{s}_2' \ \dots \ \mathbf{s}_N')'$ and $\mathbf{z} = (\mathbf{z}_1' \ \mathbf{z}_2' \ \dots \ \mathbf{z}_N')'$. The matrix $\mathbf{L}$ can be decomposed in correspondence with the vectors $\mathbf{s}$ and $\mathbf{z}$

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} & \dots & \mathbf{L}_{1N} \\ \mathbf{L}_{21} & \mathbf{L}_{22} & \dots & \mathbf{L}_{2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{L}_{N1} & \mathbf{L}_{N2} & \dots & \mathbf{L}_{NN} \end{pmatrix}. \tag{3.2.2}$$

The block $\mathbf{L}_{ij}$ describes the couplings from the jth subsystem to the ith one. If $\mathbf{L}_{ij} = \mathbf{0}$ holds, no direct coupling exists. However, the jth subsystem may influence the ith one indirectly via other subsystems.

Under what condition this roundabout way exists can be found by a qualitative analysis of the interaction relation (3.2.1), in which only the existence of couplings rather than their strength is considered. Instead of the numeric matrix $\mathbf{L}$, the structure matrix $[\mathbf{L}]$ is used (cf. Section 2.5). $[\mathbf{L}]$ is obtained from $\mathbf{L}$ after all non-vanishing elements have been replaced by the indeterminate element '$*$'. If the interconnection signals $\mathbf{s}_i$ and $\mathbf{z}_i$ are vectors rather than scalars, $\mathbf{L}_{ij}$ in eqn (3.2.2) are matrices. The same holds for $[\mathbf{L}_{ij}]$. However, since only the existence of some interconnection should be investigated, the matrices $[\mathbf{L}_{ij}]$ will be reduced to the scalar $[[\mathbf{L}_{ij}]]$. That is, the scalar $[[\mathbf{A}]]$ is defined for an (n, m) matrix $\mathbf{A} = (a_{ij})$ by

$$[[\mathbf{A}]] = \begin{cases} 0 & \text{if } \mathbf{A} = \mathbf{0} \\ * & \text{if } a_{ij} \neq 0 \text{ for at least one pair of indices } i, j. \end{cases} \tag{3.2.3}$$

For the compound matrix $\mathbf{L}$ in eqn (3.2.2), $[[\mathbf{L}]]$ is defined as the (N, N) matrix

$$[[\mathbf{L}]] = \begin{pmatrix} [[\mathbf{L}_{11}]] & [[\mathbf{L}_{12}]] & \dots & [[\mathbf{L}_{1N}]] \\ \vdots & \vdots & & \vdots \\ [[\mathbf{L}_{N1}]] & [[\mathbf{L}_{N2}]] & \dots & [[\mathbf{L}_{NN}]] \end{pmatrix}.$$

This matrix is used to describe the interconnection structure of the overall system.

An overall system with N subsystems (3.1.4) whose interconnections (3.1.6) are described by a given matrix $\mathbf{L}$ is represented by $\mathbf{S}(N, \mathbf{L})$. Then, for a given structure matrix $\mathbf{S}_1$ the class of systems (3.1.4) and (3.1.6) with structurally equivalent interactions is described by

$$\mathcal{S}_1(\mathbf{S}_1) = \{\mathbf{S}(N, \mathbf{L}): \ [[\mathbf{L}]] = \mathbf{S}_1\}. \tag{3.2.4}$$

The interconnection structure of all systems of this class can be represented by the directed graph $G(\mathbf{S}_1)$ whose N vertices visualize the subsystems and whose edges mark the direct interconnection links among the subsystems.

Example 3.1

Consider an overall system with six subsystems, $\dim \mathbf{s}_i = \dim \mathbf{z}_i = 1$ and interconnection matrix

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & l_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{25} & 0 \\ l_{31} & 0 & 0 & l_{34} & 0 & 0 \\ l_{41} & 0 & 0 & 0 & 0 & l_{46} \\ 0 & l_{52} & l_{53} & 0 & 0 & 0 \\ l_{61} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2.5}$$

The interconnections have the structure described by

$$[[\mathbf{L}]] = \begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.2.6}$$

where $[[\mathbf{L}]] = [\mathbf{L}]$ holds since the interconnection signals are scalar. Although this matrix is sparse, it cannot be immediately recognized which subsystems are coupled in both directions. The graph $G([[\mathbf{L}]])$ with the adjacency matrix $[[\mathbf{L}]]$ from eqn (3.2.6) is shown in Figure 3.2. Obviously, the overall system consists of three groups of subsystems two of which are encircled by dashed lines. Within these groups the subsystems are strongly coupled in the sense that there are direct or indirect links between each pair of subsystems. In what follows it will be explained how these groups can be found systematically. □

Definition 3.1

Consider the class $\mathscr{S}_1$ of interconnected systems. The subsystems i and j of a system $\mathbf{S}(N, \mathbf{L}) \in \mathscr{S}_1$ are called *strongly coupled* if in the graph $G(\mathbf{S}_1)$ there exist a path from vertex i to vertex j and a path from vertex j to vertex i.

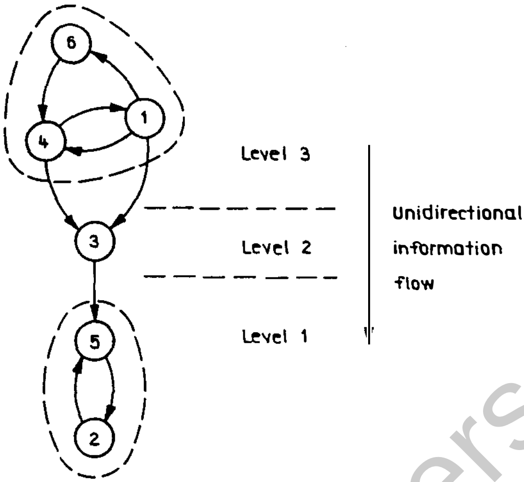


Figure 3.2 Hierarchical structure of the system in Example 3.1

That is, strongly coupled subsystems are represented by strongly connected vertices of $G(\mathbf{S}_1)$ (cf. Definition A2.1 in Appendix 2).

Decomposition of the Overall System into Groups of Strongly Coupled Subsystems

The subset of subsystems which are strongly coupled with a given subsystem i forms an equivalence class within the set of all the N subsystems of a given system $\mathbf{S}(N, \mathbf{L})$. That is, the index set

$$\mathscr{I} = \{1, 2, \ldots, N\} \tag{3.2.7}$$

which represents the numbers of the subsystems, can be uniquely decomposed into disjoint sets

$$\mathscr{I}_i = \{j_{i1}, j_{i2}, \ldots, j_{ik_i}\} \tag{3.2.8}$$

so that all pairs of subsystems of the same set $\mathscr{I}_i$ are strongly coupled whereas the subsystems of different sets $\mathscr{I}_k$, $\mathscr{I}_l$ $(k \neq l)$ do not possess this property.

Theorem 3.2

The decomposition of the overall system into strongly coupled subsystems is given by the equivalence relation on the index set $\mathscr{I}$ of the subsystems according to which $\mathcal{I}$ is decomposed into $\bar{N}$ disjoint subsets $\mathcal{I}_i$

$$\mathcal{I} = \bigcup_{i=1}^{\bar{N}} \mathcal{I}_i \qquad \mathcal{I}_i \cap \mathcal{I}_j = \emptyset \quad \text{for all } i \neq j \tag{3.2.9}$$

where all subsystems with indices of the same set $\mathcal{I}_i$ are strongly coupled with each other.

The sets $\mathcal{I}_i$ can be found by graph search algorithms. For each vertex i the set $\mathcal{R}_i$ of reachable vertices has to be determined. If $i \in \mathcal{R}_j$ and $j \in \mathcal{R}_i$ hold, then the ith and the jth subsystems belong to the same set $\mathcal{I}_k$.

The set of equivalence classes $\mathcal{I}_i$ can be renumbered in such a way that there are no interactions from subsystems of equivalence classes of lower indices towards subsystems belonging to equivalence classes of higher indices. This reordering can be represented by a permutation matrix $\mathbf{P}$. A permutation matrix is a matrix whose only non-vanishing elements are exactly one '1' in each row and each column. The new interconnection matrix $\tilde{\mathbf{L}}$, which describes the interactions after the reordering of the subsystems, is obtained from $\mathbf{L}$ according to

$$\tilde{\mathbf{L}} = \mathbf{P}'\mathbf{L}\mathbf{P}. \tag{3.2.10}$$

The matrix $\tilde{\mathbf{L}}$ is block triangular if it is decomposed according to the decomposition (3.2.9) of the index set $\mathcal{I}$:

$$\tilde{\mathbf{L}} = \begin{pmatrix} \tilde{\mathbf{L}}_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \tilde{\mathbf{L}}_{21} & \tilde{\mathbf{L}}_{22} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \tilde{\mathbf{L}}_{\bar{N}1} & \tilde{\mathbf{L}}_{\bar{N}2} & \tilde{\mathbf{L}}_{\bar{N}3} & \dots & \tilde{\mathbf{L}}_{\bar{N}\bar{N}} \end{pmatrix}. \tag{3.2.11}$$

The diagonal blocks $\tilde{\mathbf{L}}_{ii}$ describe the couplings among those subsystems that belong to the same set $\mathcal{I}_i$ and, thus, form the ith hypersubsystem (or ith cluster of subsystems). The blocks $\tilde{\mathbf{L}}_{ij}$ describe the interconnections from subsystems of $\mathcal{I}_j$ to subsystems of $\mathcal{I}_i$.

The overall system is said to have a *hierarchical structure* since the cluster of subsystems can be grouped in different levels where the information flow is unidirectional from clusters at higher levels towards clusters at lower levels (Figure 3.2).

As a consequence, the matrix $\mathbf{A}$ of the overall system (3.1.1) is block triangular too (cf. (3.1.14) with $\tilde{\mathbf{L}}$ instead of $\mathbf{L}$) if it is decomposed according to the clusters of subsystems

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{\bar{N}1} & \mathbf{A}_{\bar{N}2} & \dots & \mathbf{A}_{\bar{N}\bar{N}} \end{pmatrix}. \tag{3.2.12}$$

Therefore, the stability analysis of the overall system can be simplified as follows.

Corollary 3.1

A hierarchically structured overall system is stable if and only if all the $\bar{N}$ hypersubsystems, each of which comprises a cluster of strongly coupled subsystems, are stable.

Example 3.1 (cont.)

The example system can be decomposed into three groups of subsystems

$$\mathscr{I}_1 = \{1, 4, 6\} \qquad \mathscr{I}_2 = \{3\} \qquad \mathscr{I}_3 = \{2, 5\}.$$

The permutation matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the new interconnection matrix $\tilde{\mathbf{L}}$ is obtained from eqn (3.2.10) as

$$\tilde{\mathbf{L}} = \left[\begin{array}{ccc|c|cc} 0 & 0 & l_{14} & 0 & 0 & 0 \\ l_{61} & 0 & 0 & 0 & 0 & 0 \\ l_{41} & l_{46} & 0 & 0 & 0 & 0 \\ \hline l_{31} & 0 & l_{34} & 0 & 0 & 0 \\ \hline 0 & 0 & l_{54} & 0 & 0 & l_{52} \\ 0 & 0 & 0 & 0 & l_{52} & 0 \end{array}\right]$$

which is a lower triangular matrix. $\tilde{\mathbf{L}}$ shows that no interactions exist from subsystems 3, 5, 2 towards 1, 6, 4 or from 2, 5 toward 3. According to Corollary 3.1, the overall system is stable if and only if subsystem 3 and the clusters consisting of subsystems 1, 4, 6 or 2, 5, respectively, are stable. □

Chain-connected Systems

Systems whose subsystems are coupled as a chain are hierarchically structured (Figure 3.3). The interaction relation is

$$\mathbf{s}_1 = \mathbf{0} \qquad \mathbf{s}_{i+1} = \mathbf{L}_{i+1,i}\mathbf{z}_i \quad (i = 1, 2, \ldots, N-1) \tag{3.2.13}$$

which leads to

$$\mathbf{L} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{32} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L}_{N,N-1} & \mathbf{0} \end{pmatrix}.$$

The overall system matrix is given by

$$\mathbf{A} = (\mathbf{A}_{ij}) \quad \text{with } \mathbf{A}_{ii} = \mathbf{A}_i \tag{3.2.14}$$
$$\mathbf{A}_{i,i+1} = \mathbf{E}_{i+1}\mathbf{L}_{i+1,i}\mathbf{C}_{zi}$$
$$\mathbf{A}_{ij} = \mathbf{0} \text{ for all other } i, j$$

(cf. eqn (3.1.14)).

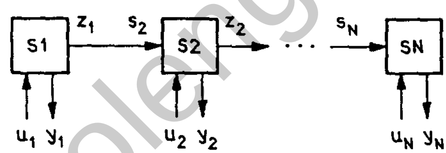


Figure 3.3 Chain-connected subsystems

Example 3.2 (*River quality control*)

The water quality of a river is mainly described by the concentrations of oxygen and pollutants. Although sewage is treated it can be carried to a natural waterway for disposal only to an amount which does not make the concentrations of the pollutants exceed prescribed bounds. In a simplified way, this problem can be stated as the task to control the sewage discharge at different places along the river in such a way that the river state remains within a given band of tolerance (Figure 3.4). A decentralized control scheme is appropriate because the variables to be controlled (concentrations of substances in different regions of the river) and the control inputs (upper bounds of the sewage discharge) are many kilometres away from each other.

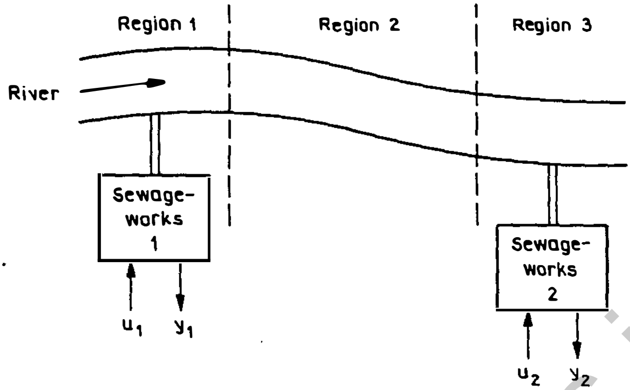


Figure 3.4 River with two sewage works

The following considerations are made under the assumption that the river is modelled with large time constants rather than delay elements since otherwise subsystem models with delays rather than eqn (3.1.4) have to be used. Owing to the unidirectional flow of the water, the different parts of the river and, thus, the different parts of the model are coupled in only one direction. Moreover, since only neighbouring regions are coupled, the interactions have the form (3.2.13). A river with three regions, only two of which have sewage stations, is described by a model of the form

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{A}_1\mathbf{x}_1 + \mathbf{B}_1\mathbf{u}_1 & \mathbf{y}_1 &= \mathbf{C}_1\mathbf{x}_1 & \mathbf{z}_1 &= \mathbf{C}_{z1}\mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{A}_2\mathbf{x}_2 + \mathbf{E}_2\mathbf{s}_2 & & & \mathbf{z}_2 &= \mathbf{C}_{z2}\mathbf{x}_2 \\ \dot{\mathbf{x}}_3 &= \mathbf{A}_3\mathbf{x}_3 + \mathbf{B}_3\mathbf{u}_2 + \mathbf{E}_3\mathbf{s}_3 & \mathbf{y}_2 &= \mathbf{C}_3\mathbf{x}_3 & & \end{aligned} \tag{3.2.15}$$

$$\mathbf{L} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix}.$$

The interaction matrix has block-triangular form. The overall system description is

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{12} & \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_3 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} \tag{3.2.16}$$

$$\mathbf{y}_1 = (\mathbf{C}_1 \;\; \mathbf{0} \;\; \mathbf{0})\mathbf{x} \qquad \mathbf{y}_2 = (\mathbf{0} \;\; \mathbf{0} \;\; \mathbf{C}_3)\mathbf{x}.$$

In the matrix **A** in the model (3.1.1) only the diagonal blocks and the blocks below the diagonal do not vanish. This model reflects the fact that the measurements are made 'near' the place where the sewage flows into the river (Figure 3.4). They are influenced only by those state variables which describe the water state in this region. □

Hierarchically Structured Systems with Input- and Output-reachable Subsystems

The division of the river system into three subsystems according to the partition of the matrices in eqn (3.2.16) highlights a problem that may arise in the decomposition of a system into hierarchically structured subsystems. For control purposes, not only the stability but also the controllability of the system is important. Obviously, the isolated subsystem 2 in Example 3.2

$$\dot{\mathbf{x}}_2 = \mathbf{A}_2\mathbf{x}_2$$

is not controllable. The question arises of how to decompose the overall system into hierarchically ordered subsystems while simultaneously ensuring the controllability and observability of all subsystems. That is, the overall system (3.1.1) should be decomposed into a model of the form

$$\begin{aligned} \dot{\mathbf{x}}_i &= \sum_{j=1}^{i} \mathbf{A}_{ij}\mathbf{x}_j + \sum_{j=1}^{i} \mathbf{B}_{ij}\mathbf{u}_j \\ \mathbf{y}_i &= \sum_{j=1}^{i} \mathbf{C}_{ij}\mathbf{x}_j \qquad (i = 1, 2, \ldots, N) \end{aligned} \tag{3.2.17}$$

where $(\mathbf{A}_{ii}, \mathbf{B}_{ii})$ is controllable and $(\mathbf{A}_{ii}, \mathbf{C}_{ii})$ observable for all $i = 1, \ldots, N$. Note that the summation is carried out only over the 'higher-level' subsystems.

Such a decomposition can be found by investigating the system structure, which is represented by the graph $G(\mathbf{Q}_0)$ with

$$\mathbf{Q}_0 = \begin{pmatrix} [\mathbf{A}] & [\mathbf{B}] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ [\mathbf{C}] & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

(cf. Section 2.5). A hierarchical decomposition should be generated by permutating the state variables until $[\mathbf{A}]$, $[\mathbf{B}]$ and $[\mathbf{C}]$ are block triagonal. Such a permutation does not change the physical meaning of the state variables and, thus, leads to subsystems that are physically interpretable.

In this structural decomposition the additional requirement of (numerical) controllability and observability of the resulting subsystems is replaced by the claim to obtain subsystems that are input- and output-connectable. Note that these structural requirements are only necessary conditions for structural controllability and observability (Theorem 2.15). They are used because they can be checked easily in the graph $G(\mathbf{Q}_0)$.

A graph-theoretic characterization of the required result can be given in terms of partitions of the graph $G(\mathbf{Q}_0)$. A partition of a given graph G into a set of $\bar{N}$ subgraphs is produced by decomposing the set $\mathcal{V}$ of vertices of G into disjoint sets $\mathcal{V}_i$

$$\mathcal{V}_i \cap \mathcal{V}_j = \emptyset \quad \text{for all } i \neq j \qquad \mathcal{V} = \bigcup_{i=1}^{\bar{N}} \mathcal{V}_i. \tag{3.2.18}$$

The partition of $G(\mathbf{Q}_0)$ is said to be an *acyclic I/O-connectable partition* if the submatrices $[\mathbf{A}]$, $[\mathbf{B}]$, $[\mathbf{C}]$ of the adjacency matrix $\mathbf{Q}_0$ are block triangular and if each subgraph is input- and output-connectable.

The conditions under which an I/O-connectable partition exists should be explained first for an overall system with two scalar inputs u_1, u_2 and two scalar outputs y_1, y_2. A hierarchical decomposition is possible if, after renumbering the state variables and the input and output signals, two conditions on the reachability matrix of the graph $G(\mathbf{Q}_0)$ are satisfied. The reachability matrix $\mathbf{R}_0$ describes which vertices are connected by a path. It can be directly determined from the adjacency matrix $\mathbf{Q}_0$ (cf. eqn (A2.1) in Appendix 2). For the graph $G(\mathbf{Q}_0)$ it has the structure

$$\mathbf{R} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} \mathcal{X} & \mathcal{U} & \mathcal{Y} \end{array} & \\ \begin{pmatrix} \mathbf{R}_{xx} & \mathbf{R}_{xu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{yx} & \mathbf{R}_{yu} & \mathbf{0} \end{pmatrix} & \begin{array}{c} \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{array} \end{array} \tag{3.2.19}$$

where $\mathbf{R}_{xx}$ described which state vertices can be reached from other state vertices, and $\mathbf{R}_{xu}$ which state vertices can be reached from some input vertices etc. (cf. the labels of the rows and columns of $\mathbf{R}$ in eqn (3.2.19).

The first condition on $\mathbf{R}$ to be introduced for the two-input two-output system claims that the output vertex y_1 must not be reachable from the input vertex u_2, that is the output y_1 does not depend, via the state variables of the whole system, on u_2. Hence, the (2, 2) matrix $\mathbf{R}_{yu}$ must have the form

$$\mathbf{R}_{yu} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \tag{3.2.20}$$

where '1' denotes reachable and '0' non-reachable connections; '*' symbolizes an arbitrary element. Eqn (3.2.20) says that there must exist paths from u_1 to y_1 and u_2 to y_2. Furthermore, there may exist a path from u_1 to y_2 but not from u_2 to y_1. As is clear from the way in which the reachability matrix $\mathbf{R}_0$ can be determined from the adjacency matrix $\mathbf{Q}_0$, this structure of $\mathbf{R}_{yu}$ ensures that the set of state vertices $\mathscr{X}$ may be partitioned into $\mathscr{X}_1$ and $\mathscr{X}_2$ so that the adjacency matrices $[\mathbf{A}]$, $[\mathbf{B}]$ and $[\mathbf{C}]$ and the reachability matrix $\mathbf{R}_{xx}, \mathbf{R}_{xu}, \mathbf{R}_{yx}$ are block triangular, that is

$$\mathbf{R}_0 = \begin{array}{c} \begin{array}{cccccc} \mathscr{X}_1 & \mathscr{X}_2 & u_1 & u_2 & y_1 & y_2 \end{array} \\ \left[\begin{array}{cc|cc|cc} \mathbf{R}_{xx11} & \mathbf{0} & \mathbf{R}_{xu11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{xx21} & \mathbf{R}_{xx22} & \mathbf{R}_{xu21} & \mathbf{R}_{xu22} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{R}_{yx11} & \mathbf{0} & \mathbf{R}_{yu11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{yx21} & \mathbf{R}_{yx22} & \mathbf{R}_{yu21} & \mathbf{R}_{yu22} & \mathbf{0} & \mathbf{0} \end{array}\right] \end{array} \begin{array}{c} \\ \mathscr{X}_1 \\ \mathscr{X}_2 \\ u_1 \\ u_2 \\ y_1 \\ y_2 \end{array} . \qquad (3.2.21)$$

Note that for the two-input two-output system $\mathbf{R}_{xu11}, \mathbf{R}_{xu21}, \mathbf{R}_{xu22}$ are column vectors, $\mathbf{R}_{yx11}, \mathbf{R}_{yx21}, \mathbf{R}_{yx22}$ row vectors, and $\mathbf{R}_{yu11}$, $\mathbf{R}_{yu21}, \mathbf{R}_{yu22}$ scalars. The labels of the (hyper) rows and (hyper) columns of $\mathbf{R}_0$ in eqn (3.2.21) refer to the sets of vertices $\mathscr{X}_1$ and $\mathscr{X}_2$ or the single vertices u_1, u_2, y_1 and y_2, respectively.

The second condition has to ensure that the partition of the set $\mathscr{X}$ of state vertices into two sets $\mathscr{X}_1$ and $\mathscr{X}_2$ is made in such a way that both subsystems are I/O-connectable to the pertinent inputs and outputs. That is, the columns $\mathbf{R}_{xu11}$ and $\mathbf{R}_{xu22}$ and the rows $\mathbf{R}_{yx11}$ and $\mathbf{R}_{yx22}$ include no zero element. This condition can be formulated by means of the Boolean AND operation '$\wedge$', which is defined as follows. The (i, j)th element of the matrix $\mathbf{A} \wedge \mathbf{B}$ is a '1' if and only if the (i, j)th elements of $\mathbf{A}$ and $\mathbf{B}$ are both '1'. The condition is stated as

$$\begin{pmatrix} \mathbf{R}_{xu11} & \mathbf{0} \\ \mathbf{R}_{xu21} & \mathbf{R}_{xu22} \end{pmatrix} \wedge \begin{pmatrix} \mathbf{R}_{yx11}' & \mathbf{R}_{yx21}' \\ \mathbf{0} & \mathbf{R}_{yx22}' \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_2 \end{pmatrix}. \qquad (3.2.22)$$

where $\mathbf{e}_1 = (1 \;\; 1 \ldots 1)'$ and $\mathbf{e}_2 = (1 \;\; 1 \ldots 1)'$ hold.

Example 3.3

Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 2 & 0 & 3 \\ 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 2 & -1 & 4 & 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad (3.2.23)$$

$$\mathbf{y} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x}$$

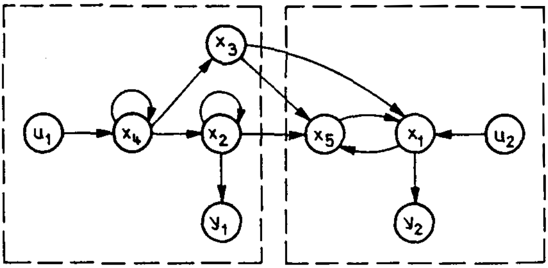


Figure 3.5 Structure of the system (3.2.23)

whose graph is shown in Figure 3.5. The reachability matrix $\mathbf{R}_{yu}$ satisfies the condition (3.2.20) since there are paths from u_1 to y_1 and y_2 and from u_2 to y_2 but not from u_2 to y_1. If the state vertices are renumbered $(1 \to 4, 4 \to 1)$ the reachability matrix is

$$\mathbf{R}_0 = \begin{array}{c} \begin{array}{ccccccccc} x_4 & x_2 & x_3 & x_1 & x_5 & u_1 & u_2 & y_1 & y_2 \end{array} \\ \left(\begin{array}{ccccc|cc|cc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ x_4 \\ x_2 \\ x_3 \\ x_1 \\ x_5. \\ u_1 \\ u_2 \\ y_1 \\ y_2 \end{array} \qquad (3.2.24)$$

For the partition $\{x_2, x_3, x_4\}$, $\{x_1, x_5\}$ of the set of state vertices the subgraphs are input- and output-connectable (Figure 3.5) and eqn (3.2.22) holds. The system can be decomposed into the form (3.2.17) with

$$\mathbf{A}_{11} = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{B}_{11} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{C}_{11} = (0 \;\; 1 \;\; 0)$$

$$\mathbf{A}_{21} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 4 \end{pmatrix} \qquad \mathbf{A}_{22} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

$$\mathbf{B}_{21} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \mathbf{B}_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{C}_{21} = (0 \;\; 0 \;\; 0) \qquad \mathbf{C}_{22} = (1 \;\; 0).$$

Another partition $\{x_2, x_4\}$, $\{x_1, x_3, x_5\}$ would lead to block-triangular matrices **A, B, C** too but the state x_3, which then belongs to subsystem 2, would not be controllable by u_2. Hence, eqn (3.2.22) would be violated. □

If the system has more than two inputs and two outputs, the sets $\mathscr{U}$ and $\mathscr{Y}$ have to be partitioned into $\mathscr{U}_1$, $\mathscr{U}_2$ and $\mathscr{Y}_1$, $\mathscr{Y}_2$ so that the reachability matrix $\mathbf{R}_{yu}$ is block triangular. If the columns of $\tilde{\mathbf{R}}_0$ corresponding to $\mathscr{U}_1$ or $\mathscr{U}_2$ and the rows corresponding to $\mathscr{Y}_1$ or $\mathscr{Y}_2$ are combined by the Boolean OR operation, a smaller matrix $\tilde{\mathbf{R}}_0$ results whose blocks $\tilde{\mathbf{R}}_{xuij}$ are single columns, $\tilde{\mathbf{R}}_{yxij}$ are single rows, and $\tilde{\mathbf{R}}_{yuij}$ are scalars. Then the decomposition exists if and only if the conditions (3.2.20) and (3.2.22) are satisfied.

The result is a system (3.2.17) with hierarchical structure. The isolated subsystems can be assumed to be structurally controllable and observable since the reachability is ensured (cf. Section 2.5). If decentralized controllers are applied, the closed-loop overall system (3.2.17) and (1.2.5) is hierarchically structured too.

3.3 DECOMPOSITION INTO DISJOINT SUBSYSTEMS

In Section 3.1 the overall system description (3.1.1) was formed from the subsystem models (3.1.4) and the interaction relation (3.1.6). This 'bottom-up' way can be used if the subsystem models are given in isolation from each other and the overall system model has to be found. In the following, the 'top-down' way from the overall system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{3.3.1}$$

to the subsystem models is considered.

The partition of the state vector $\mathbf{x}$ into subvectors $\mathbf{x}_i$ yields

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{A}_{ii}\mathbf{x}_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j(t) + \sum_{j=1}^{N} \mathbf{B}_{ij}\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i(t) &= \sum_{j=1}^{N} \mathbf{C}_{ij}\mathbf{x}_i(t) \qquad (i = 1, \ldots, N) \end{aligned} \tag{3.3.2}$$

where the matrices **B** and **C** have been partitioned into blocks $\mathbf{B}_{ij}$ and $\mathbf{C}_{ij}$ $(i = 1, \ldots, N;\ j = 1, \ldots, N)$ according to the partition of **x**, **u** and **y**. Obviously, the subsystem state $\mathbf{x}_i$ depends on all inputs $\mathbf{u}_i$, and the subsystem output $\mathbf{y}_i$ on all states $\mathbf{x}_i$. Since the aim is to get 'weakly coupled' subsystems the partition of $\mathbf{x}$ should be done in such a way that the dependencies of $\mathbf{y}_i$ on $\mathbf{u}_j$ $(i \neq j)$ are zero or weak. Systems of the form

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{A}_{ii}\mathbf{x}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i(t) &= \sum_{j=1}^{N} \mathbf{C}_{ij}\mathbf{x}_j(t) \qquad (i = 1, \dots, N) \end{aligned} \tag{3.3.3}$$

are said to be *input decentralized* and systems

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{A}_{ii}\mathbf{x}_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j(t) + \sum_{j=1}^{N} \mathbf{B}_{ij}\mathbf{u}_j(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i(t) &= \mathbf{C}_i\mathbf{x}_i(t) \qquad (i = 1, \dots, N) \end{aligned} \tag{3.3.4}$$

are called *output decentralized*. Similarly, the model (3.1.16) is called *input–output decentralized*. They are special forms of eqn (3.3.2) where $\mathbf{B}_{ij} = \mathbf{0}$ $(i \neq j)$ or $\mathbf{C}_{ij} = \mathbf{0}$ $(i \neq j)$ holds, respectively, and $\mathbf{B}_i = \mathbf{B}_{ii}$ or $\mathbf{C}_i = \mathbf{C}_{ii}$ is used. It is often assumed in the input-decentralized or the output-decentralized form that $\mathbf{u}_i$ or $\mathbf{y}_i$, respectively, are scalars. Then, the matrices $\mathbf{B}_i$ or $\mathbf{C}_i$ are replaced by the vectors $\mathbf{b}_i$ or $\mathbf{c}_i'$, respectively. Both systems can be expected to be weakly coupled since the influence of the input $\mathbf{u}_i$ on $\mathbf{y}_j$ has a 'long way' to go from $\mathbf{y}_i$ via the subsystem state $\mathbf{x}_i$ towards the subsystem state $\mathbf{x}_j$ and then towards $\mathbf{y}_i$. In particular, the system (3.3.3) or (3.3.4) is said to be weakly coupled if the matrices $\mathbf{A}_{ij}$ $(i \neq j)$ have small elements.

A reasonable way to decompose a given overall system (3.3.1) into input-decentralized subsystems starts with a transformation of the state vector

$$\tilde{\mathbf{x}} = \mathbf{Q}^{-1}\mathbf{x} \tag{3.3.5}$$

with

$$\mathbf{Q} = (\mathbf{b}_1 \;\; \mathbf{A}\mathbf{b}_1 \;\; \dots \;\; \mathbf{A}^{n_1-1}\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{A}\mathbf{b}_2 \;\; \dots \;\; \mathbf{A}^{n_m-1}\mathbf{b}_m) \tag{3.3.6}$$

where the vectors $\mathbf{b}_i$ denote the m columns of the matrix $\mathbf{B}$:

$$\mathbf{B} = (\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \dots \;\; \mathbf{b}_m).$$

Since the overall system is assumed to be controllable, the matrix $\mathbf{Q}$ can be made invertible by choosing appropriate scalars $n_1, \dots, n_m$. The transformed system (3.3.1) is described by

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \\ \mathbf{y} &= \tilde{\mathbf{C}}\tilde{\mathbf{x}} \end{aligned} \tag{3.3.7}$$

with

$$\tilde{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} \qquad \tilde{\mathbf{B}} = \text{diag } \tilde{\mathbf{b}}_i \qquad \tilde{\mathbf{C}} = \mathbf{C}\mathbf{Q} \tag{3.3.8}$$

where $\tilde{\mathbf{B}}$ has the n_i-dimensional rows

$$\tilde{\mathbf{b}}_i = (1 \;\; 0 \;\; \ldots \;\; 0)'$$

in its main diagonal.Then, the system (3.3.7) can be decomposed into

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_i &= \tilde{\mathbf{A}}_{ii}\tilde{\mathbf{x}}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \tilde{\mathbf{A}}_{ij}\tilde{\mathbf{x}}_j + \tilde{\mathbf{b}}_i\mathbf{u}_i \\ \tilde{\mathbf{y}}_i &= \sum_{j=1}^{N} \tilde{\mathbf{C}}_{ij}\tilde{\mathbf{x}}_j \qquad (i = 1, \ldots, m) \end{aligned} \tag{3.3.9}$$

where the blocks of $\tilde{\mathbf{A}} = (\tilde{\mathbf{A}}_{ij})$ and $\tilde{\mathbf{C}} = (\tilde{\mathbf{C}}_{ij})$ may not be zero but $\tilde{\mathbf{B}}$ has the structure as given above. Obviously, the overall system has been decomposed into m subsystems (3.3.9) which are input decentralized. The price for this is that the new state vector $\tilde{\mathbf{x}}$ need not be physically interpretable. This contrasts with the method for decomposing the system into hierarchical systems in the way described in Section 3.2 where the state variables have only been permutated. However, for the decomposition procedure above, in principle, no assumptions concerning the internal structure of the system have to be made.

A similar procedure can be described for the output decentralization of the subsystems, where the transformation matrix is built similarly to eqn (3.3.6) with $\mathbf{A}$ and the rows $\mathbf{c}_i'$ of $\mathbf{C}$.

The subsystems resulting from this decomposition have disjoint state vectors $\tilde{\mathbf{x}}_i$ since

$$\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1' \;\; \tilde{\mathbf{x}}_2' \;\; \ldots \;\; \tilde{\mathbf{x}}_N')' \tag{3.3.10}$$

holds. That is, the whole state space is divided into the state spaces of the subsystems (cf. (3.1.20)).

3.4 DECOMPOSITION INTO OVERLAPPING SUBSYSTEMS

The decomposition of the overall system into disjoint subsystems is not reasonable if the resulting subsystems are strongly coupled. This is particularly true if the interactions between the subsystems are mediated by a certain subsystem, as subsystem 1 in the structure of Figure 3.6. Then, overlapping decomposition can be used as an alternative way in which the resulting subsystems have some part in common. The over-

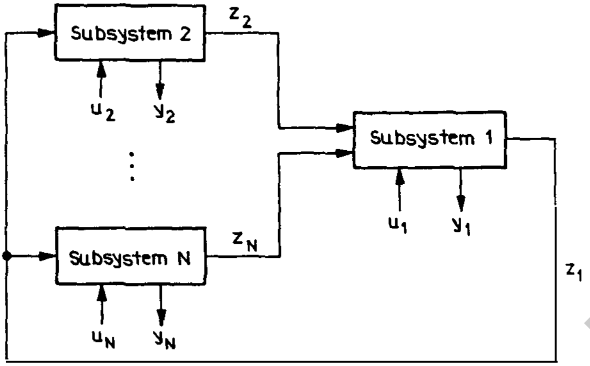


Figure 3.6 'Symmetrically' coupled system

lapping subsystems may be weakly coupled although disjoint subsystems are not.

A systematic way of overlapping decomposition starts with the expansion of the original system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{3.4.1}$$

which results in the system

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u} \qquad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \\ \mathbf{y} &= \bar{\mathbf{C}}\bar{\mathbf{x}}. \end{aligned} \tag{3.4.2}$$

Formally, the systems (3.4.1) and (3.4.2) are related by some contraction transformation

$$\mathbf{x} = \mathbf{T}^{+}\bar{\mathbf{x}} \tag{3.4.3}$$

where

$$\mathbf{T}^{+}\mathbf{T} = \mathbf{I} \tag{3.4.4}$$

holds with the superscript '+' denoting the pseudoinverse of a rectangular matrix.

Definition 3.2

A system (3.4.2) is said to *include* a system (3.4.1) if there exists an ordered pair of matrices $(\mathbf{T}, \mathbf{T}^{+})$ such that eqns (3.4.3) and (3.4.4) hold. The systems (3.4.1) or (3.4.2) are called *contraction* or *expansion*, respectively.

As the examples below will show, the expansion of (3.4.1) results in a new system (3.4.2) in which some parts of the original system (3.4.1) appear more than once. If (3.4.2) is decomposed into disjoint subsystems then these parts of (3.4.1) belong simultaneously to two or more subsystems. That is, the subsystems 'overlap'.

An expansion (3.4.2) can be found by using the matrices

$$\bar{\mathbf{A}} = \mathbf{TAT}^+ + \mathbf{M} \qquad \bar{\mathbf{B}} = \mathbf{TB} + \mathbf{N} \qquad \bar{\mathbf{C}} = \mathbf{CT}^+ + \mathbf{L} \tag{3.4.5}$$

and by choosing appropriate matrices $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{L}$.

Theorem 3.3

The system (3.4.2) and (3.4.5) is an expansion of (3.4.1) if and only if the following conditions are satisfied:

$$\begin{aligned} &\mathbf{T}^+\mathbf{M}^i\mathbf{T} = \mathbf{0} \qquad \mathbf{T}^+\mathbf{M}^{i-1}\mathbf{N} = \mathbf{0} \\ &\mathbf{LM}^{i-1}\mathbf{T} = 0 \qquad \mathbf{LM}^{i-1}\mathbf{N} = \mathbf{0} \qquad (i = 1, 2, \ldots, \dim \bar{\mathbf{x}}). \end{aligned} \tag{3.4.6}$$

This theorem can be proved by considering the equality

$$\mathbf{T}^+ \exp(\bar{\mathbf{A}}t)\mathbf{T} = \exp(\mathbf{A}t)$$

with the time series expansion of $\exp(\mathbf{A}t)$ and $\exp(\bar{\mathbf{A}}t)$.

The method of investigating a given system (3.4.1) by considering the expansion (3.4.2) and inferring the results to the contraction (3.4.1) is called the *inclusion principle*. An important fact for the application of the inclusion principle is that the stability property of the system (3.4.1) is preserved in the expansion.

Theorem 3.4

If the systems (3.4.1) and (3.4.2) are a contraction or an expansion, respectively, then the asymptotic stability of the system (3.4.2) implies the asymptotic stability of (3.4.1).

Example 3.4

Consider the system (3.4.1) with partitioned state vector $\mathbf{x} = (\mathbf{x}_1' \ \mathbf{x}_2' \ \mathbf{x}_3')'$ and structured matrices $\mathbf{A} = (\mathbf{A}_{ij})$, $\mathbf{B} = (\mathbf{B}_{ij})$. An overlapping decomposition is given by $\bar{\mathbf{x}}_1 = (\mathbf{x}_1' \ \mathbf{x}_2')'$, $\bar{\mathbf{x}}_2 = (\mathbf{x}_2' \ \mathbf{x}_3')'$ which satisfies eqn (3.4.3)

with

$$\mathbf{T} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \qquad \mathbf{T}^{+} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0.5\mathbf{I} & 0.5\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{3.4.7}$$

The matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are given by

$$\bar{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{A}_{23} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} \mathbf{0} & 0.5\mathbf{A}_{12} & -0.5\mathbf{A}_{12} & \mathbf{0} \\ \mathbf{0} & 0.5\mathbf{A}_{22} & -0.5\mathbf{A}_{22} & \mathbf{0} \\ \mathbf{0} & -0.5\mathbf{A}_{22} & 0.5\mathbf{A}_{22} & \mathbf{0} \\ \mathbf{0} & -0.5\mathbf{A}_{32} & 0.5\mathbf{A}_{32} & \mathbf{0} \end{pmatrix}$$

$$\bar{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \\ \mathbf{B}_{31} & \mathbf{B}_{32} \end{pmatrix} \qquad \mathbf{N} = \mathbf{0}.$$

Then, the isolated subsystems of the expansion (3.4.2) are

$$\begin{aligned} \dot{\bar{\mathbf{x}}}_1 &= \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \bar{\mathbf{x}}_1 + \begin{pmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{pmatrix} \mathbf{u}_1 \\ \dot{\bar{\mathbf{x}}}_2 &= \begin{pmatrix} \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix} \bar{\mathbf{x}}_2 + \begin{pmatrix} \mathbf{B}_{22} \\ \mathbf{B}_{32} \end{pmatrix} \mathbf{u}_2 \end{aligned} \tag{3.4.8}$$

A comparison with the overall system matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}$$

shows that the original system (3.4.1) has been expanded in such a way that the state $\mathbf{x}_2$ belongs to both new subsystems (3.4.8). Consequently, the matrix $\mathbf{A}_{22}$ is used twice. This explains the term 'overlapping decomposition'. The overall system state space is *not* the direct sum of

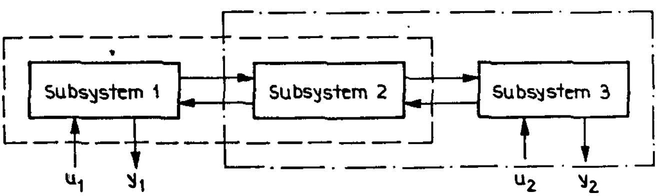


Figure 3.7 Overlapping decomposition

the subsystem state spaces. This decomposition is particularly useful for systems with $\mathbf{A}_{31} = \mathbf{0}$, $\mathbf{A}_{13} = \mathbf{0}$, $\mathbf{B}_{12} = \mathbf{0}$ and $\mathbf{B}_{21} = \mathbf{0}$, since these matrices are neglected when considering the decomposition (3.4.8) (Figure 3.7).□

Example 3.5 (*Multiarea power system*)

The control of multiarea power systems has been a major challenge for the development of the theory of large-scale systems. Power systems have nearly all the characteristics of complex systems mentioned in Section 1.1. They consist of many strongly coupled areas and have a wide geographical distribution. Their model has a large number of inputs and outputs and a long state vector. Several control layers are necessary to meet diverse operational specifications.

Power systems will be used several times in this book for illustration. First, the real power behaviour is considered by means of the model whose structure is shown in Figure 3.8. This model is valid for long time horizon investigations where all the rotating masses can be assumed to have equal velocity. Each power station together with the corresponding load is considered as an area. The control inputs $u_i = p_{si}$ are the set points of the power which is generated by the ith unit at nominal frequency ($f = 0$).

The frequency f, which is common to all areas, is determined from the difference p_b between the generated power p_g, the frequency-dependent consumed power p_p and the load p_l according to

$$f = \frac{1}{T_0} \int_0^t p_b(\tau)\, d\tau. \tag{3.4.9}$$

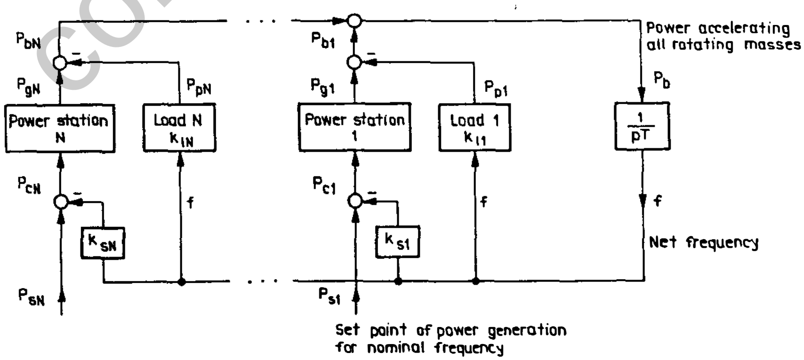


Figure 3.8 Structure of a multiarea power system

That is,

$$\dot{x}_1 = \frac{1}{T_0}\, s_1 \qquad z_1 = x_1 \tag{3.4.10}$$

and

$$s_1 = p_b = \sum_{i=1}^{N} p_{bi} = \sum_{i=1}^{N} p_{gi} - \sum_{i=1}^{N} p_{pi} - \sum_{i=1}^{N} p_{li} \tag{3.4.11}$$

hold, where $T_0 = T_1 + \cdots + T_N$ holds with T_i denoting the acceleration time constant of the rotating masses of area i. The system is now considered for constant load ($p_{li} = 0$). Each area constitutes a system with interconnection input $s_i = z_1$ and interconnection output $z_i = p_{bi}$. It can be described by a model of the form (3.1.4)

$$\begin{aligned}
\dot{\mathbf{x}}_i(t) &= \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{b}_i u_i(t) + \mathbf{e}_i s_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0}\\
\mathbf{y}_i(t) &= \mathbf{C}_i\mathbf{x}_i(t)\\
z_i(t) &= \mathbf{c}'_{zi}\mathbf{x}_i(t) \qquad (i = 2, \ldots, N).
\end{aligned} \tag{3.4.12}$$

The primary control inherent in such a power system is represented by the frequency dependence of the load and the power generation (see signal paths with k_{si}, k_{li} in Figure 3.8). The interconnection relation (3.1.6) has the form

$$s_1 = \sum_{j=2}^{N} z_j \qquad s_i = z_1 \qquad (i = 2, 3, \ldots, N). \tag{3.4.13}$$

Note that the subsystems are 'symmetrically' coupled with the first subsystem (cf. Figure 3.6). Owing to the specific interconnection structure, the matrix $\mathbf{A}$ in (3.4.1) has the form

$$\mathbf{A} = \left(\begin{array}{c|cccc}
\mathbf{O} & \mathbf{a}_{12}' & \mathbf{a}_{13}' & \cdots & \mathbf{a}_{1N}' \\
\hline
\mathbf{e}_2 & \mathbf{A}_2 & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{e}_3 & \mathbf{0} & \mathbf{A}_3 & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & & \vdots \\
\mathbf{e}_N & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_N
\end{array}\right) \tag{3.4.14}$$

where

$$\mathbf{a}_{1i} = \frac{1}{T_0}\,\mathbf{c}_{zi} \qquad (i = 2, \ldots, N)$$

holds. Notice that the non-vanishing elements in $\mathbf{A}$ form an arrow. This is typical for a system with the structure of Figure 3.6.

The given disjoint subsystems (3.4.10) and (3.4.12) have strong interactions since the couplings within the overall system are completely destroyed if the interconnection inputs s_i are set to zero (Figures 3.6 and 3.8). Therefore, the results obtained by analysing the isolated subsystems (eqns (3.4.10) and (3.4.12) with $s_i = 0$) are quite different from the result that can be received for the overall system (3.4.10), (3.4.12) and (3.4.13).

Therefore, it is reasonable to decompose the overall system model in such a way that subsystem 1 (eqn (3.4.10)) on the right-hand side of Figure 3.6 is included simultaneously in all expanded subsystems. This corresponds to using the expanded subsystem states $\bar{\mathbf{x}}_i = (f\ \mathbf{x}_i')'$ $(i = 2, \ldots, N)$. Then the behaviour of the new subsystems does represent an approximation of the performance of the areas under the influence of the whole system. Note that the expanded state vector

$$\bar{\mathbf{x}} = (f\ \mathbf{x}_2'\ f\ \mathbf{x}_3'\ \ldots\ f\ \mathbf{x}_N')'$$

and the state vector

$$\mathbf{x} = (f\ \mathbf{x}_2'\ \mathbf{x}_3'\ \ldots\ \mathbf{x}_N')'$$

of the original system (3.4.10), (3.4.12) and (3.4.13) are related by eqn (3.4.3) to

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \quad \mathbf{T}^+ = \begin{pmatrix} N^{-1} & 0 & N^{-1} & 0 & \ldots & N^{-1} & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \end{pmatrix}.$$

Then $N-1$ overlapping subsystems

$$\dot{\bar{\mathbf{x}}}_i = \bar{\mathbf{A}}_{ii}\bar{\mathbf{x}}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \bar{\mathbf{A}}_{ij}\bar{\mathbf{x}}_j + \bar{\mathbf{b}}_i u_i$$

$$\mathbf{y}_i = \bar{\mathbf{C}}_i\bar{\mathbf{x}}_i$$

are built with

$$\bar{\mathbf{A}}_{ii} = \begin{pmatrix} \mathbf{0} & \bar{\mathbf{c}}_{zi}' \\ \mathbf{e}_i & \mathbf{A}_i \end{pmatrix} \qquad \bar{\mathbf{b}}_i = \begin{pmatrix} \mathbf{0} \\ \mathbf{b}_i \end{pmatrix} \qquad \bar{\mathbf{C}}_i = (\mathbf{0}\ \ \mathbf{C}_i)$$

and some matrix $\bar{\mathbf{A}}_{ij}$ (cf. Example 3.4). Readers should find $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ from eqn (3.4.5) with appropriate matrices **M**, **N** and **L** and decompose the expansion into (3.4.15) as an exercise. Each expanded subsystem (3.4.15) represents a subsystem (3.4.12) that is directly coupled with subsystem (3.4.1) via $s_1 = z_i$ and $s_i = z_1$. Hence, the isolated systems

(3.4.15)

$$\dot{\bar{\mathbf{x}}}_i = \bar{\mathbf{A}}_{ii}\bar{\mathbf{x}}_i + \bar{\mathbf{b}}_i u_i$$
$$\mathbf{y}_i = \bar{\mathbf{C}}_i\bar{\mathbf{x}}_i$$

include the loop from s_i via subsystem 1 towards z_i (cf. Figure 3.6) and, thus, really represent approximate descriptions of the subsystem i under the influence of the subsystem surroundings. □

3.5 MULTI-TIMESCALE SYSTEMS

The decomposition methods described so far aim to divide the system into parts whose interactions have a small magnitude. These methods are sometimes called 'spatial' decomposition methods. In contrast, the following method is motivated by temporal considerations. If the overall system consists of subsystems whose main time constants are far from each other, then the fast subsystem will arrive at its final state before the slow subsystem has begun the main part of its motion. From the point of view of the slow subsystem, the fast subsystems reach their new state very quickly and behave like static systems, whereas within the time horizon of the fast part the slow subsystems seem to be quiescent.

Many practical systems have parts with quite different modes. For example, actuators or measurement devices are usually constructed so as to be much quicker than the main part of the process to be controlled. 'Parasitic elements' such as quick transmitters or short time lags appear in nearly every complex system.

Two-timescale Systems

A useful formalization of such processes is given by singularly perturbed systems. Such systems consist of two different parts

$$\begin{aligned}\dot{\mathbf{x}}_0 &= \mathbf{A}_{00}\mathbf{x}_0 + \mathbf{A}_{01}\mathbf{x}_1 + \mathbf{B}_0\mathbf{u}\\ \varepsilon\dot{\mathbf{x}}_1 &= \mathbf{A}_{10}\mathbf{x}_0 + \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{B}_1\mathbf{u}\\ \mathbf{y} &= \mathbf{C}_0\mathbf{x}_0 + \mathbf{C}_1\mathbf{x}_1\end{aligned} \tag{3.5.1}$$

where the first line represents the slow subsystem with subsystem state $\mathbf{x}_0$ and the second line describes the fast subsystem with subsystem state $\mathbf{x}_1$. Both state vectors and the input can be decomposed into a fast and a slow part

$$\mathbf{x}_0 = \mathbf{x}_{s0} + \mathbf{x}_{f0} \qquad \mathbf{x}_1 = \mathbf{x}_{s1} + \mathbf{x}_{f1} \qquad \mathbf{u} = \mathbf{u}_f + \mathbf{u}_s.$$

If the long-time system behaviour is to be investigated, the quick transition as described by $\mathbf{x}_{f0}$, $\mathbf{x}_{f1}$ and $\mathbf{u}_f$ can be neglected. Hence, eqn (3.5.1) is used with $\mathbf{u}_f = \mathbf{0}$, $\mathbf{x}_{f0} = \mathbf{0}$ and $\mathbf{x}_{f1} = \mathbf{0}$. Furthermore, since the fast subsystem is considered as static, $\varepsilon = 0$ holds. If the matrix $\mathbf{A}_{11}$ is non-singular, the second line of eqn (3.5.1) leads to the static model of the fast part:

$$\mathbf{x}_{s1} = -\mathbf{A}_{11}^{-1}\mathbf{A}_{10}\mathbf{x}_{s0} - \mathbf{A}_{11}^{-1}\mathbf{B}_1\mathbf{u}_s. \tag{3.5.2}$$

With this static description of the fast subsystem, the first and third lines of eqn (3.5.1) yield a reduced-order model of the slow system performance

$$\begin{aligned}\dot{\mathbf{x}}_{s0} &= \mathbf{A}_s\mathbf{x}_{s0} + \mathbf{B}_s\mathbf{u}_s \\ \mathbf{y}_s &= \mathbf{C}_s\mathbf{x}_{s0} + \mathbf{D}_s\mathbf{u}_s\end{aligned} \tag{3.5.3}$$

with

$$\mathbf{A}_s = \mathbf{A}_{00} - \mathbf{A}_{01}\mathbf{A}_{11}^{-1}\mathbf{A}_{10} \tag{3.5.4}$$

and

$$\begin{aligned}&\mathbf{B}_s = \mathbf{B}_0 - \mathbf{A}_{10}\mathbf{A}_{11}^{-1}\mathbf{B}_1 \qquad \mathbf{C}_s = \mathbf{C}_0 - \mathbf{C}_1\mathbf{A}_{11}^{-1}\mathbf{A}_{10} \\ &\mathbf{D}_s = -\mathbf{C}_1\mathbf{A}_{11}^{-1}\mathbf{B}_1.\end{aligned} \tag{3.5.5}$$

This model can be used to design a controller

$$\mathbf{u}_s = -\mathbf{K}_s\mathbf{x}_{s0} \tag{3.5.6}$$

for the long-term system behaviour.

On the other hand, the fast transition is described by eqn (3.5.1) for $\dot{\mathbf{x}}_{s1}(t) = \mathbf{0}$ and $\mathbf{u}_s = \mathbf{0}$. Since $\mathbf{x}_0$ is the state of the slow subsystem, $\mathbf{x}_{f0} = \mathbf{0}$ is assumed further. For $\varepsilon = 1$, eqns (3.5.1) and (3.5.2) yield the approximate model

$$\begin{aligned}\dot{\mathbf{x}}_{f1} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}_1 - \mathbf{x}_{s1}) = \dot{\mathbf{x}}_1 \\ &= \mathbf{A}_{10}\mathbf{x}_{s0} + \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{B}_1(\mathbf{u}_s + \mathbf{u}_f) \\ &= \mathbf{A}_{11}(\mathbf{x}_1 - \mathbf{x}_{s1}) + \mathbf{B}_1\mathbf{u}_f\end{aligned}$$

or

$$\begin{aligned}\dot{\mathbf{x}}_{f1} &= \mathbf{A}_{11}\mathbf{x}_{f1} + \mathbf{B}_1\mathbf{u}_f \\ \mathbf{y}_f &= \mathbf{C}_1\mathbf{x}_{f1}.\end{aligned} \tag{3.5.7}$$

This model can be used to control the fast system response by

$$\mathbf{u}_f = -\mathbf{K}_f\mathbf{x}_{f1}. \tag{3.5.8}$$

The overall feedback is received from eqns (3.5.2), (3.5.6) and (3.5.8) for $\mathbf{x}_0 = \mathbf{x}_{s0}$:

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_f + \mathbf{u}_s \\ &= -\mathbf{K}_f(\mathbf{x}_{s1} + \mathbf{x}_{f1}) - \mathbf{K}_s\mathbf{x}_{s0} \\ &= -\mathbf{K}_f\mathbf{x}_1 - [(\mathbf{I} + \mathbf{K}_f\mathbf{A}_{11}^{-1}\mathbf{B}_1)\mathbf{K}_s + \mathbf{K}_f\mathbf{A}_{11}^{-1}\mathbf{A}_{10})\mathbf{x}_0.\end{aligned} \tag{3.5.9}$$

The question of how the original system behaviour $\mathbf{x}_0(t)$, $\mathbf{x}(t)$, $\mathbf{y}(t)$ can be approximated by the slow and fast parts is answered in the following theorem.

Theorem 3.5

If the matrix $\mathbf{A}_s - \mathbf{B}_s\mathbf{K}_s$ is stable, then the following relations hold:

$$\begin{aligned}\mathbf{x}_0(t) &= \mathbf{x}_{s0}(t) + O(\varepsilon) \\ \mathbf{x}_1(t) &= -\mathbf{A}_{11}^{-1}(\mathbf{A}_{10} - \mathbf{B}_1\mathbf{K}_s)\mathbf{x}_{s0}(t) + \mathbf{x}_{f1} + O(\varepsilon) \\ \mathbf{u}(t) &= \mathbf{u}_s(t) + \mathbf{u}_f(t) + O(\varepsilon) \\ \mathbf{y}(t) &= \mathbf{y}_s(t) + \mathbf{y}_f(t) + O(\varepsilon).\end{aligned} \tag{3.5.10}$$

$O(\varepsilon)$ signifies the existence of a function $\mathbf{f}(\varepsilon)$ where for each element $f_i(\varepsilon)$ there is a constant k_i such that $f_i(\varepsilon)/\varepsilon \leqslant k_i$ holds. That is, eqn (3.5.10) says that the deviations of $\mathbf{x}_0$ from $\mathbf{x}_{s0}$ or $\mathbf{x}_1$, $\mathbf{u}$ and $\mathbf{y}$ from the given right-hand sides vanish for $\varepsilon \to 0$ and that this convergence is at least linear.

Theorem 3.5 is sometimes called a separation theorem because it states that the system performance can be decomposed into separate slow and fast parts, which are described by different models. The separation is complete for $\varepsilon = 0$ since then $\mathbf{x}_0(t) = \mathbf{x}_{s0}(t)$ holds. But $\mathbf{x}_{s0}$, $\mathbf{x}_{s1}$ and $\mathbf{x}_{f1}$ can be determined independently of one another by eqns (3.5.2), (3.5.3) and (3.5.7) to a reasonable approximation if ε is small. This implies the following assertion.

Corollary 3.2

If both the models (3.5.3) and (3.5.7) are stable then there exists some $\bar{\varepsilon} > 0$ such that the system (3.5.1) is asymptotically stable for all $\varepsilon \in (0, \bar{\varepsilon})$.

Multi-timescale Systems

The method of decomposition described so far is known from multivariable control theory. It can be directly extended for large-scale systems in which a single slow subsystem is coupled with several fast subsystems. Formally, this extension leads from eqn (3.5.1) to

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \mathbf{A}_{00}\mathbf{x}_0 + \sum_{j=1}^{N} \mathbf{A}_{0j}\mathbf{x}_j + \sum_{j=1}^{N} \mathbf{B}_{0j}\mathbf{u}_j \\ \varepsilon_i\dot{\mathbf{x}}_i &= \mathbf{A}_{i0}\mathbf{x}_0 + \mathbf{A}_{ii}\mathbf{x}_i + \mathbf{B}_{ii}\mathbf{u}_i \qquad (i = 1, \ldots, N) \\ \mathbf{y}_i &= \mathbf{C}_{i0}\mathbf{x}_0 + \mathbf{C}_{ii}\mathbf{x}_i \end{aligned} \tag{3.5.11}$$

where the first line describes the slow subsystem and the second line the ith fast subsystem. The fast subsystems are assumed to have their own subsystem inputs $\mathbf{u}_i$, which all together act on the slow subsystem too. The parameters ε_i are independent of each other. No assumptions are made concerning the relation between the speeds of the fast subsystems. The state, output and control input are decomposed as above:

$$\begin{aligned} \mathbf{x}_i &= \mathbf{x}_{si} + \mathbf{x}_{fi} \qquad \mathbf{y}_i = \mathbf{y}_{si} + \mathbf{y}_{fi} \\ \mathbf{u}_i &= \mathbf{u}_{si} + \mathbf{u}_{fi} \qquad (i = 0, 1, \ldots, N). \end{aligned}$$

The slow part of the overall system can be approximately described by an analogy of eqn (3.5.3) as follows. For $\varepsilon_i = 0$

$$\begin{aligned} \mathbf{x}_{si} &= -\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0}\mathbf{x}_{s0} - \mathbf{A}_{ii}^{-1}\mathbf{B}_{ii}\mathbf{u}_{si} \quad (i = 1, \ldots, N) \\ \mathbf{y}_{si} &= (\mathbf{C}_{i0} - \mathbf{C}_{ii}\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0})\mathbf{x}_{s0} - \mathbf{C}_{ii}\mathbf{A}_{ii}^{-1}\mathbf{B}_{ii}\mathbf{u}_{si} \end{aligned} \tag{3.5.12}$$

is obtained. With this static approximation, eqn (3.5.11) yields

$$\dot{\mathbf{x}}_{s0} = \mathbf{A}_s\mathbf{x}_{s0} + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_{si} \tag{3.5.13}$$

with

$$\begin{aligned} \mathbf{A}_s &= \mathbf{A}_{00} - \sum_{i=1}^{N} \mathbf{A}_i\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0} \\ \mathbf{B}_{si} &= \mathbf{B}_{0i} - \mathbf{A}_{0i}\mathbf{A}_{11}^{-1}\mathbf{B}_{ii} \qquad (i = 1, \ldots, N). \end{aligned} \tag{3.5.14}$$

The slow part of the controller

$$\mathbf{u}_{si} = -\mathbf{K}_{si}\mathbf{x}_{s0} \qquad (i = 1, \ldots, N) \tag{3.5.15}$$

can be chosen for this model.

The fast part of the overall system performance is different if the system is investigated from the point of view of different subsystems. From the point of view of the ith subsystem it is reasonable to ignore all the other fast subsystems and to assume the slow subsystem to be static:

$$\dot{\mathbf{x}}_0 = \mathbf{0} \qquad \mathbf{x}_0 = \mathbf{x}_{s0} \qquad \mathbf{u}_s = \mathbf{0}.$$

Then, similarly to eqn (3.5.7), the model

$$\begin{aligned} \dot{\mathbf{x}}_{fi} &= \mathbf{A}_{ii}\mathbf{x}_{fi} + \mathbf{B}_{ii}\mathbf{u}_{fi} \\ \mathbf{y}_{fi} &= \mathbf{C}_{ii}\mathbf{x}_{fi} \end{aligned} \tag{3.5.16}$$

results. A controller

$$\mathbf{u}_{fi} = -\mathbf{K}_{fi}\mathbf{x}_{fi} \tag{3.5.17}$$

can be designed using this model. Note that for the fast response, N different approximate models (3.5.16) of the same overall system (3.5.11) are used. This is referred to as *multimodelling*.

The overall system performance is approximately described by the separate models (3.5.13) and (3.5.16), the solution of which compose the system outputs as

$$\begin{aligned} \mathbf{y}_i &= \mathbf{y}_{si} + \mathbf{y}_{fi} \\ &= (\mathbf{C}_{i0} - \mathbf{C}_{ii}\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0})\mathbf{x}_{s0} + \mathbf{C}_{ii}\mathbf{x}_{fi} - \mathbf{C}_{ii}\mathbf{A}_{ii}^{-1}\mathbf{B}_{ii}\mathbf{u}_{si}. \end{aligned}$$

The overall controller consists of the two parts described by eqns (3.5.15) and (3.5.17), which amount to

$$\begin{aligned} \mathbf{u}_i &= \mathbf{u}_{si} + \mathbf{u}_{fi} \\ &= -\mathbf{K}_{fi}\mathbf{x}_i - [(\mathbf{I} + \mathbf{K}_{fi}\mathbf{A}_{ii}^{-1}\mathbf{B}_{ii})\mathbf{K}_{si} + \mathbf{K}_{fi}\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0})\mathbf{x}_0. \end{aligned} \tag{3.5.18}$$

This controller feeds back the overall system state

$$\mathbf{x} = (\mathbf{x}_0' \;\; \mathbf{x}_1' \;\; \dots \;\; \mathbf{x}_N')'.$$

BIBLIOGRAPHICAL NOTES

The relations between the unstructured overall model and the interaction-oriented model have been investigated in detail by Ikeda and Kodama (1973). They gave the original proof of Theorem 3.1. Singh and Liu (1973) answered the question of under what conditions the state vector of the overall system must contain all subsystem states. The German names for the different models summarized in Section 3.1 have been introduced by Litz (1983). Although they are not yet commonly used in the literature, they are mentioned here to shorten references to specific forms of the model.

Hierarchical decomposition has been introduced by Özgüner and Perkins (1975) in connection with the stability analysis of large-scale systems. Since the strongly connected components may be unrelated to the control inputs and outputs, Sezer and Šiljak (1981c) and Pichai *et al.* (1983) elaborated graph-theoretic algorithms for hierarchical decomposition into I/O-reachable subsystems. These methods are powerful since they give the possibility of utilizing well-established graph-theoretic means for investigating large-scale systems. On the other hand, graph representations are useful means of representing complex systems *before* setting up their differential equations. This has been discussed by, for example, Waller (1979).

The disjoint decomposition into input-decentralized or output-decentralized subsystems has been proposed by Šiljak and Vukčević (1976). Sezer and Šiljak (1984) gave a decomposition method where the interactions of the resulting subsystems are lower than a prescribed threshold.

Overlapping decomposition has been described for a traffic control example by Isaksen and Payne (1973) and generalized to the relation of expansion and contraction by Ikeda and Šiljak (1980). An extension of the inclusion principle to non-linear systems was given by Ohta and Šiljak (1984). Iftar (1990) generalized this approach for overlapping inputs and outputs. The model structure used in Example 3.5 for the power–frequency behaviour of multiarea power systems was proposed by Küßner and Uhlig (1984).

The timescale separation was investigated extensively in the 1970s. A survey has been given by Kokotović *et al.* (1976) and Saksena *et al.* (1984). Two-timescale systems have two different clusters of eigenvalues. This has already been investigated for linear dynamical systems by Milne (1965) who gave quantitative bounds on the separation of the eigenvalue clusters so that the spectrum can be approximated with reasonable accuracy by the union of the clusters of $\mathbf{A}_{11}$ and $\mathbf{A}_s$ (cf. eqns (3.5.1) and (3.5.4)). Özgüner (1975b) used the Gershgorin theorem to establish similar bounds on the interconnection matrices $\mathbf{A}_{12}$ and $\mathbf{A}_{21}$ so that the couplings between two subsystems in eqn (3.5.1) for $\varepsilon = 1$ are weak enough to be ignored during the analysis of the overall system. All these conditions are satisfied by singularly perturbed systems if ε is small enough. Theorem 3.5 is due to Chow and Kokotović (1976), whereas Corollary 3.2 was proved earlier by Klimushev and Krasovskii (1961).

Multimodelling as a means of describing large-scale systems with slow and fast modes from the point of view of the different decision makers was introduced by Khalil and Kokotović (1978, 1979).

The decomposition of the overall system into composite systems with a specific interconnection structure is an important means of deriving special analytical and design methods, which are tailored to particular classes of systems and are more efficient than general-purpose methods. Apart from systems with hierarchically structured, weakly coupled or temporarily separated subsystems, systems with symmetric interactions are of particular importance. Voicu (1980) investigated systems which have the so-called Kirchhoff interconnections. Another generalization will be investigated in detail in Chapter 12.

4

Decentralized Stabilizability

4.1 DECENTRALIZED FIXED MODES

The results on stabilizability and pole assignability by means of a linear feedback have been summarized in Section 2.2 for centralized control. Both problems will now be tackled under the structural constraints on the control law, which are imposed in decentralized control. The definition of fixed modes under decentralized control and the algebraic characterization of such modes will be formulated in the best possible analogy of Definition 2.3 and Theorem 2.3.

Consider the overall system in I/O-oriented description (3.1.2)

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}_i(t) &= \mathbf{C}_{si}\mathbf{x}(t) \qquad (i = 1, 2, \ldots, N) \end{aligned} \tag{4.1.1}$$

under static decentralized feedback

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \end{pmatrix} = \begin{pmatrix} -\mathbf{K}_{y1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_{y2} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{K}_{yN} \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} \tag{4.1.2}$$

which will be considered as output feedback

$$\mathbf{u} = -\mathbf{K}_y\mathbf{y} \quad \text{with} \quad \mathbf{K}_y \in \mathcal{K} = \{\mathbf{K}_y\colon\ [\mathbf{K}_y] = \mathbf{K}\} \tag{4.1.3}$$

where the structural constraints are prescribed by the structure matrix $\mathbf{K}$ (cf. Section 2.5). The matrices $\mathbf{B}$ and $\mathbf{C}$ are composed of $\mathbf{B}_{si}$ and $\mathbf{C}_{si}$ as in eqn (3.1.3).

The fixed modes under decentralized control can be defined analogously to their centralized counterparts as those eigenvalues of the matrix $\mathbf{A}$ which appear unchanged in every closed-loop system (4.1.1) and (4.1.2).

Definition 4.1

The elements of the set

$$\Lambda_{df} = \bigcap_{\mathbf{K}_y \in \mathcal{K}} \sigma[\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C}] \tag{4.1.4}$$

are called *decentralized fixed modes* (or *decentralized fixed eigenvalues*) under the structural constraint $\mathcal{K}$.

Note that the definition is related to the structural constraints given by $\mathcal{K}$ although this relation will not always be explicitly stated. Because of $\mathcal{K} \subseteq \mathcal{R}^{m \times r}$ all centralized fixed modes are decentralized fixed modes of the plant (4.1.1)

$$\Lambda_f \subseteq \Lambda_{df}. \tag{4.1.5}$$

Decentralized fixed modes may occur even if the overall system (4.1.1) is completely controllable through the whole input vector $\mathbf{u}$ and completely observable via the output $\mathbf{y}$.

Existence of Decentralized Fixed Modes

In the following a necessary and sufficient condition for the existence of decentralized fixed modes will be derived step by step while simultaneously revealing the reasons for the existence of such modes. First, the system (4.1.1) has obviously no decentralized fixed modes if it is controllable and observable by one channel $(\mathbf{u}_i, \mathbf{y}_i)$ only. Then all eigenvalues of $\mathbf{A}$ can be changed by the single control station

$$\mathbf{u}_i = -\mathbf{K}_{yi}\mathbf{y}_i. \tag{4.1.6}$$

The number i of the control station (4.1.6) belongs to the index set

$$\mathcal{I} = \{1, 2, \ldots, N\}. \tag{4.1.7}$$

Lemma 4.1

The system (4.1.1) has no decentralized fixed modes if it is completely controllable and completely observable by a single channel $(\mathbf{u}_i, \mathbf{y}_i)$, that is if there exists some index $i \in \mathcal{I}$ such that the pairs $(\mathbf{A}, \mathbf{B}_{si})$ and $(\mathbf{A}, \mathbf{C}_{si})$ are controllable or observable, respectively.

If this fact is transmitted to a single system mode or eigenvalue, it becomes evident that only that eigenvalue λ may be decentralized fixed which is not both controllable and observable from the same channel or, conversely, which is either uncontrollable by $\mathbf{u}_i$

$$\operatorname{rank}(\mathbf{A} - \lambda\mathbf{I} \;\; \mathbf{B}_{si}) < n \tag{4.1.8}$$

or unobservable from $\mathbf{y}_i$

$$\operatorname{rank}\begin{pmatrix} \mathbf{A} - \lambda\mathbf{I} \\ \mathbf{C}_{si} \end{pmatrix} < n \tag{4.1.9}$$

(cf. Theorem 2.3). Inequality (4.1.8) or (4.1.9) has to hold for all $i \in \mathscr{I}$.

Lemma 4.2

A necessary condition for the eigenvalue $\lambda[\mathbf{A}]$ to be decentralized fixed is that there exists a disjoint partition of the index set $\mathscr{I}$

$$\mathscr{D} = \{i_1, i_2, \ldots, i_k\} \qquad \mathscr{H} = \{i_{k+1}, i_{k+2}, \ldots, i_N\} \tag{4.1.10}$$

with

$$\mathscr{D} \cup \mathscr{H} = \mathscr{I} \qquad \mathscr{D} \cap \mathscr{H} = \emptyset \tag{4.1.11}$$

such that condition (4.1.8) holds for all $i \in \mathscr{D}$ and (4.1.9) for all $i \in \mathscr{H}$.

The motivation for the introduction of the disjoint partition of $\mathscr{I}$ is provided by the fact that it does not matter whether the eigenvalue λ is not controllable nor observable from the ith channel, although at least one of these properties has to occur. Consequently, the channels can be grouped so that λ is not controllable from $\mathbf{u}_i$ with $i \in \mathscr{D}$ and not observable from $\mathbf{y}_i$ with $i \in \mathscr{H}$. The corresponding matrices $\mathbf{B}_{si}$ and $\mathbf{C}_{si}$ enter into the conditions (4.1.8) or (4.1.9), respectively.

This assertion can be reformulated if the matrices

$$\mathbf{B}_\mathrm{D} = (\mathbf{B}_{si_1} \;\; \mathbf{B}_{si_2} \;\; \ldots \;\; \mathbf{B}_{si_k}) \qquad \mathbf{B}_\mathrm{H} = (\mathbf{B}_{si_{k+1}} \;\; \mathbf{B}_{si_{k+2}} \;\; \ldots \;\; \mathbf{B}_{si_N})$$
$$\mathbf{C}_\mathrm{D} = \begin{pmatrix} \mathbf{C}_{si_1} \\ \vdots \\ \mathbf{C}_{si_k} \end{pmatrix} \qquad \mathbf{C}_\mathrm{H} = \begin{pmatrix} \mathbf{C}_{si_{k+1}} \\ \vdots \\ \mathbf{C}_{si_N} \end{pmatrix} \tag{4.1.12}$$

are formed. The conditions (4.1.8) for all $i \in \mathscr{D}$ together are equivalent to the single condition

$$\operatorname{rank}(\mathbf{A} - \lambda\mathbf{I} \;\; \mathbf{B}_\mathrm{D}) < n. \tag{4.1.13}$$

Analogously, the conditions (4.1.9) for all indices $i \in \mathscr{H}$ can be lumped together to form the equivalent condition

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}-\lambda\mathbf{I}\\ \mathbf{C}_{\mathrm{H}}\end{pmatrix} < n. \tag{4.1.14}$$

That this is really true should be explained for the case when the scalar λ is a unique eigenvalue of the matrix $\mathbf{A}$. Then (4.1.8) reads as

$$\operatorname{rank}(\mathbf{A}-\lambda\mathbf{I}\;\; \mathbf{B}_{si}) = n-1$$

and says that the columns of the matrix $\mathbf{B}_{si}$ are linearly dependent upon the columns of $\mathbf{A}-\lambda\mathbf{I}$. According to Lemma 4.2 this is true for all matrices $\mathbf{B}_{si}$ with $i \in \mathscr{D}$. Therefore, all columns of $\mathbf{B}_{\mathrm{D}}$ are linearly dependent on the columns of $\mathbf{A}-\lambda\mathbf{I}$, and conditions (4.1.8) for $i \in \mathscr{D}$ can really be written as

$$\operatorname{rank}(\mathbf{A}-\lambda\mathbf{I}\;\; \mathbf{B}_{\mathrm{D}}) = n-1.$$

With the same argument, the equivalence of conditions (4.1.8) for $i \in \mathscr{D}$ or (4.1.9) for $i \in \mathscr{H}$ and (4.1.13) or (4.1.14), respectively, can be shown for multiple eigenvalues.

The conditions of Lemma 4.2 have to be made more restrictive to ensure that λ may not be made controllable through $\mathbf{u}_i$ ($i \in \mathscr{D}$) by choosing appropriate decentralized control stations (4.1.6) for $i \in \mathscr{H}$. These control stations are described by

$$\begin{pmatrix}\mathbf{u}_{i_{k+1}}\\ \vdots\\ \mathbf{u}_{i_N}\end{pmatrix} = \begin{pmatrix}-\mathbf{K}_{yi_{k+1}} & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{K}_{yi_N}\end{pmatrix}\begin{pmatrix}\mathbf{y}_{i_{k+1}}\\ \vdots\\ \mathbf{y}_{i_N}\end{pmatrix}. \tag{4.1.15}$$

The controller matrix in eqn (4.1.15) is denoted by $\mathbf{K}_{\mathrm{H}}$. The eigenvalue λ cannot be made controllable through some input $\mathbf{u}_i$ ($i \in \mathscr{D}$) by using an appropriate feedback (4.1.15) if and only if

$$\operatorname{rank}(\mathbf{A}-\mathbf{B}_{\mathrm{H}}\mathbf{K}_{\mathrm{H}}\mathbf{C}_{\mathrm{H}}-\lambda\mathbf{I}\;\; \mathbf{B}_{\mathrm{D}}) < n \tag{4.1.16}$$

is valid for an arbitrary matrix $\mathbf{K}_{\mathrm{H}}$ with the structure given in eqn (4.1.15). The matrix occurring in (4.1.16) can be represented as the product

$$(\mathbf{A}-\mathbf{B}_{\mathrm{H}}\mathbf{K}_{\mathrm{H}}\mathbf{C}_{\mathrm{H}}-\lambda\mathbf{I}\;\; \mathbf{B}_{\mathrm{D}}) = (\mathbf{I}\;\; \mathbf{0})\begin{pmatrix}\mathbf{I} & -\mathbf{B}_{\mathrm{H}}\mathbf{K}_{\mathrm{H}}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}_{\mathrm{D}}\\ \mathbf{C}_{\mathrm{H}} & \mathbf{0}\end{pmatrix}.$$

The matrix $(\mathbf{I}\;\; \mathbf{0})$ includes an (n, n) identity matrix and, thus, has rank n. The rank of the matrix in the middle of the product is greater than or equal to n. Therefore, the condition (4.1.16) is satisfied if

$$\operatorname{rank}\begin{pmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}_{\mathrm{D}}\\ \mathbf{C}_{\mathrm{H}} & \mathbf{0}\end{pmatrix} < n \tag{4.1.17}$$

is valid. This inequality is independent of $\mathbf{K}_H$ and represents a property that the system (4.1.1) should possess.

In the same way it can be shown that for systems satisfying (4.1.17) it is impossible to make the eigenvalue λ observable from $\mathbf{y}_i$ ($i \in \mathcal{H}$) by means of decentralized control stations (4.1.6) with $i \in \mathcal{D}$. Consequently, (4.1.17) is a sufficient condition for the existence of the decentralized fixed eigenvalue λ. It can be proved that this condition is necessary as well.

Theorem 4.1

A necessary and sufficient condition for a complex number λ to be a decentralized fixed mode of the system (4.1.1) is that there exists a scalar k and a disjoint partition $\mathcal{D}$, $\mathcal{H}$ of the index set $\mathcal{I}$ according to eqns (4.1.10) and (4.1.11) such that the inequality (4.1.17) holds, where $\mathbf{B}_D$ and $\mathbf{C}_H$ are defined in eqns (4.1.12).

Consequently, the set of decentralized fixed modes is given by

$$\Lambda_{df} = \left\{ \lambda_i[\mathbf{A}] : \operatorname{rank}\begin{pmatrix} \mathbf{A} - \lambda\mathbf{I} & \mathbf{B}_D \\ \mathbf{C}_H & \mathbf{0} \end{pmatrix} < n \text{ for some partition } \mathcal{D}, \mathcal{H} \right\}. \tag{4.1.18}$$

Theorem 4.1 and eqn (4.1.18) are very similar to Theorem 2.3 and eqn (2.2.14). In the course of deriving Theorem 4.1 it becomes clear in which way the structural constraints inherent in decentralized control make the test more complicated than that of Theorem 2.3.

Discussion

Condition (4.1.17) refers to the auxiliary plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_D\mathbf{u}_D \\ \mathbf{y}_H &= \mathbf{C}_H\mathbf{x}. \end{aligned} \tag{4.1.19}$$

Note that this plant has inputs and outputs from different subsystems (cf. sets $\mathcal{D}, \mathcal{H}$). It is, therefore, called a *complementary system*. Eqn (4.1.18) states that a decentralized fixed mode is equivalent to a mode of the complementary system (4.1.19) which is neither controllable nor observable (in the centralized sense) or, equivalently, represents an invariant zero of (4.1.19) (cf. eqn (2.1.16)). The relation between this auxiliary plant and the original system (4.1.1) can be easily seen if the channels are renumbered so that those with the original indices $i \in \mathcal{D}$

get the new indices $1, \dots, k$. After the matrices $\mathbf{B}_{si}$ and $\mathbf{C}_{si}$ have been reordered accordingly, the model (4.1.1) has the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + (\mathbf{B}_\mathrm{D}\ \ \mathbf{B}_\mathrm{H})\begin{pmatrix}\mathbf{u}_\mathrm{D}\\ \mathbf{u}_\mathrm{H}\end{pmatrix}$$
$$\begin{pmatrix}\mathbf{y}_\mathrm{D}\\ \mathbf{y}_\mathrm{H}\end{pmatrix} = \begin{pmatrix}\mathbf{C}_\mathrm{D}\\ \mathbf{C}_\mathrm{H}\end{pmatrix}\mathbf{x} \tag{4.1.20}$$

where $\mathbf{u}_\mathrm{D}, \mathbf{u}_\mathrm{H}$ or $\mathbf{y}_\mathrm{D}, \mathbf{y}_\mathrm{H}$ result from the decomposition of the whole reordered input or output, respectively, into their first k and the remaining $N-k$ elements. The auxiliary plant (4.1.19) is obtained from (4.1.20) if $\mathbf{u}_\mathrm{H}$ and $\mathbf{y}_\mathrm{H}$ are removed and

$$\mathbf{u}_\mathrm{D} = \begin{pmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_k\end{pmatrix} \qquad \mathbf{y}_\mathrm{H} = \begin{pmatrix}\mathbf{y}_{k+1}\\ \vdots\\ \mathbf{y}_N\end{pmatrix}$$

remain. For further investigations the systems (4.1.19) and (4.1.20) are transformed according to (2.2.4) into

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}_\mathrm{D}\mathbf{u}_\mathrm{D} \qquad \mathbf{y}_\mathrm{H} = \tilde{\mathbf{C}}_\mathrm{H}\tilde{\mathbf{x}}$$

and

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + (\tilde{\mathbf{B}}_\mathrm{D}\ \tilde{\mathbf{B}}_\mathrm{H})\begin{pmatrix}\mathbf{u}_\mathrm{H}\\ \mathbf{u}_\mathrm{D}\end{pmatrix} \qquad \begin{pmatrix}\mathbf{y}_\mathrm{D}\\ \mathbf{y}_\mathrm{H}\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{C}}_\mathrm{D}\\ \tilde{\mathbf{C}}_\mathrm{H}\end{pmatrix}\tilde{\mathbf{x}}(t) \tag{4.1.21}$$

respectively, with

$$\tilde{\mathbf{A}} = \mathrm{diag}\,\lambda_i \qquad \tilde{\mathbf{B}}_{si} = \mathbf{V}^{-1}\mathbf{B}_{si} \qquad \tilde{\mathbf{C}}_{si} = \mathbf{C}_{si}\mathbf{V}$$
$$\tilde{\mathbf{B}}_\mathrm{D} = (\tilde{\mathbf{B}}_{s1}\ \dots\ \tilde{\mathbf{B}}_{sk}) \qquad \tilde{\mathbf{B}}_\mathrm{H} = (\tilde{\mathbf{B}}_{s,k+1}\ \dots\ \tilde{\mathbf{B}}_{sN})$$
$$\tilde{\mathbf{C}}_\mathrm{D} = \begin{pmatrix}\tilde{\mathbf{C}}_{s1}\\ \vdots\\ \tilde{\mathbf{C}}_{sk}\end{pmatrix} \qquad \tilde{\mathbf{C}}_\mathrm{H} = \begin{pmatrix}\tilde{\mathbf{C}}_{s,k+1}\\ \vdots\\ \tilde{\mathbf{C}}_{sN}\end{pmatrix}. \tag{4.1.22}$$

The application of Theorem 4.1 to the transformed systems reveals that the decentralized fixed modes are exactly those modes which are not controllable through $\tilde{\mathbf{B}}_\mathrm{D}$ nor observable through $\tilde{\mathbf{C}}_\mathrm{H}$. That is, the eigenvalue λ_i of $\mathbf{A}$ is decentralized fixed if and only if the ith row of $\tilde{\mathbf{B}}_\mathrm{D}$ and the ith column of $\tilde{\mathbf{C}}_\mathrm{H}$ have only zero elements.

If these zero rows or columns are brought to the bottom of $\tilde{\mathbf{B}}_\mathrm{D}$ or to the right of $\tilde{\mathbf{C}}_\mathrm{H}$ by reordering the state variables $\tilde{\mathbf{x}}_i$, the reordered state vector can be decomposed into the two parts $\tilde{\mathbf{x}}_\mathrm{c}$ and $\tilde{\mathbf{x}}_\mathrm{f}$ so that zero blocks appear in the matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ after corresponding decompositions. Hence, the model (4.1.21) has the form

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}\tilde{\mathbf{x}}_c\\ \tilde{\mathbf{x}}_f\end{pmatrix} &= \begin{pmatrix}\tilde{\mathbf{A}}_c & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{A}}_{df}\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{x}}_c\\ \tilde{\mathbf{x}}_f\end{pmatrix} + \begin{pmatrix}\tilde{\mathbf{B}}_{D1} & \tilde{\mathbf{B}}_{H1}\\ \mathbf{0} & \tilde{\mathbf{B}}_{H2}\end{pmatrix}\begin{pmatrix}\mathbf{u}_D\\ \mathbf{u}_H\end{pmatrix}\\ \begin{pmatrix}\mathbf{y}_D\\ \mathbf{y}_H\end{pmatrix} &= \begin{pmatrix}\tilde{\mathbf{C}}_{D1} & \tilde{\mathbf{C}}_{D2}\\ \tilde{\mathbf{C}}_{H1} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{x}}_c\\ \tilde{\mathbf{x}}_f\end{pmatrix}.\end{aligned} \tag{4.1.23}$$

The structure of this model is shown in Figure 4.1. Due to the transformation (2.2.4) every state variable $\tilde{\mathbf{x}}_i$ is associated with one system mode. The overall system is completely controllable and completely observable if the input matrix $\tilde{\mathbf{B}}$ and output matrix $\tilde{\mathbf{C}}$ as a whole have no zero rows or columns, respectively. Although the system (4.1.23) usually satisfies these conditions, fixed modes may occur under decentralized control because the second part of the state denoted by $\tilde{\mathbf{x}}_f$ cannot be controlled by $\mathbf{u}_D$ nor observed from $\mathbf{y}_H$ (cf. Theorem 4.1). In fact the fixed eigenvalues are exactly those that belong to the state vector $\tilde{\mathbf{x}}_f$. This fact reveals the effects of the structural constraints, which are imposed by the decentralization of the control law.

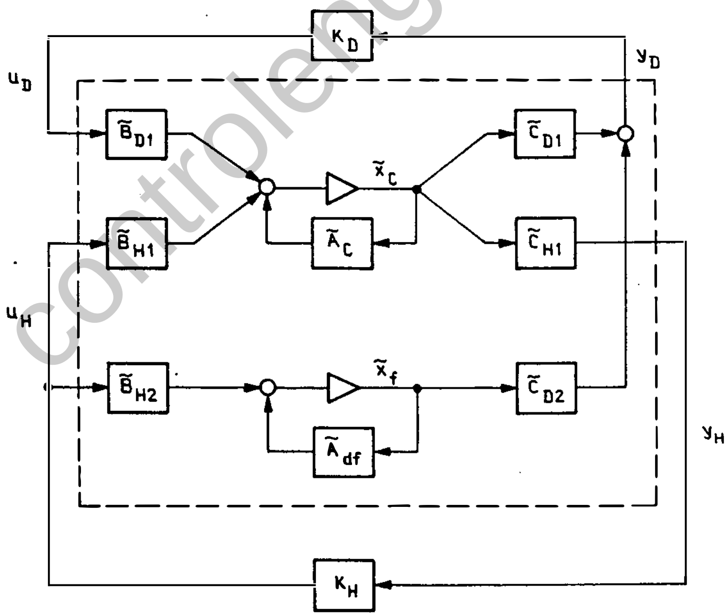


Figure 4.1 Structure of the transformed model (4.1.23) and the decomposed decentralized controller

Example 4.1

A simple example is given by

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 2 & 3 \\ a & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.1.24}$$

$$\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

The system is completely controllable and completely observable independently of the parameter a. However, for $a = 0$ the eigenvalue -2 is decentralized fixed, because with $\mathscr{D} = \{1\}$, $\mathscr{H} = \{2\}$ and $\lambda = -2$ the condition (4.1.17) is satisfied

$$\operatorname{rank} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array} \right) = 2 < 3.$$

For $a \neq 0$ condition (4.1.17) is violated for all partitions $\mathscr{D}, \mathscr{H}$ and the system has no decentralized fixed modes. □

Example 4.2

Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & a & 3 \\ 0 & b & 4 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

$$\mathbf{y}_1 = (1 \;\; 2 \;\; 0)\mathbf{x}$$

$$\mathbf{y}_2 = (0 \;\; 0 \;\; 1)\mathbf{x}.$$

It is completely controllable and completely observable for $a \neq -2b$. The condition (4.1.17)

$$\operatorname{rank} \left(\begin{array}{ccc|c} 0-\lambda & a & 3 & 1 \\ 0 & b-\lambda & 4 & 1 \\ 0 & 0 & 3-\lambda & 0 \\ \hline 0 & 0 & 1 & 0 \end{array} \right) = 2 < 3.$$

is satisfied for $\lambda = 0$ if $a = b$ holds. That is, there exist fixed eigenvalues $\lambda = 0$ although the system is controllable and observable. The existence of the decentralized fixed mode at $\lambda = 0$ is restricted to exceptional parameter values where the 'exceptions' are described by the equation $a = b$. For $a = -2b$, $\lambda = b$ is another fixed eigenvalue. It is also a centralized fixed mode because this eigenvalue is not observed through the overall system output $\mathbf{y} = (y_1\ \ y_2)'$. □

Example 4.3 (*River quality control*)

If the water quality in the control problem described in Example 3.2 is to be controlled along the whole river all modes must be open to influence by decentralized control. However, the system (3.2.16)

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{12} & \mathbf{A}_2 & \mathbf{0} \\ 0 & \mathbf{A}_{32} & \mathbf{A}_3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_3 \end{pmatrix}\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} \tag{4.1.25}$$

$$\mathbf{y}_1 = (\mathbf{C}_1\ \ \mathbf{0}\ \ \mathbf{0})\mathbf{x}$$

$$\mathbf{y}_2 = (\mathbf{0}\ \ \mathbf{0}\ \ \mathbf{C}_3)\mathbf{x}$$

has fixed modes. The spectrum of the closed-loop system (4.1.2) and (4.1.25)

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A}_1 - \mathbf{B}_1\mathbf{K}_{y1}\mathbf{C}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{12} & \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_3 - \mathbf{B}_3\mathbf{K}_{y2}\mathbf{C}_3 \end{pmatrix}\mathbf{x}$$

consists of the eigenvalues of the diagonal blocks. Since $\mathbf{A}_2$ does not depend on the controller parameters its eigenvalues are decentralized fixed. That is, all the modes of region 2 (Figure 4.2) cannot be changed by any decentralized feedback (4.1.2).

The determination of these decentralized fixed modes by means of Theorem 4.1 is straightforward if the structure of the matrix $\mathbf{A}$ is exploited. With $\mathcal{D} = \{2\}$ and $\mathcal{H} = \{1\}$ condition (4.1.17) is

$$\operatorname{rank}\left(\begin{array}{ccc|c} \mathbf{A}_1 - \lambda\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{12} & \mathbf{A}_2 - \lambda\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{32} & \mathbf{A}_3 - \lambda\mathbf{I} & \mathbf{B}_3 \\ \hline \mathbf{C}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right) < n$$

with $n = \dim \mathbf{A}_1 + \dim \mathbf{A}_2 + \dim \mathbf{A}_3$. If $(\mathbf{A}_1, \mathbf{C}_1)$ is completely observable and $(\mathbf{A}_3, \mathbf{B}_3)$ completely controllable then the rank of the test matrix equals

$$\dim \mathbf{A}_1 + \dim \mathbf{A}_3 + \operatorname{rank}(\mathbf{A}_2 - \lambda\mathbf{I})$$

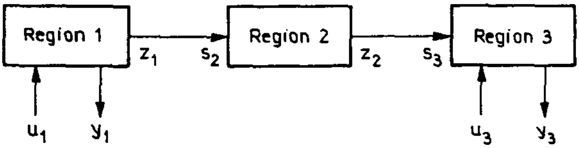


Figure 4.2 Decentralized river control

(cf. Theorem 2.2(i)) which is less than n for all λ that are eigenvalues of $\mathbf{A}_2$. Hence, all eigenvalues of $\mathbf{A}_2$ are decentralized fixed.

The modes of region 2 can only be influenced if a centralized controller is used. Then the controllability and observability of the plant ensure that it possesses no centralized fixed mode. The decentralized fixed modes appear as a result of structural constraints imposed on the control law in order to avoid long-distance data transmissions for the implementation of the feedback. □

Decentralized Fixed Modes in Interconnected Systems

The results presented thus far describe conditions under which decentralized fixed modes exist in direct analogy to well-known results from multivariable system theory. The exploitation of these conditions for the determination of fixed modes necessitates arithmetic operations with the whole model (4.1.1). The following investigations show the way in which the fixed modes depend on the properties of the subsystems

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i + \mathbf{E}_i\mathbf{s}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i &= \mathbf{C}_i\mathbf{x}_i \\ \mathbf{z}_i &= \mathbf{C}_{zi}\mathbf{x}_i \qquad (i = 1, \dots, N) \end{aligned} \tag{4.1.26}$$

and the interactions

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{4.1.27}$$

in which the overall system may be decomposed as explained in Chapter 3. They provide ways for testing the existence of fixed modes by considering the subsystems independently of each other.

If all the subsystems (4.1.26) are completely controllable and observable via $\mathbf{u}_i$ and $\mathbf{y}_i$ then all modes of the isolated subsystems (4.1.26) can be changed by means of a decentralized output feedback (4.1.6). Therefore, it is reasonable to conjecture that the overall system (4.1.26) and (4.1.27) has no decentralized fixed modes if all the subsystems (4.1.26) and the overall system are completely controllable and observable. However, the following counterexample shows that this conjecture is not true.

Example 4.4

Consider a system (4.1.26) and (4.1.27) with

$$N=2 \qquad \mathbf{A}_1=\begin{pmatrix} 2 & 1 \\ -30 & -9 \end{pmatrix} \qquad \mathbf{B}_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{E}_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{C}_1=\mathbf{I} \qquad \mathbf{C}_{z1}=(1 \quad 1) \qquad \mathbf{A}_2=\begin{pmatrix} -4 & 1 \\ -6 & 1 \end{pmatrix} \qquad \mathbf{B}_2=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{E}_2=\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{C}_2=(1 \quad 0) \qquad \mathbf{C}_{z2}=(0 \quad 1) \qquad \mathbf{L}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

All subsystems are completely controllable and observable from both $\mathbf{u}_i$ and $\mathbf{s}_i$ or $\mathbf{y}_i$ and $\mathbf{z}_i$, respectively. However, the overall system

$$\dot{\mathbf{x}}=\begin{pmatrix} 2 & 1 & 0 & 1 \\ -30 & -9 & 0 & 0 \\ 1 & 1 & -4 & 1 \\ 0 & 0 & -6 & 1 \end{pmatrix}\mathbf{x}+\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}u_1+\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}u_2$$

$$\mathbf{y}_1=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\mathbf{x}$$

$$\mathbf{y}_2=(0 \quad 0 \quad 1 \quad 0)\,\mathbf{x}$$

has a decentralized fixed eigenvalue at $+1$ because the condition (4.1.17) is satisfied for $\mathscr{D}=\{1\}$, $\mathscr{H}=\{2\}$

$$\operatorname{rank}\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ -30 & -10 & 0 & 0 & 1 \\ 1 & 1 & -5 & 1 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \end{array}\right)=3<4.$$

Obviously, the first and second columns of the test matrix are both linearly dependent upon the last two columns. Incidentally, the couplings between the subsystems are so 'strong' that the overall system is unstable with eigenvalues at $+1$, $-2.37 \pm 3.27i$ and -6.25, none of which coincides with some subsystem eigenvalue. □

None the less, some criteria for the existence of fixed modes in the overall system can be formulated in terms of the subsystems and their interactions. The first is based on the observation that a subsystem eigenvalue $\lambda[\mathbf{A}_i]$ cannot be influenced by the attached controller as well as by all the other subsystems, including their control stations, if $\lambda[\mathbf{A}_i]$ is not controllable or is not observable through both the inputs $\mathbf{u}_i$ and $\mathbf{s}_i$ or both the outputs $\mathbf{y}_i$ and $\mathbf{z}_i$, respectively.

Lemma 4.3

A subsystem eigenvalue $\lambda[\mathbf{A}_i]$ is a decentralized fixed eigenvalue of the overall system (4.1.26) and (4.1.27) if either

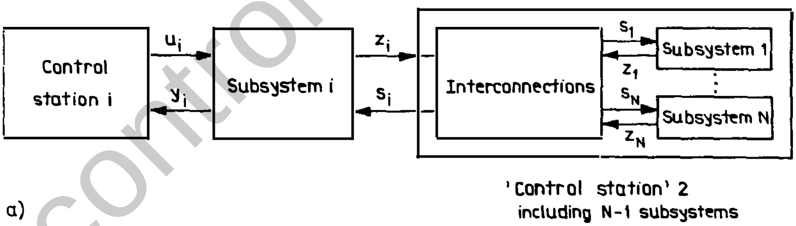
$$\operatorname{rank}(\mathbf{A}_i - \lambda\mathbf{I} \quad \mathbf{B}_i) < n_i \quad \text{and} \quad \operatorname{rank}(\mathbf{A}_i - \lambda\mathbf{I} \quad \mathbf{E}_i) < n_i \tag{4.1.28}$$

or

$$\operatorname{rank}\begin{pmatrix} \mathbf{A}_i - \lambda\mathbf{I} \\ \mathbf{C}_i \end{pmatrix} < n_i \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} \mathbf{A}_i - \lambda\mathbf{I} \\ \mathbf{C}_{zi} \end{pmatrix} < n_i \tag{4.1.29}$$

holds.

A refinement of this condition can be obtained if the ith subsystem is interpreted as a plant with two decentralized controllers. The first controller is the ith control station and the second 'controller' is represented by all other subsystems including their controllers (Figure 4.3(a)). Then



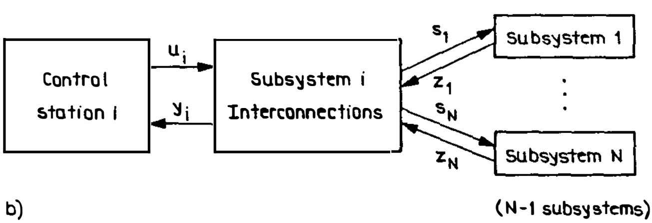


Figure 4.3 Decomposition of the plant: (a) decomposition used in Theorem 4.2; (b) decomposition used in Theorem 4.3

a subsystem eigenvalue $\lambda[\mathbf{A}_i]$ is decentralized fixed if the inequality (4.1.17) is valid.

Theorem 4.2

A subsystem eigenvalue $\lambda[\mathbf{A}_i]$ is a decentralized fixed eigenvalue of the overall system (4.1.26) and (4.1.27) if at least one of the following conditions is satisfied:

$$\operatorname{rank}\begin{pmatrix} \mathbf{A}_i - \lambda\mathbf{I} & \mathbf{E}_i \\ \mathbf{C}_i & \mathbf{0} \end{pmatrix} < n_i \qquad \operatorname{rank}\begin{pmatrix} \mathbf{A}_i - \lambda\mathbf{I} & \mathbf{B}_i \\ \mathbf{C}_{zi} & \mathbf{0} \end{pmatrix} < n_i. \tag{4.1.30}$$

Both conditions are sufficient but not necessary because they prove a subsystem eigenvalue $\lambda[\mathbf{A}_i]$ to be fixed. As Example 4.4 has shown, all overall system eigenvalues may differ from the subsystem eigenvalues but some of them may be decentralized fixed.

Theorem 4.2 is not influenced by the interaction relation (4.1.27). A subsystem eigenvalue which is proved to be decentralized fixed has this property for arbitrary interactions between the subsystems. The peculiarity of such eigenvalues is that they remain fixed even if some subsystems are disconnected from the overall system.

The conditions of Theorem 4.2 can be related to the interaction description (4.1.27) if the ith subsystem and the interaction relations are used as a common block within the overall system (Figure 4.3(b))

$$\begin{aligned} \dot{\mathbf{x}}_i &= (\mathbf{A}_i + \mathbf{E}_i\mathbf{L}_{ii}\mathbf{C}_{zi})\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i + \sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{E}_i\mathbf{L}_{ij}\mathbf{z}_j \\ \mathbf{y}_i &= \mathbf{C}_i\mathbf{x}_i \\ \mathbf{s}_j &= \mathbf{L}_{ji}\mathbf{C}_{zi}\mathbf{x}_i \qquad (j = 1, 2, \ldots, N;\ j \neq i). \end{aligned} \tag{4.1.31}$$

Then, all other subsystems can be interpreted as 'decentralized feedbacks' from $\mathbf{s}_j$ to $\mathbf{z}_j$. If eqn (4.1.31) is expressed as

$$\begin{aligned} \dot{\mathbf{x}} &= \tilde{\mathbf{A}}\mathbf{x} + \sum_{j=1}^{N} \tilde{\mathbf{B}}_{sj}\tilde{\mathbf{u}}_j \\ \tilde{\mathbf{y}}_j &= \tilde{\mathbf{C}}_{sj}\mathbf{x} \qquad (j = 1, 2, \ldots, N) \end{aligned} \tag{4.1.32}$$

with

$$\begin{aligned} \tilde{\mathbf{u}}_j &= \mathbf{z}_j \quad \text{for } j \neq i \text{ and } \tilde{\mathbf{u}}_i = \mathbf{u}_i \\ \tilde{\mathbf{y}}_j &= \mathbf{s}_j \quad \text{for } j \neq i \text{ and } \tilde{\mathbf{y}}_i = \mathbf{y}_i \end{aligned}$$

where

$$\begin{aligned} &\tilde{\mathbf{A}} = \mathbf{A}_i + \mathbf{E}_i\mathbf{L}_{ii}\mathbf{C}_{zi} \\ &\tilde{\mathbf{B}}_{sj} = \mathbf{E}_i\mathbf{L}_{ij} \qquad \tilde{\mathbf{C}}_{sj} = \mathbf{L}_{ji}\mathbf{C}_{zi} \quad \text{for } j \neq i \\ &\tilde{\mathbf{B}}_{si} = \mathbf{B}_i \qquad \tilde{\mathbf{C}}_{si} = \mathbf{C}_i \end{aligned} \tag{4.1.33}$$

hold, the applicability of Theorem 4.1 becomes evident.

Theorem 4.3

A sufficient condition for a subsystem eigenvalue $\lambda[\mathbf{A}_i]$ to be a decentralized fixed mode of the overall system (4.1.26) and (4.1.27) is that there exists a scalar k and a disjoint partition $\mathcal{D}, \mathcal{H}$ of the index set $\mathcal{I}$ according to eqns (4.1.10) and (4.1.11) such that the inequality (4.1.17) holds where $\mathbf{B}_\mathrm{D}$ and $\mathbf{C}_\mathrm{H}$ are formed according to eqn (4.1.12) with $\tilde{\mathbf{B}}_{si}$ and $\tilde{\mathbf{C}}_{si}$ from eqn (4.1.33) instead of $\mathbf{B}_{si}$ and $\mathbf{C}_{si}$. That is, the sufficient condition is given by

$$\operatorname{rank}\begin{pmatrix} \tilde{\mathbf{A}} - \lambda\mathbf{I} & \tilde{\mathbf{B}}_\mathrm{D} \\ \tilde{\mathbf{C}}_\mathrm{H} & \mathbf{0} \end{pmatrix} < n_i.$$

Example 4.3 (cont.)

For the river model (4.1.25) the matrices in eqn (4.1.32) for $i = 2$ are

$$\begin{aligned} &\tilde{\mathbf{A}} = \mathbf{A}_2 \qquad \tilde{\mathbf{B}}_{s1} = \mathbf{E}_2 \qquad \tilde{\mathbf{B}}_{s2} = \mathbf{0} \qquad \tilde{\mathbf{B}}_{s3} = \mathbf{0} \\ &\tilde{\mathbf{C}}_{s1} = \mathbf{0} \qquad \tilde{\mathbf{C}}_{s2} = \mathbf{0} \qquad \tilde{\mathbf{C}}_{s3} = \mathbf{C}_{z2} \end{aligned}$$

(do not confuse the different indices for $\mathbf{u}_2$, $\mathbf{y}_2$ and subsystem 3; cf. eqn (3.2.15)). The condition of Theorem 4.3 is satisfied with $\mathcal{D} = \{2, 3\}$, $\mathcal{H} = \{1\}$ for $\lambda = \lambda_j[\mathbf{A}_2]$ $(j = 1, \ldots, \dim \mathbf{A}_2)$,

$$\operatorname{rank}\begin{pmatrix} \mathbf{A}_2 - \lambda\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} < \dim \mathbf{A}_2.$$

That is, all eigenvalues of $\mathbf{A}_2$ are decentralized fixed and remain fixed if subsystems 1 or 3 are disconnected from the overall system (while leaving two subsystems with centralized control). □

If, for the system (4.1.26), the relations

$$\mathbf{E}_i = \mathbf{B}_i \qquad \mathbf{C}_{zi} = \mathbf{C}_i \tag{4.1.34}$$

hold, the system (4.1.26) and (4.1.27) is called a *system with* *input–output interconnections*, since the coupling signals act on the subsystems in the same way as the control inputs. For this special class of systems,

$$\Lambda_{\mathrm{df}} = \Lambda_{\mathrm{f1}} \cup \Lambda_{\mathrm{f2}} \cup \cdots \cup \Lambda_{\mathrm{f}N} \tag{4.1.35}$$

holds, that is the set of decentralized fixed modes consists of the sets of the centralized fixed modes of the isolated subsystems. Hence, for this particular class of systems the conjecture stated prior to Example 4.4 is true. The same holds true for systems with hierarchical structure.

4.2 STRUCTURALLY FIXED MODES

In centralized control, complete controllability and observability of the system turned out to be structural properties. That is, these properties are based on the interactions between the external and internal signals $\mathbf{u}_i$, $\mathbf{y}_i$ or $\mathbf{x}_i$, respectively. If this structure is *s*-complete according to Definition 2.6 only specific parameter combinations can make the system uncontrollable or unobservable. These results will now be extended to decentralized control where the decentralized fixed modes represent the analogue of the non-observable or non-controllable modes under centralized control.

The class $\mathcal{S}_{\mathrm{d}}$ of the systems under consideration is defined in terms of the I/O-oriented model

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}_i(t) &= \mathbf{C}_{si}\mathbf{x}(t) \qquad (i = 1, 2, \ldots, N) \end{aligned} \tag{4.2.1}$$

where the decompositions of $\mathbf{u}$ into $\mathbf{u}_i$ and $\mathbf{y}$ into $\mathbf{y}_i$ correspond to the structural constraints on the feedback, which are given by the restriction $\mathbf{K} \in \mathcal{K}$. $\mathcal{S}_{\mathrm{d}}$ is described in terms of structure matrices $\mathbf{S}_a$, $\mathbf{S}_{bi}$ and $\mathbf{S}_{ci}$ by

$$\begin{aligned} &\mathcal{S}_{\mathrm{d}}(\mathbf{S}_a, \mathbf{S}_{b1}, \ldots, \mathbf{S}_{bN}, \mathbf{S}_{c1}, \ldots, \mathbf{S}_{cN}) \\ &\quad = \Bigg\{(\mathbf{A}, \mathbf{B}, \mathbf{C})\colon\ [\mathbf{A}] = \mathbf{S}_a,\ \mathbf{B} = (\mathbf{B}_{s1}\ \ldots\ \mathbf{B}_{sN}) \text{ with } [\mathbf{B}_{si}] = \mathbf{S}_{bi}, \\ &\qquad \mathbf{C} = \begin{pmatrix} \mathbf{C}_{s1} \\ \vdots \\ \mathbf{C}_{sN} \end{pmatrix} \text{ with } [\mathbf{C}_{si}] = \mathbf{S}_{ci} \Bigg\}. \end{aligned} \tag{4.2.2}$$

Definition 4.2

A class $\mathscr{S}_d$ of systems is said to have *structurally fixed modes* with respect to the feedback structure constraints $\mathscr{K}$ if every system $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathscr{S}_d$ has decentralized fixed modes with respect to $\mathscr{K}$.

Corollary 4.1

A given system (4.2.1) is known to have decentralized fixed modes if the class

$$\mathscr{S}_d([\mathbf{A}], [\mathbf{B}_{s1}], \ldots, [\mathbf{B}_{sN}], [\mathbf{C}_{s1}], \ldots, [\mathbf{C}_{sN}])$$

has structurally fixed modes.

As with centralized fixed modes, structurally fixed modes can be detected by investigating the graph of the closed-loop system. Because of the structural feedback constraints the graph has to be drawn for the adjacency matrix

$$\mathbf{Q}_{\mathbf{K}} = \begin{pmatrix} \mathbf{S}_a & \mathbf{S}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_k \\ \mathbf{S}_c & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{4.2.3}$$

where $\mathbf{S}_k$ describes the structure of the decentralized feedback

$$\mathbf{K} \in \mathscr{K} = \{\mathbf{K}\colon\ [\mathbf{K}] = \mathbf{S}_k\}$$

and

$$\mathbf{S}_b = (\mathbf{S}_{b1}\ \ \mathbf{S}_{b2}\ \ \ldots\ \ \mathbf{S}_{bN}) \qquad \mathbf{S}_c = \begin{pmatrix} \mathbf{S}_{c1} \\ \vdots \\ \mathbf{S}_{cN} \end{pmatrix} \tag{4.2.4}$$

hold. For example of the graph $G(\mathbf{Q}_{\mathbf{K}})$ see Example 4.1 below and Figure 4.4.

The development of Theorem 4.1 has shown that fixed modes are those eigenvalues which are not simultaneously controllable and observable from a single channel or which can be made to possess this property by selecting the appropriate control stations at other channels. Therefore, the reachability of the states as defined by Definition 2.7 has to be referred to the channels $(\mathbf{u}_i, \mathbf{y}_i)$. A vertex is said to belong to $\mathbf{u}_i$ or $\mathbf{y}_i$ if it represents a signal $\mathbf{u}_{ij}$ or $\mathbf{y}_{ij}$ included in the vector

$$\mathbf{u}_i = (u_{i1}\ \ u_{i2}\ \ \ldots\ \ u_{im_i})' \quad \text{or} \quad \mathbf{y}_i = (y_{i1}\ \ y_{i2}\ \ \ldots\ \ y_{ir_i})'$$

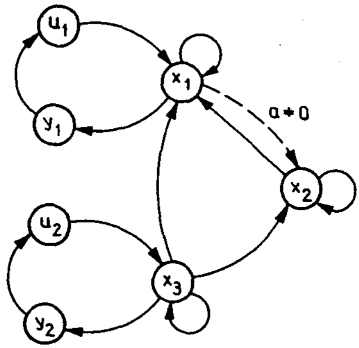


Figure 4.4 Structure of the system (4.1.24)

respectively. For dim $\mathbf{u}_i$ = dim $\mathbf{y}_i = 1$ only one vertex belongs to $\mathbf{u}_i$ or $\mathbf{y}_i$, but if the channel has vector-valued signals, then $\mathbf{u}_i$ and $\mathbf{y}_i$ correspond to more than one vertex.

Definition 4.3

A state vertex v of the graph $G(\mathbf{Q}_K)$ is said to be *connectable to a channel* $(\mathbf{u}_i, \mathbf{y}_i)$ if there exist paths from at least one of the vertices belonging to $\mathbf{u}_i$ to v and from v to at least one of the vertices belonging to $\mathbf{y}_i$.

Since a state can be directly influenced by the decentralized control station i described by (4.1.6) only if it is connectable to channel i, the existence of a state vertex which is not connectable to any channel is sufficient for a decentralized fixed mode to exist in all systems with a given structure. A second sufficient condition is similar to Theorem 2.16(ii).

Theorem 4.4

For the class $\mathscr{S}_d$ of systems there exist structurally fixed modes if and only if at least one of the following conditions is satisfied for the graph $G(\mathbf{Q}_K)$.

(i) There exists a state vertex which is not connectable to a channel.
(ii) There does not exist a cycle family of width n.

Structurally fixed modes that occur due to condition (i) or due to condition (ii) are called *structurally fixed modes of type I* or *structurally fixed modes of type II*, respectively. As in centralized control, fixed modes of type II that occur for structural reasons are known to be fixed at zero.

Example 4.1 (cont.)

The structure of the system (4.1.24) under decentralized control is shown in Figure 4.4. For $a = 0$ the vertex x_2 is connectable neither to channel 1 nor to channel 2. Therefore, the decentralized fixed eigenvalue at -2 revealed in Section 4.1 occurs for structural reasons. It can only be avoided if the reachability of x_2 is improved. This can be done if a connection between the internal signals x_1 and x_2 of the system is created ($a \neq 0$, cf. Figure 4.4), if a new input is created that belongs to channel 1 and reaches x_2, or if a new measurement is introduced in channel 2 so that x_2 is connectable to the expanded channel 2.

The application of the first measure is questionable because it concerns a modification of the internal structure of the plant. The other changes refer to the location of measurement devices or actuators, to the introduction of new inputs and outputs, or to their grouping into channels. They avoid the structurally fixed mode by extending the influence that the controller can have on the plant.

Since it is known from Example 2.1 that the system (4.1.24) for $a = 0$ is structurally complete for centralized control, all vertices can be included in some centralized feedback loop and none of them is structurally fixed. The appearance of structurally fixed modes under decentralized control is based on the absence of the edges $y_2 \to u_1$ and $y_1 \to u_2$ (cf. Figures 2.4 and 4.4) and, consequently, the restriction of the reachability property to the channels (u_i, y_i) (cf. Definitions 2.7 and 4.3). Therefore, the structurally fixed mode detected above can also be avoided by relaxing the structural constraints on the feedback law, that is by introducing new information links into the controller. □

Example 4.2 (cont.)

The graph $G(\mathbf{Q_K})$ of this example system is depicted in Figure 4.5. If a and b are known to be zero the corresponding edges do not exist (dashed lines). Then, although all state vertices are connectable to a channel, no cycle family of width 3 can be found. A structurally fixed mode of type II occurs. It has value zero for all actual systems that have the same

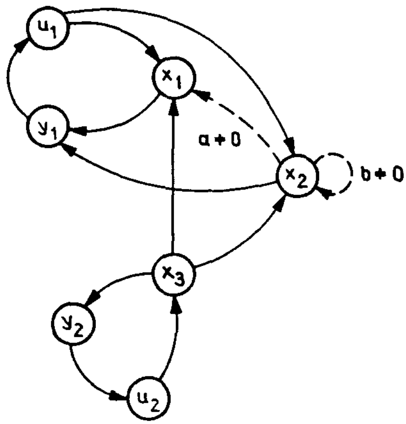


Figure 4.5 Example 4.2

structure as the example. However, if $a \neq b$ and $b \neq 0$ hold, no structurally fixed modes exist. The decentralized fixed mode at $\lambda = 0$ found in Section 4.1 for $a = b$ can only occur for this specific combination of the parameters a and b but not for the whole class

$$\mathcal{S}_{\mathrm{d}}([\mathbf{A}], [\mathbf{B}_{s1}], [\mathbf{B}_{s2}], [\mathbf{C}_{s1}], [\mathbf{C}_{s2}])$$

with $\mathbf{A}, \mathbf{B}_{s1}, \ldots$ as above. □

Structurally Fixed Modes in Interconnected Systems

The question is now considered of which structurally fixed modes are based on the interconnection structure of the overall system. This can be done at two different levels of abstraction. First, the global structure of the overall system is considered. Second, analogous results to those of Section 4.1 are obtained by investigating subsystem i in its surroundings.

Investigation of the global system structure

On the first level of abstraction the overall system is considered according to its global structure, that is the interactions between the subsystems. A graph that represents this global structure is constructed according to the model (3.1.16)

$$\dot{\mathbf{x}}_i = \mathbf{A}_{ii}\mathbf{x}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{B}_i\mathbf{u}_i \tag{4.2.5}$$

$$\mathbf{y}_i = \mathbf{C}_i\mathbf{x}_i \qquad (i = 1, \ldots, N)$$

in the following steps. Each subsystem is associated with a (hyper)vertex of the system graph (as in Section 3.2). Edges from the vertex of subsystem i to the vertex of subsystem j exist if the matrix $\mathbf{A}_{ji}$ is not the zero matrix. The matrices $\mathbf{A}_{ii} \neq \mathbf{0}$ create self-circles. Each channel $(\mathbf{u}_i, \mathbf{y}_i)$ is represented by one input and one output vertex. Edges to the subsystem vertex exist whenever $\mathbf{B}_i \neq \mathbf{0}$ and $\mathbf{C}_i \neq \mathbf{0}$. For the decentralized controller edges are drawn from all the $\mathbf{y}_i$ vertices towards the corresponding $\mathbf{u}_i$ vertices. Since all edges represent connections between the hypervertices of the overall system graph $G(\mathbf{Q}_0)$ defined in Section 2.5 the edges are drawn with double lines. The resulting graph is called $G(\mathbf{Q}_\mathrm{L})$ with

$$\mathbf{Q}_\mathrm{L} = \left(\begin{array}{ccc|ccc|ccc}
[[\mathbf{A}_{11}]] & \ldots & [[\mathbf{A}_{1N}]] & [[\mathbf{B}_1]] & \ldots & 0 & & & \\
\vdots & & \vdots & \vdots & & \vdots & & \mathbf{0} & \\
[[\mathbf{A}_{N1}]] & \ldots & [[\mathbf{A}_{NN}]] & 0 & \ldots & [[\mathbf{B}_N]] & & & \\
\hline
 & & & & & & * & \ldots & 0 \\
 & \mathbf{0} & & & \mathbf{0} & & \vdots & & \vdots \\
 & & & & & & 0 & \ldots & * \\
\hline
[[\mathbf{C}_1]] & \ldots & 0 & & & & & & \\
\vdots & & \vdots & & \mathbf{0} & & & \mathbf{0} & \\
0 & \ldots & [[\mathbf{C}_N]] & & & & & &
\end{array}\right)$$

(for the definition of [[.]] see Section 3.2). The class of all systems that have a given graph $G(\mathbf{Q}_\mathrm{L})$ is called $\mathcal{S}_\mathrm{L}$.

Example 4.5

Figure 4.6 shows the graph $G(\mathbf{Q}_\mathrm{L})$ for a system (4.2.5) with $N = 4$ and

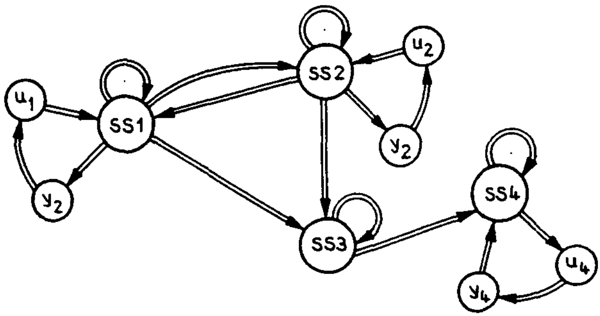


Figure 4.6 Global graph of an interconnected system

vanishing matrices $\mathbf{A}_{13}$, $\mathbf{A}_{14}$, $\mathbf{A}_{23}$, $\mathbf{A}_{24}$, $\mathbf{A}_{34}$, $\mathbf{A}_{41}$, $\mathbf{A}_{42}$, $\mathbf{B}_3$ and $\mathbf{C}_3$. Hence,

$$
\mathbf{Q}_L = \left[\begin{array}{cccc|cccc|cccc}
* & * & 0 & 0 & * & 0 & 0 & 0 & & & & \\
* & * & 0 & 0 & 0 & * & 0 & 0 & \multicolumn{4}{c}{\mathbf{0}} \\
* & * & * & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & * & * & 0 & 0 & 0 & * & & & & \\
\hline
 & & & & & & & & * & 0 & 0 & 0 \\
 & & & & & & & & 0 & * & 0 & 0 \\
\multicolumn{4}{c|}{\mathbf{0}} & \multicolumn{4}{c|}{\mathbf{0}} & 0 & 0 & * & 0 \\
 & & & & & & & & 0 & 0 & 0 & * \\
\hline
* & 0 & 0 & 0 & & & & & & & & \\
0 & * & 0 & 0 & \multicolumn{4}{c|}{\mathbf{0}} & \multicolumn{4}{c}{\mathbf{0}} \\
0 & 0 & * & 0 & & & & & & & & \\
0 & 0 & 0 & * & & & & & & & &
\end{array}\right]
$$

holds. Since $\mathbf{B}_3$ and $\mathbf{C}_3$ are zero, y_3 and u_3 do not exist and are, thus, omitted in Figure 4.6. □

For the graph $G(\mathbf{Q}_L)$ a reachability property similar to that of Definition 4.3 can be defined for the whole subsystem state $\mathbf{x}_i$, which is represented by the subsystem vertex. A subsystem state $\mathbf{x}_j$ is said to be connectable to a channel $(\mathbf{u}_i, \mathbf{y}_i)$ if there exist paths from $\mathbf{u}_i$ towards $\mathbf{x}_j$ and $\mathbf{x}_j$ towards $\mathbf{y}_i$. According to Theorem 4.4, if some subsystem vertex is not connectable to a channel structurally fixed modes exist in the class $\mathscr{S}_L$. The second condition of Theorem 4.4 has its analogy too.

Theorem 4.5

For the class $\mathcal{S}_L$ of interconnected systems (4.2.5) there exist structurally fixed modes if at least one of the following conditions is satisfied for the graph $G(\mathbf{Q}_L)$.

(i) There exists a subsystem vertex which is not connectable to a channel.
(ii) There does not exist a cycle family of width N.

Example 4.3 (cont.)

The global graph $G(\mathbf{Q}_L)$ of the river control system is shown in Figure 4.7. Obviously, subsystem 2 is not connectable to a channel. Therefore, the decentralized fixed modes that have been detected in Section 4.1 exist due to structural properties of the system. Since Theorem 4.5 regards the subsystems as a whole, *all* subsystem eigenvalues coincide with the eigenvalues of the overall system and are structurally fixed. The same holds true for subsystem 3 in Figure 4.6. □

The correspondences between the graph $G(\mathbf{Q}_K)$ used in Theorem 4.4 and the global structure represented by the graph $G(\mathbf{Q}_L)$ reveal another result which concerns the conjecture that an interconnected system has no decentralized fixed modes if all subsystems are completely controllable and observable (cf. Section 4.1). If each subsystem is structurally controllable and observable then all state variables included in the vector $\mathbf{x}_i$ are connectable to the ith channel. Moreover, the subgraph of $G(\mathbf{Q}_K)$, which belongs to the ith subsystem, includes a cycle family of width n_i (cf. Section 2.5). This holds true for all subsystems ($i = 1, \ldots, N$). Consequently, both conditions of Theorem 4.5 are violated and the overall system does not possess *structurally* fixed modes.

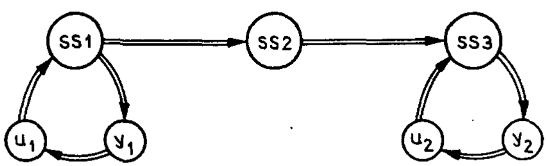


Figure 4.7 Global structure of the river control system

Corollary 4.2

If all subsystems of the overall system (4.1.26) and (4.1.27) are completely controllable and completely observable then the class of structurally equivalent systems has no structurally fixed modes.

Hence, an overall system that consists of controllable and observable subsystems has decentralized fixed modes only for exceptional parameter combinations. The conjecture stated prior to Example 4.4 is true for 'almost all' systems.

Investigation of a single subsystem

In analogy to the considerations at the end of Section 4.1 the couplings to and from a single subsystem i will now be investigated. As in Figure 4.3(a) the subsystem i (eqn (4.1.26) for fixed i) can be considered as a system with two channels $(\mathbf{u}_i, \mathbf{y}_i)$ and $(\mathbf{s}_i, \mathbf{z}_i)$. Theorem 4.4 can be applied to the 'local subsystem graph' $G(\mathbf{Q}_{1i})$ with

$$\mathbf{Q}_{1i} = \left(\begin{array}{c|c|c|cc} [\mathbf{A}_i] & [\mathbf{B}_i] & [\mathbf{E}_i] & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & * & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & * \\ \hline [\mathbf{C}_i] & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ [\mathbf{C}_{zi}] & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right) \tag{4.2.6}$$

in order to investigate whether a subsystem eigenvalue is a structurally fixed eigenvalue of the overall system. In this graph, all other subsystems are considered as one control station between $\mathbf{z}_i$ and $\mathbf{s}_i$.

Theorem 4.6

Consider the class of systems whose ith subsystem has the structure described by $\mathbf{Q}_{1i}$ $(i = 1, \ldots, N)$. This class has structurally fixed modes if there exists some index $i \in \mathcal{I}$ such that at least one of the following conditions is satisfied in the graph $G(\mathbf{Q}_{1i})$.

(i) There exists a state vertex which is not connectable to a channel.
(ii) There does not exist a cycle family of width n_i.

On the other hand, if the ith subsystem (4.1.26) is considered as an N-channel system as in Figure 4.3(b) it can be described by a model of the form (4.1.32) and (4.1.33). The subsystem and the interactions with its surroundings are described by the 'global subsystem graph' $G(\mathbf{Q}_{gi})$ with

$$\mathbf{Q}_{gi} = \left[\begin{array}{c|ccc|ccc} [\bar{\mathbf{A}}] & [\tilde{\mathbf{B}}_{s1}] & \dots & [\tilde{\mathbf{B}}_{sN}] & & \mathbf{0} & \\ \hline \mathbf{0} & & & & * & \dots & 0 \\ \vdots & & \mathbf{0} & & \vdots & & \vdots \\ \mathbf{0} & & & & 0 & \dots & * \\ \hline [\tilde{\mathbf{C}}_{s1}] & & & & & & \\ \vdots & & \mathbf{0} & & & \mathbf{0} & \\ [\tilde{\mathbf{C}}_{sN}] & & & & & & \end{array}\right].$$

Theorem 4.5 leads to the following sufficient condition.

Theorem 4.7

Consider the class of systems (4.1.26) and (4.1.27) whose ith subsystem (4.1.26) and interaction relation (4.1.27) together have the structure described by $\mathbf{Q}_{gi}$ $(i = 1, \dots, N)$. This class has structurally fixed modes if there exists some index $i \in \mathcal{I}$ such that at least one of the following conditions is satisfied in the graph $G(\mathbf{Q}_{gi})$.

(i) There exists a state vertex which is not connectable to a channel.
(ii) There does not exist a cycle family of width n_i.

In systems satisfying Theorem 4.6 or 4.7 the decentralized fixed modes are centralized fixed modes of the isolated ith subsystem.

4.3 STABILIZABILITY AND POLE ASSIGNABILITY BY DECENTRALIZED FEEDBACK

From Definition 4.1 all modes of the plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}_i(t) &= \mathbf{C}_{si}\mathbf{x}(t) \qquad (i = 1, 2, \dots, N) \end{aligned} \tag{4.3.1}$$

can be changed by some static decentralized feedback

$$\mathbf{u} = -\mathbf{K}_y\mathbf{y} \quad \text{with} \quad \mathbf{K}_y \in \mathcal{K} \tag{4.3.2}$$

if and only if the system (4.3.1) has no decentralized fixed modes. So far this assertion says merely that all eigenvalues of $\mathbf{A}$ are 'movable'. However, stabilizability and pole assignability presuppose more because the eigenvalues have to be moved simultaneously into the left-half complex plane or to prescribed positions. Therefore the importance of the fixed modes became clear only after it could be proved that all non-fixed modes can be assigned *simultaneously arbitrarily* prescribed values by some dynamic decentralized controller

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{F}_i \mathbf{x}_{ri} + \mathbf{G}_i \mathbf{y}_i + \mathbf{H}_i \mathbf{v}_i \\ \mathbf{u}_i &= -\mathbf{K}_{xi} \mathbf{x}_{ri} - \mathbf{K}_{yi} \mathbf{y}_i + \mathbf{K}_{vi} \mathbf{v}_i. \end{aligned} \tag{4.3.3}$$

Theorem 4.8

The following assertions hold for a system (4.3.1) whose set of decentralized fixed modes is denoted by Λ_{df}.

(i) There exists a decentralized controller (4.3.3) such that the closed-loop system (4.3.1) and (4.3.3) is stable if and only if all decentralized fixed eigenvalues have negative real parts:

$$\text{Re}[\lambda] < 0 \quad \text{for all } \lambda \in \Lambda_{df}. \tag{4.3.4}$$

(ii) There exists a decentralized controller (4.3.3) such that the closed-loop system (4.3.1) and (4.3.3) has an arbitrarily prescribed set σ_0 of eigenvalues if and only if the system (4.3.1) has no decentralized fixed modes:

$$\Lambda_{df} = \emptyset. \tag{4.3.5}$$

This result is the direct analogy of Theorem 2.5.

One answer to the question of how to find a stabilizing decentralized controller (4.3.3) can be found from the considerations of Section 4.1. Accordingly, all non-fixed eigenvalues are controllable and observable through a single channel ($\mathbf{u}_i$, $\mathbf{y}_i$) or can be made so by choosing appropriate static control stations

$$\mathbf{u}_j = -\mathbf{K}_{yj} \mathbf{y}_j \tag{4.3.6}$$

for $j \neq i$. If this is done, a centralized dynamical controller (eqn (4.3.3) for fixed i) can be designed for the remaining channel i by methods known from multivariable control. This design method will be explained in detail in Section 6.1.

For centralized control, pole assignment is possible without any controller dynamics if the system state $\mathbf{x}$ can be measured. Then, the controller represents a state feedback

$$\mathbf{u} = -\mathbf{K}\mathbf{x}.$$

An analogy to this is given by decentralized state feedback

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i. \tag{4.3.7}$$

However, an analogy to Theorem 2.6 does not hold. That is, decentralized state feedback (4.3.7) may not be sufficient to place the closed-loop system eigenvalues at arbitrarily prescribed positions in the complex plane, and a dynamical controller (4.3.3) may be necessary. This becomes clear from the following example.

Example 4.6

The subsystems (4.1.26) with

$$\mathbf{A}_1 = \mathbf{A}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{B}_1 = \mathbf{B}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{E}_1 = \mathbf{E}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{C}_{z1} = \mathbf{C}_{z2} = (0 \;\; 1)$$

are coupled according to eqn (4.1.27) with

$$\mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, both isolated subsystems are controllable. The overall system is described by eqn (3.1.1) with

$$\mathbf{A} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \mathbf{B} = \left(\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \\ 0 & 1 \end{array}\right).$$

It does not possess decentralized fixed modes, which can be checked by means of Theorem 4.1. However, the system under decentralized state feedback

$$u_i = -(k_{i1} \;\; k_{i2})\mathbf{x}_i$$

has the characteristic equation $\det(\bar{\mathbf{A}} - \lambda \mathbf{I}) = 0$ with

$$\bar{\mathbf{A}} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ -k_{11} & -k_{12} & 0 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & -k_{21} & -k_{22} \end{array}\right)$$

which is equivalent to

$$\lambda^4 + \lambda^3(k_{12} + k_{22}) + \lambda^2(k_{12}k_{22} + k_{21}k_{11}) + \lambda(k_{11}k_{22} + k_{12}k_{21}) = 0.$$

This system has a closed-loop eigenvalue fixed at zero. It is unstable irrespective of the controller parameters.

BIBLIOGRAPHICAL NOTES

The importance of decentralized fixed modes for decentralized stabilizability was recognized by Wang and Davison as early as 1973, although the stabilizability problem for a specific class of systems had been investigated earlier by Aoki (1972) and Corfmat and Morse (1973). They introduced Definition 4.1 and proved the main result formulated in Theorem 4.8. The algebraic characterization of fixed modes given in Theorem 4.1 is due to Anderson and Clements (1981). The complementary system (4.1.19) used there had been introduced earlier by Corfmat and Morse (1976a). The way in which Theorem 4.1 was derived in Section 4.1 was suggested by the idea of Corfmat and Morse (1976a) to make all modes controllable and observable from a single channel; a thorough explanation of this idea appears in the monograph by Litz (1983). Anderson (1982) gave a necessary and sufficient condition for the existence of fixed modes in terms of the matrix fraction description of the plant.

Investigations of the dependencies of fixed modes upon the subsystem and interaction properties of interconnected systems are rare. Since the controllability and the observability of the overall system are prerequisites for the absence of fixed modes, Davison (1977b) considered these properties and showed that connectability as defined in his paper describes under what conditions on the interconnections the overall system is controllable and observable through the whole vectors $\mathbf{u}$ and $\mathbf{y}$. Contrary to the investigations of Corfmat and Morse, connectability refers to the internal structure of the interconnected system rather than to its I/O behaviour.

Fessas (1979) published the conjecture that observable and controllable subsystems never yield an overall system with fixed modes, but Ikeda and Šiljak (1979) gave counterexamples. The counterexample represented in Example 4.4 is taken from Litz (1983). Several authors have tried to find specific kinds of interconnected systems for which this conjecture is true, among them Šiljak and Vukčević (1977), Sezer and Hüsein (1978), Saeks (1979), Davison (1979) and Lunze (1986). In particular, it has been shown that for systems with input–output interconnections the set of decentralized fixed modes comprises exactly those modes which are not controllable within an isolated subsystem (eqn (4.1.34)). Linnemann (1984) extended this result to dynamically interconnected systems.

Conditions on the subsystems and interconnection properties under which the overall system can be stabilized by decentralized feedback have been elaborated, for example, by Davison and Özgüner (1983) or Wu and Mansour (1989). Theorems 4.2 and 4.3 are due to Litz (1983). Hassan *et al.* (1989) gave an example of the fact that fixed modes may be caused by overlapping decomposition if the original system has no such modes.

Structural investigations of decentralized fixed modes began with the work of Sezer and Šiljak (1981b) and Pichai *et al.* (1984). Their main result is equivalent to Theorem 4.4, the second part of which has been formulated in terms of the cycle families as used by Reinschke (1984, 1988). Theorems 4.5–4.7 have been derived here as structural analogies of Theorems 4.2 and 4.3.

If the plant has unstable decentralized fixed modes then the restriction of the control law to be linear and decentralized has to be removed. One way is to allow information exchange among the control stations. Wang and Davison (1978) showed how communication links could be chosen so as to minimize the transmission cost, which is linear with respect to the number of signals transmitted. A similar procedure, which concerns structurally fixed modes of type I, was proposed by Travé *et al.* (1984). Another way is to retain the decentralization of the control law but allow non-linear or time-varying controller gains as proposed by Anderson and Moore (1981). This method of stabilization, in spite of decentralized fixed modes, is possible only if the fixed modes are not structurally fixed. The river control example was suggested by a similar example described by Singh (1981).

5

The Decentralized Servomechanism Problem

This chapter presents conditions under which decentralized controllers exist that ensure asymptotic regulation within the closed-loop system. It continues the investigations concerning the existence of suitable decentralized control laws but, in addition to the claim of closed-loop stability discussed in Chapter 4, it extends the control aim to that of asymptotic tracking of a given command signal $\mathbf{v}(t)$ despite unmeasurable disturbances $\mathbf{d}(t)$.

For the sake of comparison, the solution to the servomechanism problem of centralized control will be briefly reviewed in Section 5.1. Its extension to decentralized control is given in Section 5.2. Specific results for asymptotic regulation under step disturbances and step commands are summarized in Section 5.3.

5.1 THE CENTRALIZED SERVOMECHANISM PROBLEM

Consider the plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{M}\mathbf{d} \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{N}\mathbf{d} \end{aligned} \tag{5.1.1}$$

with $m = \operatorname{rank} \mathbf{B} = \dim \mathbf{u}$ and $r = \operatorname{rank} \mathbf{C} = \dim \mathbf{y}$. The disturbances are unmeasurable but are assumed to belong to the class of signals that are described by the disturbance model

$$\begin{aligned} \dot{\mathbf{x}}_d &= \mathbf{A}_d\mathbf{x}_d \qquad \mathbf{x}_d(0) = \mathbf{x}_{d0} \\ \mathbf{d} &= \mathbf{C}_d\mathbf{x}_d \end{aligned} \tag{5.1.2}$$

with unknown initial state $\mathbf{x}_{d0}$. Since $\mathbf{x}_{d0}$ is unknown the actual disturbance $\mathbf{d}(t)$ is not known (Figure 5.1).

The same assumption can be made concerning the command signal $\mathbf{v}(t)$ which belongs to the class of signals described by

$$\begin{aligned} \dot{\mathbf{x}}_v &= \mathbf{A}_v\mathbf{x}_v \qquad \mathbf{x}_v(0) = \mathbf{x}_{v0} \\ \mathbf{v} &= \mathbf{C}_v\mathbf{x}_v \end{aligned} \tag{5.1.3}$$

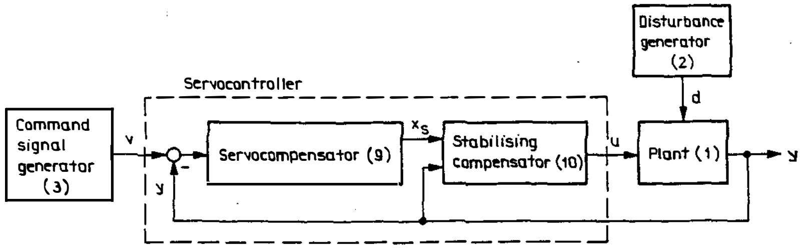


Figure 5.1 Centralized servocontroller configuration

Problem 5.1 (*Robust centralized servomechanism problem*)

For a given plant (5.1.1) and given signal generators (5.1.2) and (5.1.3) find a centralized controller

$$\begin{aligned} \dot{\mathbf{x}}_r &= \mathbf{F}\mathbf{x}_r + \mathbf{G}\mathbf{y} + \mathbf{H}\mathbf{v} \qquad \mathbf{x}_r(0) = \mathbf{x}_{r0} \\ \mathbf{u} &= -\mathbf{K}_x\mathbf{x}_r - \mathbf{K}_y\mathbf{y} + \mathbf{K}_v\mathbf{v} \end{aligned} \tag{5.1.4}$$

such that the following requirements are satisfied:

(i) The closed-loop system (5.1.1) and (5.1.4) is stable.
(ii) The closed loop system (5.1.1) and (5.1.4) ensures asymptotic regulation

$$\mathbf{y}(t) - \mathbf{v}(t) \xrightarrow[t\to\infty]{} \mathbf{0} \tag{5.1.5}$$

for all initial states $\mathbf{x}_0$, $\mathbf{x}_{r0}$, $\mathbf{x}_{d0}$ and $\mathbf{x}_{v0}$ and for all parameter variations of the plant (5.1.1) which do not destabilize the closed-loop system (5.1.1) and (5.1.4).

In order to avoid trivial solutions it is assumed that all eigenvalues of the matrices $\mathbf{A}_d$ and $\mathbf{A}_v$ have positive real parts and that rank $(\mathbf{M}'\ \mathbf{N}') = \text{rank}\ \mathbf{C}_d$ holds. Furthermore, the systems (5.1.2) and (5.1.3) are assumed to be observable.

The requirement (ii) of Problem 5.1 excludes the possibility of utilizing specific properties of the plant which are due to particular parameter values. This is the reason why the matrices **M** and **N** from eqn (5.1.1) do not influence the solution to this problem as will be seen below. The controller has to be robust in the sense that eqn (5.1.5) holds even for large variations of the parameters of the plant model (5.1.1) which do not destabilize the closed loop. Problem 5.1 is, therefore, called the robust centralized servomechanism problem.

The existence of a solution to Problem 5.1 depends on two properties of the plant (5.1.1). The first is the set Λ_f of centralized fixed modes

(Definition 2.3). The second is the set of transmission zeros of the plant which are defined to be values of λ and for which the inequality

$$\operatorname{rank}\begin{pmatrix} \mathbf{A}-\lambda\mathbf{I} & B \\ \mathbf{C} & \mathbf{D} \end{pmatrix} < n+r \tag{5.1.6}$$

with $n = \dim \mathbf{x}$ and $r = \dim y \leqslant \dim \mathbf{u}$ holds. Furthermore, the solution depends on the minimal polynomial $m_d(p)$ and $m_v(p)$ of $\mathbf{A}_d$ or $\mathbf{A}_v$, respectively. The monic least common multiple of m_d and m_v is called m_c and written as

$$m_c(p) = \prod_{i=1}^{q} (p-\lambda_i) = p^q + k_1 p^{q-1} + \cdots + k_q. \tag{5.1.7}$$

The set Λ is defined by $\Lambda = \{\lambda_1, \ldots, \lambda_q\}$ with λ_i as in eqn (5.1.7). Roughly speaking, it includes all eigenvalues of $\mathbf{A}_d$ and $\mathbf{A}_v$.

Lemma 5.1

There exists a solution to the robust centralized servomechanism problem if and only if the following conditions are satisfied:

(i) The plant (5.1.1) has no unstable centralized fixed mode.
(ii) $m \geqslant r$ holds.
(iii) The elements of Λ are not transmission zeros of the plant, that is

$$\operatorname{rank}\begin{pmatrix} \mathbf{A}-\lambda_i\mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = n+r \quad \text{for all } \lambda_i \in \Lambda. \tag{5.1.8}$$

Lemma 5.2

The solution to the robust centralized servomechanism problem consists of the servocompensator

$$\dot{\mathbf{x}}_s = \mathbf{T}\ \text{diag}\ \mathbf{A}_s\mathbf{T}^{-1}\mathbf{x}_s + \mathbf{T}\ \text{diag}\ \mathbf{B}_s(\mathbf{y}-\mathbf{v}) \tag{5.1.9}$$

and the stabilizing compensator

$$\begin{aligned} \dot{\mathbf{x}}_c &= \mathbf{F}_c\mathbf{x}_c + \mathbf{G}_c\mathbf{y} + \mathbf{H}_c\mathbf{x}_s \\ \mathbf{u} &= -\mathbf{K}_1\mathbf{x}_c - \mathbf{K}_2\mathbf{y} - \mathbf{K}_3\mathbf{x}_s. \end{aligned} \tag{5.1.10}$$

The parameters of the servocompensator can be determined from the parameters of the signal generators (5.1.2) and (5.1.3) as follows. $\mathbf{T}$ is an arbitrary non-singular real matrix, and diag $\mathbf{A}_s$ and diag $\mathbf{b}_s$ consist of r matrices $\mathbf{A}_s$ and vectors $\mathbf{b}_s$, respectively, with

$$\mathbf{A}_s = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -k_q & -k_{q-1} & -k_{q-2} & \dots & -k_1 \end{pmatrix} \qquad \mathbf{b}_s = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{5.1.11}$$

The parameters of the stabilizing compensator have to be chosen so that the closed-loop system (5.1.1), (5.1.9) and (5.1.10) is stable.

The feedback controller (5.1.4), which consists of the servocompensator and the stabilizing compensator, is also called the *servocontroller* (Figure 5.1).

The servocompensator consists of r identical parts each of which includes the q unstable eigenvalues of the set Λ. It can be considered as a model of the external signals $\mathbf{d}$ and $\mathbf{v}$. The necessity of including the signal generators in the feedback controller (5.1.4) is referred to as the *internal model principle.*

Since the stability of the closed-loop system (5.1.1), (5.1.9) and (5.1.10) is independent of the matrices $\mathbf{N}, \mathbf{M}, \mathbf{A}_d, \mathbf{C}_d, \mathbf{A}_v, \mathbf{C}_v$, these matrices need not be known and may even be arbitrary with the restriction that $\mathbf{A}_d$ and $\mathbf{A}_v$ produce a fixed set Λ. Asymptotic regulation occurs for all plant parameter variations which do not violate the conditions (i)–(iii) of Lemma 5.1 and do not destabilize the closed-loop system. Moreover, asymptotic regulation occurs despite parameter variations in the stabilizing compensator and in the matrix $\mathbf{T}$, which do not destabilize the closed-loop system.

In an extension of Problem 5.1 the closed-loop system is required to have a prescribed set of eigenvalues. Then, condition (i) in Lemma 5.1 has to be replaced by the statement that the plant (5.1.1) has no centralized fixed modes.

5.2 THE DECENTRALIZED SERVOMECHANISM PROBLEM

In order to extend Problem 5.1 to decentralized control, the input and output vectors of the plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{M}\mathbf{d} \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{N}\mathbf{d} \end{aligned} \tag{5.2.1}$$

have to be partitioned as $\mathbf{u} = (\mathbf{u}_1', \dots, \mathbf{u}_N')'$, $\mathbf{y} = (\mathbf{y}_1', \dots, \mathbf{y}_N')'$. For a compatible partition of $\mathbf{B}$ and $\mathbf{C}$

$$\mathbf{B} = (\mathbf{B}_{s1} \ \dots \ \mathbf{B}_{sN}) \qquad \mathbf{C} = \begin{pmatrix} \mathbf{C}_{s1} \\ \vdots \\ \mathbf{C}_{sN} \end{pmatrix}$$

the scalars $m_i = \text{rank } \mathbf{B}_{si} = \dim \mathbf{u}_i$ and $r_i = \text{rank } \mathbf{C}_{si} = \dim \mathbf{y}_i$ are defined. Likewise it is assumed that the vectors $\mathbf{d}$ and $\mathbf{v}$ as generated by

$$\begin{aligned} \dot{\mathbf{x}}_d &= \mathbf{A}_d \mathbf{x}_d \qquad \mathbf{x}_d(0) = \mathbf{x}_{d0} \\ \mathbf{d} &= \mathbf{C}_d \mathbf{x}_d \end{aligned} \tag{5.2.2}$$

and

$$\begin{aligned} \dot{\mathbf{x}}_v &= \mathbf{A}_v \mathbf{x}_v \qquad \mathbf{x}_v(0) = \mathbf{x}_{v_0} \\ \mathbf{v} &= \mathbf{C}_v \mathbf{x}_v \end{aligned} \tag{5.2.3}$$

consist of the subsystem vectors $\mathbf{d}_i$ or $\mathbf{v}_i$, respectively.

Problem 5.2 (*Robust decentralized servomechanism problem*)

For a given plant (5.2.1) with partitioned input and output vectors and given signal generators (5.2.2) and (5.2.3) find a decentralized controller

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{F}_i \mathbf{x}_{ri} + \mathbf{G}_i \mathbf{y}_i + \mathbf{H}_i \mathbf{v}_i \qquad \mathbf{x}_{ri}(0) = \mathbf{x}_{ri0} \\ \mathbf{u}_i &= -\mathbf{K}_{xi} \mathbf{x}_{ri} - \mathbf{K}_{yi} \mathbf{y}_i + \mathbf{K}_{vi} \mathbf{v}_i \qquad (i = 1, \dots, N) \end{aligned} \tag{5.2.4}$$

such that the following requirements are satisfied:

(i) The closed-loop system (5.2.1) and (5.2.4) is stable.
(ii) The closed-loop system (5.2.1) and (5.2.4) ensures asymptotic regulation

$$\mathbf{y}_i(t) - \mathbf{v}_i(t) \xrightarrow[t \to \infty]{} \mathbf{0} \qquad (i = 1, \dots, N)$$

for all initial states $\mathbf{x}_0$, $\mathbf{x}_{ri0}$, $\mathbf{x}_{d0}$ and $\mathbf{x}_{v_0}$ and for all parameter variations of the plant (5.2.1) which do not destabilize the closed-loop system (5.2.1) and (5.2.4).

Existence of a Decentralized Servocontroller

The solution to Problem 5.2 is similar to the centralized case but has to satisfy the structural restrictions on the control law. Therefore, condition (iii) of Lemma 5.1 becomes more involved. An auxiliary system

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \lambda_j \mathbf{I} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathbf{B} \\ \mathbf{D} \end{pmatrix} \mathbf{u}$$

$$\begin{pmatrix} \bar{\mathbf{y}}_1 \\ \bar{\mathbf{y}}_2 \\ \vdots \\ \bar{\mathbf{y}}_N \end{pmatrix} = \left[\begin{array}{ccccc} \mathbf{C}_{s1} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{I}_{r1} & 0 & \dots & 0 \\ \hline \mathbf{C}_{s2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \mathbf{I}_{r2} & \dots & 0 \\ \hline \vdots & \vdots & \vdots & & \vdots \\ \mathbf{C}_{sN} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \mathbf{I}_{rN} \end{array}\right] \mathbf{x} \tag{5.2.5}$$

is defined for every $\lambda_j \in \Lambda$, where $\mathbf{I}_{ri}$ denotes an r_i-dimensional identity matrix. The output vectors $\bar{\mathbf{y}}_i$ have twice the dimension of the plant outputs $\mathbf{y}_i$. The output $\mathbf{u}$ is partitioned as in eqn (5.2.1).

Theorem 5.1

There exists a solution to the robust decentralized servomechanism problem if and only if the following conditions are satisfied:

(i) The plant (5.2.1) has no unstable decentralized fixed modes.
(ii) $m_i \geqslant r_i$ $(i = 1, \dots, N)$ hold.
(iii) The elements of Λ are not decentralized fixed modes of the q systems described by (5.2.5) for $j = 1, \dots, q$.

For $m_i = r_i$, condition (iii) has a similar form as in Lemma 5.1 since then it claims λ_i not to be a transmission zero of the plant, that is

$$\operatorname{rank}\begin{pmatrix} \mathbf{A} - \lambda_i \mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = n + r \quad \text{for all } \lambda_i \in \Lambda. \tag{5.2.6}$$

Example 5.1

To illustrate the use of Theorem 5.1 consider the simple system given in Example 4.1

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 2 & 3 \\ 1 & -2 & 4 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{5.2.7}$$

$$\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

As shown in Section 4.1 this system has no decentralized fixed modes. If it is assumed that the set Λ consists merely of a single element λ, then the auxiliary system (5.2.5) has the form

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & 2 & 3 & 0 & 0 \\ 1 & -2 & 4 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & 0 & \lambda \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} \bar{\mathbf{y}}_1 \\ \bar{\mathbf{y}}_2 \end{pmatrix} = \left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \mathbf{x}.$$

According to Theorem 4.1, λ is a decentralized fixed mode if and only if

$$\operatorname{rank} \left[\begin{array}{ccccc|c} -1-\lambda & 2 & 3 & 0 & 0 & 0 \\ 1 & -2-\lambda & 4 & 0 & 0 & 0 \\ 0 & 0 & -3-\lambda & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] < 5$$

or

$$\operatorname{rank} \left[\begin{array}{ccccc|c} -1-\lambda & 2 & 3 & 0 & 0 & 1 \\ 1 & -2-\lambda & 4 & 0 & 0 & 0 \\ 0 & 0 & -3-\lambda & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] < 5$$

holds where the lines indicate the partition of the matrices which corresponds to that in eqn (4.1.17). Since both matrices have one pair of identical rows, one of these rows can be deleted without changing the rank. Then the matrix is quadratic and simple determinant manipulations show that no λ with a positive real part is a decentralized fixed mode of the auxiliary system. Hence, the decentralized servomechanism problem has a solution. □

As the example shows, the application of Theorem 5.1 to high-dimensional systems leads to extensive manipulations with high-dimensional matrices. These matrices are even larger than those of the plant (5.2.1). Therefore, it is helpful that for specific structures of the plant Theorem 5.1 can be replaced by simpler tests, although not all of them are necessary *and* sufficient for the existence of the solution.

The servomechanism problem for interconnected systems

If the plant (5.2.1) consists of different subsystems it can be described by a model of the form (3.1.16)

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}_{ii}\mathbf{x}_i + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{B}_i\mathbf{u}_i + \mathbf{M}_i\mathbf{d} \\ \mathbf{y}_i &= \mathbf{C}_i\mathbf{x}_i \qquad\qquad\qquad\qquad + \mathbf{N}_i\mathbf{d} \end{aligned} \tag{5.2.8}$$

where $\mathbf{A}_{ij} = \mathbf{E}_i\mathbf{L}_{ij}\mathbf{C}_{zj}$ holds (cf. eqn (3.1.14)). For $\mathbf{A}_{ij} = 0$ $(i \neq j)$ eqn (5.2.8) describes the isolated subsystems $(i = 1, \ldots, N)$ for which the centralized servomechanism problem can be solved.

Theorem 5.2

If for each isolated subsystem (eqn (5.2.8) for $\mathbf{A}_{ij} = 0$ for $i \neq j$) there exists a solution to the centralized servomechanism problem with respect to the signal models (5.2.2) and (5.2.3) then there exists a solution to the decentralized servomechanism problem for almost all sufficiently small interaction matrices $\mathbf{L}_{ij}$.

This theorem does not give bounds for the elements of $\mathbf{L}_{ij}$ nor does it describe the exceptional matrices which do not belong to 'almost all' matrices. However, it shows that for a system whose isolated subsystems can locally attenuate its disturbances and ensure command tracking the overall servomechanism problem can, in general, be solved in a decentralized way if the interactions of the subsystems are not too strong.

Subsystems with input–output interconnections

For plants with specific properties ($\mathbf{E}_i = \mathbf{B}_i$, $\mathbf{C}_{zi} = \mathbf{C}_i$) the stabilizability conditions have been shown to be reducible to the corresponding conditions of centralized control of the isolated subsystems (cf. Section 4.1). The same thing happens with the solvability of Problem 5.2.

Theorem 5.3

If the plant (5.2.8) has the properties $\mathbf{E}_i = \mathbf{B}_i$ and $\mathbf{C}_{zi} = \mathbf{C}_i$ then the decentralized servomechanism problem has a solution if and only if there exist solutions for the centralized servomechanism problems for all isolated subsystems. That is,

(i) if all subsystems are completely controllable and observable,
(ii) if $m_i \geqslant r_i$ holds for all subsystems, and
(iii) if

$$\text{rank}\begin{pmatrix} \mathbf{A}_{ii} - \lambda_j \mathbf{I} & \mathbf{B}_i \\ \mathbf{C}_i & \mathbf{0} \end{pmatrix} = n_i + r_i \tag{5.2.9}$$

holds for all $\lambda_j \in \Lambda$ $(j = 1, \ldots, q)$.

Structure of the Servocontroller

The solution to Problem 5.2 is similar to that of Problem 5.1. Owing to the structural constraints on the control law the servocompensator and the stabilizing compensator consist of independent parts for each control station (Figure 5.2).

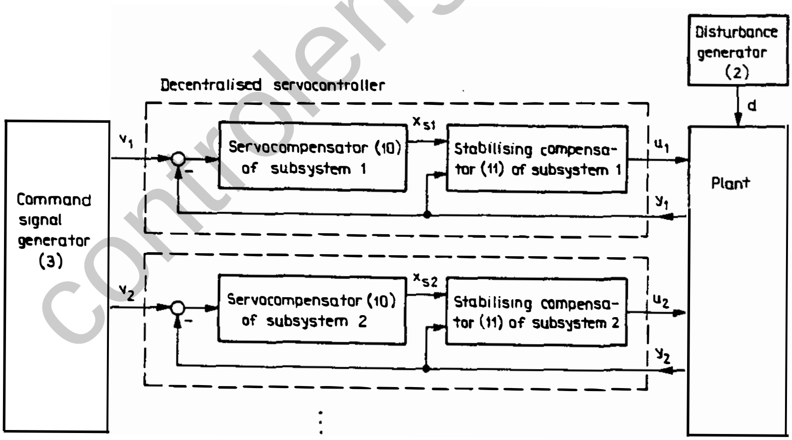


Figure 5.2 Decentralized servocontroller configuration

Theorem 5.4

The solution to the robust decentralized servomechanism problem consists of N control stations each of which includes a servocompensator

$$\dot{\mathbf{x}}_{si} = \mathbf{T}_i \text{ diag } \mathbf{A}_s \mathbf{T}_i^{-1} \mathbf{x}_{si} + \mathbf{T}_i \text{ diag } \mathbf{B}_{si}(\mathbf{y}_i - \mathbf{v}_i) \tag{5.2.10}$$

and a stabilizing compensator

$$\begin{aligned} \dot{\mathbf{x}}_{ci} &= \mathbf{F}_{ci}\mathbf{x}_{ci} + \mathbf{G}_{ci}\mathbf{y}_i + \mathbf{H}_{ci}\mathbf{x}_{si} \\ \mathbf{u}_i &= -\mathbf{K}_{xi}\mathbf{x}_{ci} - \mathbf{K}_{yi}\mathbf{y}_i + \mathbf{K}_{vi}\mathbf{x}_{si}. \end{aligned} \tag{5.2.11}$$

The parameters of the servocompensators can be determined from the parameters of the signal generators (5.2.2) and (5.2.3) as follows. $\mathbf{T}_i$ is an arbitrary non-singular real matrix, and diag $\mathbf{A}_s$ and diag $\mathbf{b}_s$ consist of r_i matrices $\mathbf{A}_s$ or vectors $\mathbf{b}_s$, respectively

$$\mathbf{A}_s = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -k_q & -k_{q-1} & -k_{q-2} & \dots & -k_1 \end{pmatrix} \qquad \mathbf{b}_s = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{5.2.12}$$

The parameters of the stabilizing compensators have to be chosen so that the closed-loop system (5.2.1), (5.2.10), (5.2.11) and (5.2.12) is stable.

The decentralized controller (5.2.4), which consists of the servocompensators (5.2.10) and the stabilizing compensators (5.2.11), is called the *decentralized servocontroller*.

Like the solution to the centralized tracking problem, the controller (5.2.10) and (5.2.11) ensures asymptotic regulation in spite of all variations of the plant parameters and the parameters of the stabilizing compensator that do not destabilize the closed-loop system (5.2.1), (5.2.10) and (5.2.11) and that do not violate the conditions stated in Theorem 5.1.

Theorems 5.1 and 5.2 have two important consequences. First, each control station of the decentralized servocontroller is known to consist of two parts, the first being the servocompensator and the second the stabilizing compensator. The first part can be determined as soon as the set Λ is known. If all servocompensators are connected to the plant, an unstable extended plant results (Figure 5.3). Whenever this extended plant is stabilized by some stabilizing compensators the overall closed-loop system is known to ensure asymptotic regulation. Hence, the tracking problem is induced in the stabilization problem for this extended plant. If there are even prescriptions on the dynamical I/O performance, the stabilization can be followed by a step for shaping the I/O behaviour. Hence, the design process has the following structure.

Algorithm 5.1 (*Basic design steps*)

Given: Plant model (5.2.1); design requirements (1)–(3) including the signal models (5.2.2) and (5.2.3) (cf. Section 1.2).

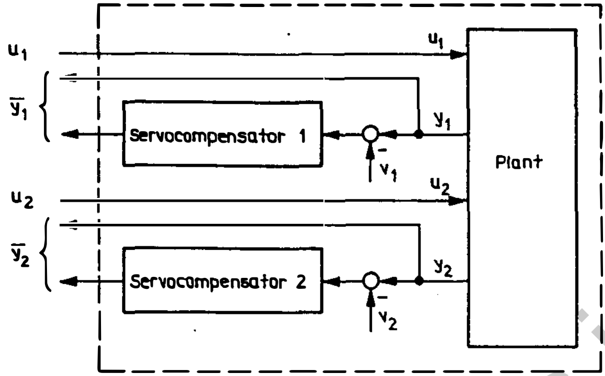


Figure 5.3 The extended plant

1. Extend the plant (5.2.1) by the decentralized servocompensators (5.2.10) and (5.2.12).
2. Stabilize the extended plant (5.2.1), (5.2.10) and (5.2.12) by means of decentralized stabilizing compensators (5.2.11) in order to satisfy the design requirements (1) and (2).
3. Improve the I/O behaviour of the closed-loop system according to the design requirements (3) by means of additional decentralized compensators (5.2.11) in parallel with the stabilizing compensator or by changing the parameters of the stabilizing compensator (5.2.11).

Result: Decentralized servocontroller (5.2.10)–(5.2.12), for which the closed-loop system (5.2.1), (5.2.10)–(5.2.12) satisfies the design requirements (1)–(3).

This design scheme has two consequences. First, for step 1 the solution is completely described by Theorem 5.4. The decentralized servocompensators can be determined as soon as the disturbance and command signal models (5.2.2) and (5.2.3) are known. The main problem consists of the determination of stabilizing compensators. Therefore, the following chapters of the book can be restricted to the problems of stabilizing and shaping the I/O behaviour of the plant or the extended plant.

Second, the extended plant is unstable due to the unstable parts of the servocompensators and, additionally, due to the possibly unstable parts of the plant. Therefore, the stabilization problem is not trivial. The conditions of Theorem 5.1 ensure that the extended plant has no decentralized fixed modes where condition (i) claims the absence of fixed modes for the plant and condition (ii) for the series connection of the plant and the servocompensator. The problem remains how to find parameters for the N stabilizing compensators such that the overall closed-loop system is stable. The methods described in Chapters 6–9 are generally applicable, but for PI controllers simpler methods will be developed in Chapter 11.

Sequential stability

A practically important extension of the assertions of Theorems 5.1 and 5.2 can be made if the plant (5.2.1) is stable. Then the control stations of the decentralized servocontroller (5.2.10)–(5.2.12) can be designed and applied one after the other in such a way that the stability of the closed-loop system remains unstable after all steps.

Theorem 5.5

If the plant (5.2.1) is stable and there exists a solution to the decentralized servomechanism problem then there is a sequence in which appropriately chosen control stations (5.2.10) and (5.2.11) can be applied one after the other while ensuring the stability of the resulting closed-loop system.

That is, the unstable parts of the servocompensator of the ith decentralized control station can be locally stabilized by the ith stabilizing compensator (5.2.11). This is a practically important result which will be further exploited in the tuning rules for decentralized PI controllers in Chapter 11.

5.3 SERVOCONTROLLERS FOR POLYNOMIAL COMMAND AND DISTURBANCE SIGNALS

The results of Section 5.2 can be specified for polynomial command

$$v(t) = k_0 + k_1 t + k_2 t^2 + \cdots + k_q t^q \tag{5.3.1}$$

and similar disturbance signals which are produced by signal generators

(5.2.3) of the form

$$\dot{\mathbf{x}}_v = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \mathbf{x}_v \qquad \mathbf{x}_v(0) = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{q0} \\ x_{q+1,0} \end{pmatrix} \tag{5.3.2}$$

$$v = (1 \;\; 0 \;\; \dots \;\; 0)x_v$$

where k_i in eqn (5.3.1) depends upon the initial values x_{i0} in eqn (5.3.2). In particular, the practically important step signal $v(t) = k_0$ is generated by

$$\begin{aligned} \dot{\mathbf{x}}_v &= 0 \qquad \mathbf{x}_v(0) = k_0 \\ v &= x_v \end{aligned} \tag{5.3.3}$$

where $\Lambda = \{0\}$ and $q = 1$ hold (cf. eqn (5.1.7)). According to Theorem 5.1, $\Lambda = 0$ must not be a decentralized fixed mode of the auxiliary system (5.2.5). For $m_i = r_i$ the corresponding condition (5.2.6) reads as

$$\operatorname{rank} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = n + r \tag{5.3.4}$$

which for stable systems can be reformulated as

$$\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det \mathbf{A} \, \det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}) = \det \mathbf{K}_s \neq 0 \tag{5.3.5}$$

where

$$\mathbf{K}_s = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B} \tag{5.3.6}$$

denotes the static transmission matrix of the plant (5.2.1). Theorems 5.1 and 5.4 yield the following result.

Corollary 5.1

For the plant (5.2.1) with $m_i = r_i$ there exists a decentralized controller (5.2.4) that ensures asymptotic regulation for step disturbances and step commands (5.3.3) if and only if the following conditions are satisfied:

(i) The plant has no unstable decentralized fixed modes.
(ii) The condition (5.3.4) is satisfied.

Asymptotic regulation is ensured if all control stations consist of the servocompensator

$$\dot{\mathbf{x}}_{si} = \mathbf{y}_i - \mathbf{v}_i \tag{5.3.7}$$

and a stabilizing compensator (5.2.11), whose parameters are chosen so as to ensure closed-loop stability.

The controller (5.2.11) and (5.3.7) has PI characteristics. This becomes clear if the stabilizing compensator (5.2.11) represents merely a static feedback

$$\mathbf{u}_i = -\mathbf{K}_{yi}\mathbf{y}_i - \mathbf{K}_{xi}\mathbf{x}_{si}$$

so that each control station has the form

$$\begin{aligned} \dot{\mathbf{x}}_{si} &= \mathbf{y}_i - \mathbf{v}_i \\ \mathbf{u}_i &= -\mathbf{K}_{yi}\mathbf{y}_i - \mathbf{K}_{xi}\mathbf{x}_{si}. \end{aligned} \tag{5.3.8}$$

If polynomial signals with $q > 1$ are used the servocompensator has q integrators for each of the $r_i = \dim \mathbf{y}_i$ output signals of subsystem i.

Methods for determining the parameters of the PI controller (5.3.8) will be presented in Chapter 11. It will also be shown there under what conditions a static stabilizing feedback suffices to make the extended plant stable. For the purpose of improving the dynamical properties of the closed-loop system, the control stations (5.3.8) are often extended by an additional feedforward part so that instead of (5.3.8) the controller

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{y}_i - \mathbf{v}_i \\ \mathbf{u}_i &= -\mathbf{K}_{Pi}(\mathbf{y}_i - \mathbf{v}_i) - \mathbf{K}_{Ii}\mathbf{x}_{ri} \end{aligned} \tag{5.3.9}$$

is used.

BIBLIOGRAPHICAL NOTES

The solution of the centralized servomechanism problem was investigated by Davison (1976b). It has been included in modern textbooks such as, for example, that by Patel and Munro (1982).

Davison also investigated thoroughly the robust decentralized servomechanism problem. He presented (Davison 1976a) the solution (Theorems 5.1 and 5.2) to Problem 5.2 in a slightly more general form, where the measurable outputs $\mathbf{y}_i^{\mathrm{m}}$ were distinguished from the outputs $\mathbf{y}_i$ to be controlled. The equivalence of condition (iii) in Theorem 5.1 with eqn (5.2.6) for systems with like numbers of inputs and outputs was published in 1978. Theorems 5.3 and 5.5 are due to Davison (1979) and Davison and Gesing (1979), respectively. Davison and Özgüner (1982) showed that for the sequential design in each design step only a local model is necessary. This is a model which describes that part of the plant and the preceding control stations which are controllable and observable from the channel $(\mathbf{u}_i, \mathbf{y}_i)$ under consideration. The closed-loop system eigenvalues are elements of a given region $\mathscr{C}$ of the complex plane, if each control station is chosen so as to move all controllable and observable eigenvalues into $\mathscr{C}$.

Several further extensions have been made. For example, Davison (1977a) considered the servomechanism problem in which some outputs need not be asymptotically regulated. The corresponding control stations can be used for the stabilization problem but they do not include a servocompensator. Therefore, conditions (ii) and (iii) can be weakened. Vaz and Davison (1989) considered the servomechanism problem under the circumstances that different signal generators can be assigned to the subsystems and that certain elements of the interconnection matrix $\mathbf{L}$ are fixed to zero and, thus, cannot be altered by parameter variations within the plant. They developed specific existence conditions and simplifications of the controller structure.

6

Decentralized Stabilization and Pole Assignment

The principle of designing feedback controllers by assigning the eigenvalues of the closed-loop system certain values is known from multivariable control theory. If the task is to stabilize a given unstable system then the eigenvalues should be simply moved to some position within the left-half complex plane. If additional requirements exist on the I/O behaviour then these requirements have to be formulated as prescriptions of the values of the closed-loop system eigenvalues, and the task is to find a feedback controller which changes the eigenvalues accordingly.

This design principle is now extended for decentralized control. Section 6.1 presents a control scheme in which $N-1$ static control stations are used to make the overall system controllable and observable via the remaining channel $(\mathbf{u}_k, \mathbf{y}_k)$. The remaining control station is chosen so as to place all closed-loop system eigenvalues in prescribed positions. Section 6.2 briefly describes the extension of dynamic compensation to decentralized control. In Section 6.3, a method is presented in which a centralized controller is replaced by an appropriate decentralized feedback.

6.1 STABILIZATION AND POLE ASSIGNMENT THROUGH A SPECIFIC CHANNEL

The plant is described by the I/O-oriented model

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i \\ \mathbf{y}_i &= \mathbf{C}_{si}\mathbf{x} \qquad (i = 1, \ldots, N). \end{aligned} \tag{6.1.1}$$

The aim is to stabilize this system and to give the closed-loop system a prescribed set σ_0 of eigenvalues. In the method to be presented here, the overall decentralized controller should consist of $N-1$ static output feedbacks

$$\mathbf{u}_i = -\mathbf{K}_{yi}\mathbf{y}_i \qquad (i = 1, \ldots, N;\ i \neq k) \tag{6.1.2}$$

and a dynamic control station

$$\begin{aligned} \dot{\mathbf{x}}_{rk} &= \mathbf{F}_k\mathbf{x}_{rk} + \mathbf{G}_k\mathbf{y}_k \\ \mathbf{u}_k &= -\mathbf{K}_{xk}\mathbf{x}_{rk} - \mathbf{K}_{yk}\mathbf{y}_k. \end{aligned} \tag{6.1.3}$$

The static control stations have to be chosen so as to make the closed-loop system (6.1.1) and (6.1.2)

$$\begin{aligned} \dot{\mathbf{x}} &= \left(\mathbf{A} - \sum_{\substack{i=1 \\ i\neq k}}^{N} \mathbf{B}_{si}\mathbf{K}_{yi}\mathbf{C}_{si}\right)\mathbf{x} + \mathbf{B}_{sk}\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}_{sk}\mathbf{x} \end{aligned} \tag{6.1.4}$$

completely controllable via $\mathbf{u}_k$ and completely observable via $\mathbf{y}_k$. Then the remaining control station (6.1.3) is used in order to stabilize the system (6.1.4) or move the eigenvalues to prescribed positions within the complex plane, respectively. This second step is a problem in the centralized control of the system (6.1.4).

The motivation for this design method becomes clear from the way in which the existence condition of decentralized fixed modes (Theorem 4.1) was derived in Section 4.1. It was explained there that a fixed eigenvalue λ is controllable but not observable from the channels $(\mathbf{u}_i, \mathbf{y}_i)$, $i \in \mathcal{H}$ and cannot be made so by appropriately choosing a static decentralized feedback

$$\mathbf{u}_D = -\mathbf{K}_D\mathbf{y}_D.$$

In addition it is observable but not controllable via the channels $i \in \mathcal{D}$ and cannot be made so by decentralized controllers at the channels $i \in \mathcal{H}$.

If the system has no decentralized fixed modes then the converse is true. That is, for every eigenvalue λ there exists a partition $\mathcal{D}$, $\mathcal{H}$ of the index set so that λ can be made controllable and observable from a channel $i \in \mathcal{H}$ or $i \in \mathcal{D}$ by appropriately choosing decentralized controllers at the channels $i \in \mathcal{D}$ or $i \in \mathcal{H}$, respectively. The development of Theorem 4.1, in particular the relation between eqns (4.1.16) and (4.1.17), shows that the partition $\mathcal{H}$, $\mathcal{D}$ depends on the eigenvalue λ under consideration. Therefore, conditions have to be derived under which the partition is independent of λ and, moreover, all eigenvalues can be made controllable and observable by a single channel. Conditions (6.1.5) and (6.1.6) below, which are more restrictive than the converse of (4.1.17), ensure that there are no uncontrollable and unobservable modes in *any* complementary system (4.1.19) and, thus, the aim of the first design step can be reached for an arbitrary channel k.

Theorem 6.1

For an arbitrary index k there exist feedback matrices $\mathbf{K}_{yi}$ $(i \neq k)$ for the control stations (6.1.2) such that the system (6.1.4) is completely controllable and completely observable if and only if at least one of the following conditions is satisfied for all disjoint partitions $\mathscr{H}$, $\mathscr{D}$ of the index set $\mathscr{I} = \{1, 2, \ldots, N\}$:

$$\operatorname{rank}(\mathbf{A} - \lambda \mathbf{I} \quad \mathbf{B}_{\mathrm{D}}) = n \quad \text{for all } \lambda \in \sigma[\mathbf{A}] \tag{6.1.5}$$

$$\operatorname{rank}\begin{pmatrix} \mathbf{A} - \lambda I \\ \mathbf{C}_{\mathrm{H}} \end{pmatrix} = n \quad \text{for all } \lambda \in \sigma[\mathbf{A}] \tag{6.1.6}$$

(For the definition of $\mathscr{H}$, $\mathscr{D}$ see Lemma 4.2.) This theorem leads to the following design procedure.

Algorithm 6.1

Given: Plant (6.1.1) which satisfies the conditions stated in Theorem 6.1; set σ_0.

1. Choose a channel number k. Determine a decentralized feedback controller (6.1.2) such that the system (6.1.1) and (6.1.2) is completely controllable and completely observable via $(\mathbf{u}_k, \mathbf{y}_k)$.
2. Design a centralized controller (6.1.3) such that the closed-loop system (6.1.1)–(6.1.3) has the prescribed set σ_0 of eigenvalues.

Result: Decentralized controller for which the closed-loop system has a prescribed set σ_0 of eigenvalues.

The second step can be carried out by means of well-known methods from multivariable feedback control.

The design method discussed above leads to a decentralized controller of dynamical order n if the second design step is accomplished by designing a state feedback controller and an observer. That is, the controller has the same dynamical order as the plant. In practical situations the order may be lower if, for example, a reduced-order observer is used. None the less, the dynamical order is very high if the plant is a large-scale system.

Example 6.1

Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\mathbf{y} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x} \tag{6.1.7}$$

which is controllable and observable via the whole vectors $\mathbf{u}$ and $\mathbf{y}$ but not via a single channel $(\mathbf{u}_i, \mathbf{y}_i)$. Condition (6.1.5) is satisfied for both complementary systems

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u_1$$
$$y_2 = (1 \ \ 0 \ \ 0)\mathbf{x}$$

and

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2$$
$$y_1 = (0 \ \ 1 \ \ 0)\mathbf{x}.$$

Hence, the design method can be applied. This will be illustrated for $k = 1$. The eigenvalue $\lambda = 0$ is not observable through y_1 but can be made observable by means of an appropriate control station

$$u_2 = -k_{y2} y_2 \tag{6.1.8}$$

(step 1 of Algorithm 6.1). The system (6.1.7) and (6.1.8)

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 4 \\ -k_{y2} & 1 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u_1$$
$$y_1 = (0 \ \ 1 \ \ 0)\mathbf{x}$$

is completely controllable if $k_{y2} \neq 0.3$ and $k_{y2} \neq -30.3$ hold and completely observable for $k_{y2} \neq 0$. After such a controller parameter k_{y2} has been chosen, the stabilization or pole assignment problem for the system (6.1.7) and (6.1.8) can be solved in step 2 of Algorithm 6.1, for example by using the dyadic state feedback in connection with an observer (cf. Section 2.4) which together form a dynamic control station for channel

$k = 1$. For this example with $k_{y2} = 1$, a static output feedback $u_1 = -y_1$ at channel (u_1, y_1) suffices to assign the values -4.91 and $-0.54 \pm 1.53i$ to the closed-loop eigenvalue. □

Example 6.2

In order to show that the conditions of Theorem 6.1 determine more than the absence of decentralized fixed modes consider the plant from Example 4.2 with $a = 2$ and $b = -2$

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x}.$$

This system has no decentralized fixed modes. None the less, since the complementary system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}u_1$$

$$y_2 = (0 \;\; 0 \;\; 1)\mathbf{x}$$

is neither completely controllable nor completely observable, the system cannot be made completely controllable and observable via a single channel. For example, there is no scalar control station 2 such that all eigenvalues of the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 3 - k_{y2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}u_1$$

$$y_1 = (1 \;\; 2 \;\; 0)\mathbf{x}$$

are controllable and observable. Therefore, Algorithm 6.1 cannot be applied. However, due to the absence of decentralized fixed modes the plant can be stabilized by decentralized feedback. For this simple example, which has a hierarchical structure (cf. Section 3.2), control station 2 can be used in order to move the plant eigenvalue $\lambda = 3$, and the remaining two eigenvalues can be changed arbitrarily by control station 1. □

An advantage of this design method in relation to those which will be presented in Sections 6.2 and 6.3 is the straightforward way in which all decentralized control stations can be found. In particular, the second design step can be solved by means of methods for pole assignment by multivariable feedback controllers.

The main practical difficulties of this design method result from the concentration of the control efforts and design freedom at a single control station. If some modes are weakly observable or controllable from the kth channel, impractically large controller gains are required. Moreover, because of the 'unsymmetry' of the control law, disturbances have to propagate through the overall system until they affect the kth channel and are compensated there. Therefore, this way of designing decentralized controllers provides a first impression of sequential design and gives an interesting insight into the possibility of assigning the eigenvalues by decentralized controllers. However, for practical applications this method will usually not be preferred to other design methods.

6.2 DECENTRALIZED DYNAMIC COMPENSATION

In centralized control, dynamic compensation was developed as an alternative way of realizing state feedback controllers by means of observers. It follows the philosophy of extending the freedom inherent in output feedback control by introducing n_r integrators in the given plant. That is, instead of the plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{6.2.1}$$

the extended plant

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \bar{\mathbf{x}} + \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{y}} &= \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \bar{\mathbf{x}} \end{aligned} \tag{6.2.2}$$

is used to design an output feedback controller

$$\bar{\mathbf{u}} = -\bar{\mathbf{K}}\bar{\mathbf{y}} \tag{6.2.3}$$

such that the closed-loop system (6.2.2) and (6.2.3) has $n + n_r$ prescribed eigenvalues. Whereas, in general, no output feedback $\mathbf{u} = -\mathbf{K}\mathbf{y}$ can be found for the plant (6.2.1) to move definitely all closed-loop eigenvalues, an extended output feedback (6.2.3) can be found if the number n_r of integrators is large enough.

An upper bound for n_r can be found from the controllability index

n_c and observability index n_o, which are defined as the smallest integers for which

$$\text{rank}(\mathbf{B}\ \ \mathbf{AB}\ \ \ldots\ \ \mathbf{A}^{n_c-1}\mathbf{B}) = n$$

or

$$\text{rank}(\mathbf{C}'\ \ \mathbf{A}'\mathbf{C}'\ \ \ldots\ \ \mathbf{A}^{n_o-1}\mathbf{C}') = n$$

holds, respectively.

Lemma 6.1

Consider a linear system (6.2.1) which is completely controllable and completely observable. For

$$n_r = \min(n_c, n_o) \tag{6.2.4}$$

there exists an output feedback (6.2.3) for the extended plant (6.2.2) such that the closed-loop system (6.2.2) and (6.2.3) has $n + n_r$ arbitrarily prescribed eigenvalues.

Note that the closed-loop system (6.2.2) and (6.2.3) with

$$\bar{\mathbf{K}} = \begin{pmatrix} -\mathbf{K}_y & -\mathbf{K}_x \\ \mathbf{G} & \mathbf{F} \end{pmatrix} \tag{6.2.5}$$

is equivalent to the closed loop consisting of the original plant (6.2.1) and the controller

$$\begin{aligned} \dot{\mathbf{x}}_r &= \mathbf{F}\mathbf{x}_r + \mathbf{G}\mathbf{y} \\ \mathbf{u} &= -\mathbf{K}_x\mathbf{x}_r - \mathbf{K}_y\mathbf{y}. \end{aligned} \tag{6.2.6}$$

In the following, this result will be extended to decentralized control. As a result of the structural constraints the control law (6.2.6) is split into independent controllers

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{F}_i\mathbf{x}_{ri} + \mathbf{G}_i\mathbf{y}_i \\ \mathbf{u}_i &= -\mathbf{K}_{xi}\mathbf{x}_{ri} - \mathbf{K}_{yi}\mathbf{y}_i. \end{aligned} \tag{6.2.7}$$

That is, all four parts of $\bar{\mathbf{K}}$ in eqn (6.2.5) are restricted to be block-diagonal matrices

$$\bar{\mathbf{K}} = \begin{pmatrix} \text{diag}\ -\mathbf{K}_{yi} & \text{diag}\ -\mathbf{K}_{xi} \\ \text{diag}\ \mathbf{G}_i & \text{diag}\ \mathbf{F}_i \end{pmatrix} \tag{6.2.8}$$

where the matrices $\mathbf{K}_{yi}$, $\mathbf{K}_{xi}$, $\mathbf{G}_i$ and $\mathbf{F}_i$ have dimensions $m_i \times r_i$, $m_i \times n_{ri}$, $n_{ri} \times r_i$, and $n_{ri} \times n_{ri}$. The following theorem states that Lemma 6.1 can be straightforwardly extended to decentralized control.

Theorem 6.2

Consider a linear controllable and observable system (6.2.1) under decentralized feedback (6.2.7) with $n_{ri} \geqslant 0$:

$$\sum_{i=1}^{N} n_{ri} = n_r = \min(n_c, n_o). \tag{6.2.9}$$

There exist feedback gain matrices $\mathbf{K}_{yi}$, $\mathbf{K}_{xi}$, $\mathbf{F}_i$ and $\mathbf{G}_i$ such that the closed-loop system (6.2.1) and (6.2.7) has $n + n_r$ arbitrarily prescribed eigenvalues.

Algorithm 6.2

Given: Plant (6.2.1); set σ_0.

1. Determine n_r from eqn (6.2.4).
2. Extend the plant (6.2.1) as described in eqn (6.2.2).
3. Determine the controller matrix $\bar{\mathbf{K}}$, which satisfies the structural constraints given in eqn (6.2.8), such that the closed-loop system eigenvalues belong to σ_0.

Result: Decentralized controller for which the closed-loop system has the prescribed set σ_0 of eigenvalues.

Theorem 6.2 states only an existence condition for a set of N decentralized dynamic compensators which all together solve the pole placement problem. Constructive procedures for step 3 in Algorithm 6.2 are rare. Most of them use a sequential method in which one control station is designed after another in order to move a subset of the closed-loop system spectrum. Since all of them are based on lengthy manipulations with the overall model, and sequential design procedures will be considered later in Chapters 10 and 11, the design of dynamic compensators should be adequately illustrated here by a simple example.

Example 6.3

Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$$

for which $n_r = 1$ holds. If a decentralized controller with $n_{r1} = 1$ and $n_{r2} = 0$ is used then the matrix $\bar{\mathbf{K}}$ has the structure

$$\bar{\mathbf{K}} = \left(\begin{array}{cc|c} -k_{y1} & 0 & k_{x1} \\ 0 & -k_{y2} & 0 \\ \hline g_1 & 0 & f_1 \end{array}\right).$$

The system matrix of the closed-loop system (6.2.1), (6.2.7) and (6.2.8) for the given parameters is

$$\left(\begin{array}{cc|c} 0 & 2-k_{y1} & k_{x1} \\ 1-k_{y2} & 0 & 0 \\ \hline 0 & g_1 & f_1 \end{array}\right).$$

With a static decentralized controller ($k_{x1} = g_1 = f_1 = 0$ and, hence, $n_{r1} = 0$) the closed-loop system cannot be stabilized since the closed-loop system characteristic polynomial

$$\lambda^2 - (1 - k_{y2})(2 - k_{y1}) = 0$$

yields at least one unstable eigenvalue for arbitrary controller parameters. For the dynamical controller ($n_{r1} = 1$) the characteristic polynomial of the closed-loop system is

$$\lambda^3 - f_1\lambda^2 - (1 - k_{y2})(2 - k_{y1})\lambda - (1 - k_{y2})k_x g_1 \\ + (1 - k_{y2})(2 - k_{y1})f_1 = 0.$$

The eigenvalues can be placed arbitrarily. The closed-loop spectrum $\sigma_0 = \{-1, -2, -3\}$ is received for the controller parameters $f_1 = -6$, $g_1 = 1$, $k_x = -10$, $k_{y1} = 0.167$, $k_{y2} = 7$. □

6.3 REPLACEMENT OF A CENTRALIZED STATE FEEDBACK BY A DECENTRALIZED OUTPUT FEEDBACK

In the third method of decentralized stabilization and pole placement, which will be explained now, it is assumed that for the plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i \\ \mathbf{y}_i = \mathbf{C}_{si}\mathbf{x} \qquad (i = 1, \ldots, N) \tag{6.3.1}$$

a centralized state feedback

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{6.3.2}$$

has already been designed. This could be done by methods for centralized state feedback which uses the system (6.3.1) as the centralized plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

with

$$\mathbf{B} = (\mathbf{B}_{s1} \;\; \mathbf{B}_{s2} \;\; \dots \;\; \mathbf{B}_{sN})$$
$$\mathbf{C} = (\mathbf{C}'_{s1} \;\; \mathbf{C}'_{s2} \;\; \dots \;\; \mathbf{C}'_{sN})'$$

and assigns the eigenvalues of the closed-loop system matrix

$$\bar{\mathbf{A}}^{c} = \mathbf{A} - \mathbf{B}\mathbf{K} \tag{6.3.3}$$

values of a prescribed set σ_0. The aim is to replace the centralized controller (6.3.2) by a decentralized controller

$$\mathbf{u} = -\operatorname{diag} \mathbf{K}_{yi}\mathbf{y} \tag{6.3.4}$$

which locates the eigenvalues of the system matrix

$$\bar{\mathbf{A}}^{d} = \mathbf{A} - \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{K}_{yi}\mathbf{C}_{si} \tag{6.3.5}$$

of the decentralized system (6.3.1) and (6.3.4) as near as possible to the eigenvalues of $\bar{\mathbf{A}}^{c}$.

In order to compare the eigenvalues of the centralized and the decentralized system, the eigenvector of $\bar{\mathbf{A}}^{c}$ belonging to the eigenvalue $\lambda_j[\bar{\mathbf{A}}^{c}]$ is denoted by $\mathbf{u}_j$, that is

$$\bar{\mathbf{A}}^{c}\mathbf{u}_j = \lambda_j\mathbf{u}_j \tag{6.3.6}$$

holds. The centralized feedback matrix $\mathbf{K}$ is partitioned according to the input vector $\mathbf{u}$

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \\ \vdots \\ \mathbf{K}_N \end{pmatrix} \tag{6.3.7}$$

so that eqn (6.3.2) is equivalent to

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}.$$

Lemma 6.2

Consider the eigenvalue $\lambda_j[\bar{\mathbf{A}}^{c}]$ and denote the corresponding eigenvector by $\mathbf{u}_j$. If the decentralized controller satisfies the relation

$$\mathbf{K}_{yi}\mathbf{C}_{si}\mathbf{u}_j = \mathbf{K}_i\mathbf{u}_j \qquad (i = 1, \ldots, N) \tag{6.3.8}$$

then λ_j is an eigenvalue of the matrix $\bar{\mathbf{A}}^{\mathrm{d}}$ as well.

Proof

Because of eqns (6.3.6) and (6.3.8) the equation

$$\bar{\mathbf{A}}^{\mathrm{d}}\mathbf{u}_j = \left(\mathbf{A} - \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{K}_{yi}\mathbf{C}_{si}\right)\mathbf{u}_j = \left(\mathbf{A} - \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{K}_i\right)\mathbf{u}_j$$
$$= \bar{\mathbf{A}}^{\mathrm{c}}\mathbf{u}_j = \lambda_j\mathbf{u}_j$$

holds which proves that $\lambda_j[\bar{\mathbf{A}}^{\mathrm{c}}]$ is an eigenvalue of $\bar{\mathbf{A}}^{\mathrm{d}}$. □

Note that each eigenvalue to be retained imposes a restriction (6.3.8) on the choice of the matrices $\mathbf{K}_{yi}$ of all control stations. If q eigenvalues λ_j $(j = 1, \ldots, q)$ should be retained in the decentralized system, eqn (6.3.8) must be satisfied for all of them. By using the matrix

$$\mathbf{U}_q = (\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \ldots \;\; \mathbf{u}_q) \tag{6.3.9}$$

the conditions (6.3.8) for $j = 1, \ldots, q$ can be represented as the single condition

$$\mathbf{K}_{yi}\mathbf{C}_{si}\mathbf{U}_q = \mathbf{K}_i\mathbf{U}_q \qquad (i = 1, \ldots, N). \tag{6.3.10}$$

Corollary 6.1

Let $\{\lambda_1, \ldots, \lambda_q\}$ be a subset of the spectrum $\sigma[\bar{\mathbf{A}}^{\mathrm{c}}]$. It is a subset of $\sigma[\bar{\mathbf{A}}^{\mathrm{d}}]$ as well if eqn (6.3.10) is satisfied for $i = 1, \ldots, N$.

Since, in general, the centralized controller yields

$$\mathrm{rank}(\mathbf{K}_i\mathbf{U}_q) = \min(m_i, q)$$

for the right-hand side of eqn (6.3.10), whereas

$$\mathrm{rank}(\mathbf{K}_{yi}\mathbf{C}_{si}\mathbf{U}_q) \leqslant \min(m_i, r_i, q)$$

holds for the left-hand side, the sufficient condition (6.3.10) can be satisfied by appropriately choosing the gain matrix $\mathbf{K}_{yi}$ only if

$$q \leqslant \bar{q} = \min(m_1, r_1, m_2, r_2, \ldots, m_N, r_N)$$

holds. Under this condition, the controller matrices $\mathbf{K}_{yi}$ which satisfy eqn

(6.3.10) can be explicitly represented by

$$\mathbf{K}_{yi} = \mathbf{K}_i\mathbf{U}_q(\mathbf{C}_{si}\mathbf{U}_q)^+ \tag{6.3.11}$$

where $(.)^+$ denotes the (right) pseudoinverse. The right pseudoinverse of some matrix $\mathbf{T}$ is defined by $\mathbf{T}^+\mathbf{T} = \mathbf{I}$. For (r_i, q) matrices $\mathbf{T}$ with

$$\text{rank } \mathbf{T} = q \leqslant r_i,$$

$\mathbf{T}^+$ can be determined according to

$$\mathbf{T}^+ = (\mathbf{T}'\mathbf{T})^{+1}\mathbf{T}'$$

(cf. Theorem A1.8). That is, eqn (6.3.11) is identical to

$$\mathbf{K}_{yi} = \mathbf{K}_i\mathbf{U}_q(\mathbf{U}_q'\mathbf{C}_{si}'\mathbf{C}_{si}\mathbf{U}_q)^{-1}\mathbf{U}_q'\mathbf{C}_{si}'.$$

Since $\bar{q}$ may be very small there is a need to generalize condition (6.3.10) for a large number q of eigenvalues to be retained. Since for large q there is no solution to eqn (6.3.10), the differences $\mathbf{K}_{yi}\mathbf{C}_{si}\mathbf{u}_j - \mathbf{K}_i\mathbf{u}_j$ are weighted by factors w_{ij}, and the controller parameters are to be chosen so as to make the sum of these differences minimal

$$\min_{\mathbf{K}_{yi}} \| (\mathbf{K}_{yi}\mathbf{C}_{si}\mathbf{U}_q - \mathbf{K}_i\mathbf{U}_q)\mathbf{W} \|. \tag{6.3.12}$$

In eqn (6.3.12), $\| . \|$ denotes a matrix norm and $\mathbf{W}$ is a matrix with the weighting factors w_{ij}: $\mathbf{W} = (w_{ij})$. The solution to the optimization problem (6.3.12) is given by

$$\mathbf{K}_{yi} = \mathbf{K}_i\mathbf{U}_q\mathbf{W}(\mathbf{C}_{si}\mathbf{U}_q\mathbf{W})^+. \tag{6.3.13}$$

The decentralized closed-loop system (6.3.1), (6.3.4) and (6.3.12) has a spectrum which approximates the prescribed set σ_0 of the eigenvalues. How large the differences between $\lambda_j[\bar{\mathbf{A}}^c] \in \sigma_0$ and $\lambda_j[\bar{\mathbf{A}}^d]$ are depends on the relations between the plant properties, the structural constraints of the control law and the design requirements σ_0. If, on the one hand, all those entries of the centralized controller matrix $\mathbf{K}$ that are fixed to zero in the decentralized controller are negligible, then the differences between corresponding eigenvalues will be very small. On the other hand, if the structural constraints on the control law refer to large entries of $\mathbf{K}$, then the differences will be large. However, the formulation of the design task as an optimization problem (6.3.12) enables the control engineer to influence the way in which all the eigenvalues of $\bar{\mathbf{A}}^c$ are approximated by those of $\bar{\mathbf{A}}^d$. It can be proved that for $\mathbf{W} = \text{diag } w_{ii}$ an increase of the weighting factor w_{kk}, while leaving all other factors w_{ii} $(i \neq k)$ constant, yields a better approximation of $\lambda_k[\bar{\mathbf{A}}^c]$ by $\lambda_k[\bar{\mathbf{A}}^d]$. This possibility of influencing the accuracy of the approximation in a systematic way can be used in a trial-and-error procedure for determining a decentralized controller (6.3.4) which suitably replaces the centralized feedback (6.3.2).

Algorithm 6.3

Given: Plant (6.3.1); set σ_0.

1. Determine a centralized state feedback (6.3.2) for which the closed loop (6.3.1) and (6.3.2) has the prescribed eigenvalue set σ_0.
2. Choose a weighting matrix **W** and determine the controller matrices (6.3.13).
3. Determine the spectrum $\sigma_d = \{\lambda_i[\bar{\mathbf{A}}^d],\ i = 1, \ldots, n\}$. If σ_d is a reasonable approximation of σ_0, stop; otherwise continue with step 2.

Result: Decentralized controller (6.3.4) which assigns the closed-loop system eigenvalues approximately the values of σ_0.

Clearly, the algorithm can be modified in step 1, where an arbitrary method for centralized control can be applied, which is not necessarily based on the pole assignment principle. Then the set σ_0 is not prescribed but obtained after step 1 as the set of the eigenvalues of the closed-loop centralized system (6.3.1) and (6.3.2). In the following example, the **LQ** design will be used in step 1 in order to get a reasonable centralized solution. Then the algorithm proceeds as above.

Example 6.4 (*Load–frequency control of a multiarea power system*)

The design algorithm explained above will be illustrated by designing a controller for the power system whose model was discussed in Example 3.5. The feedback controller has to ensure constant net frequency ($f = 0$) and constant tie line power ($p_{ti} = 0$) in spite of changing power demands p_{li}. Control inputs are the set points p_{si} of the internal controllers of the power stations (Figure 3.8).

The controller must be decentralized since the measurement signals f and p_{ti} and the input signals p_{si} are distributed over a large geographical area and since the plant is subjected to structural perturbations, which include the connection and disconnection of power stations or areas during normal operation.

The structure of the overall plant is shown in Figure 6.1(a). It can be described by an unstructured model of the form (2.1.4)

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} + \mathbf{Md}\\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} + \mathbf{Nd}\end{aligned} \tag{6.3.14}$$

with $\mathbf{u} = (u_1\ u_2\ u_3)'$, $\mathbf{y} = (\mathbf{y}_1'\ \mathbf{y}_2'\ \mathbf{y}_3')'$. $u_i = p_{si}$ $(i = 1, 2, 3)$ are the scalar control inputs of the ith area and $\mathbf{y}_i = (f\ p_{ti})'$ the output vectors. The

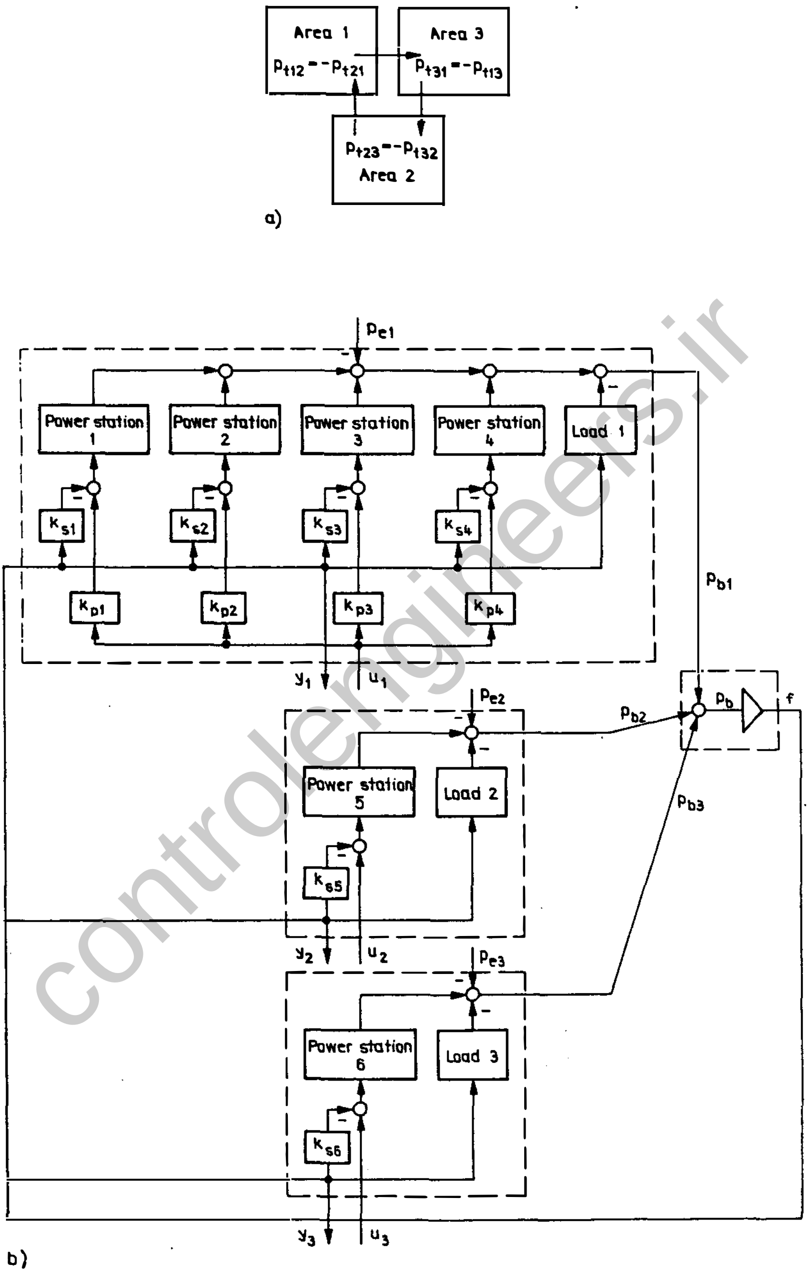


Figure 6.1 A three-area power system: (a) global structure of the system; (b) structure of the model

disturbance $d = p_{e1}$ represents load changes in area 1. As explained in Example 3.5, the overall system consists of overlapping subsystems. Therefore, the model (6.3.14) has the structure depicted in Figures 3.6 and 3.8 (cf. Figure 6.1(b)) and the following parameters:

$$\mathbf{A} = \begin{bmatrix} -0.109 & 0.0001 & 0 & 0.000\,05 & 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.0001 \\ -84.3 & -0.167 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -66.25 & 0 & 0 & -0.125 & -0.25 & 0 & 0 & 0 & 0 & 0 \\ -332 & 0 & 1.25 & -1.875 & -1.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.063 & -0.125 & 0 & 0 & 0 & 0 & 0 \\ -2.1 & 0 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0 & 0 \\ -1.47 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.034 & 0 & 0 & 0 & 0 & 0 & -0.002 & -0.059 & 0 & 0 \\ -225 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 \\ -152 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 \end{bmatrix}$$

$$\mathbf{B}' = \begin{pmatrix} 0 & 0 & 0.243 & 0.243 & 0 & 0.0067 & 0.0048 & 0.0001 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \end{pmatrix}$$

$$\mathbf{M}' = (-0.0001 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 35 & 0.658 & 0 & 0.329 & 0.658 & 0.658 & 0.658 & 0 & -0.342 & -0.342 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -0.410 & 0 & -0.205 & -0.410 & -0.410 & -0.410 & 0 & 0.590 & -0.410 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & -0.248 & 0 & -0.124 & -0.248 & -0.248 & -0.248 & 0 & -0.248 & 0.752 \end{pmatrix} \qquad (6.3.15)$$

$$\mathbf{D} = 0$$

$$\mathbf{N}' = (0 \quad -0.658 \quad 0 \quad 0.410 \quad 0 \quad 0.248).$$

The frequency is measured in hertz, the power in megawatts and the time in seconds. This model was obtained in the same way as the model of Example 3.5 with the following extensions:

- The overall system has six power stations. The first four are considered as area 1, the fifth as area 2, and the sixth as area 3. Each area includes a load of appropriate value (Figure 6.1(b)).
- The control inputs u_1, u_2, u_3 are the set points of the power stations where in area 1 the signal u_1 is distributed among the four power stations in proportion to the nominal power generation (cf. weights k_{pi} in Figure 6.1(b)).
- The net frequency f, which represents the first state variable of the model, and the tie line power can be measured locally (although in the model they depend on the whole state vector). The power flow p_{ti} from area i into all the other areas depends on the distribution of the synchronously rotating masses along the areas. It can be determined from

$$p_{ti} = p_{bi} - T_i/T\, p_b \tag{6.3.16}$$

  where p_b, p_{bi} and T are explained in Figures 3.8 and 6.1(b) and in eqn (3.4.11). For example, for area 1 eqn (6.3.16) yields

$$p_{t1} = p_{t21} + p_{t31} = p_{b1} - T_1/T\, p_b = 0.658 p_{b1} - 0.342 p_b.$$

- Disturbances of the system occur as load changes p_{li} of area i. The model reflects the system subject to load changes in area 1 only ($d = p_{l1}$).
- The frequency-dependent part of the load is assumed to have a proportional part so that the third line in the subsystem model (3.4.12) has direct throughput from $\mathbf{s}_i$ to $\mathbf{z}_i$ as described by

$$z_i = \mathbf{c}'_{zi}\mathbf{x}_i + f_{zi}s_i.$$

  This is the reason why the upper left element a_{11} of the matrix $\mathbf{A}$ is not zero, in contrast to eqn (3.4.14).

The control aims comprise $f = 0$ and $p_{ti} = 0$. Owing to the structural properties of power systems, these aims can be lumped together and replaced by the single area control error $k_i f + p_{ti} = 0$, where k_i is a known scalar describing the 'stiffness' of area i in case of load changes.

The term $k_i f$ describes the power which will be consumed by the rotating masses of area i in case of frequency deviation f. Therefore, decentralized PI controllers

$$\begin{aligned} \dot{x}_{ri} &= k_i f + p_{ti} \\ u_i &= -\mathbf{k}'_{Pi}\mathbf{y}_i - k_{Ii}x_{ri}. \end{aligned} \tag{6.3.17}$$

are used with $k_1 = 1162$, $k_2 = 1555$ and $k_3 = 1052$ for the example system. In the design procedure, the first line of eqn (6.3.17) is used to extend the plant model with the integral parts of all three control stations. The extended plant has the model

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\bar{\mathbf{u}} + \bar{\mathbf{M}}\mathbf{d} \\ \bar{\mathbf{y}} &= \bar{\mathbf{C}}\bar{\mathbf{x}} \qquad\quad + \bar{\mathbf{N}}\mathbf{d} \end{aligned} \tag{6.3.18}$$

with $\bar{\mathbf{x}} = (\mathbf{x}' \;\; \mathbf{x}'_r)'$, $\bar{\mathbf{y}} = (\bar{\mathbf{y}}'_1 \;\; \bar{\mathbf{y}}'_2 \;\; \bar{\mathbf{y}}'_3)'$ and $\bar{\mathbf{y}}_i = (f \;\; p_{ti} \;\; x_{ri})'$. The second line in eqn (6.3.17) is equivalent to

$$\mathbf{u} = -\operatorname{diag} \mathbf{K}_{yi}\bar{\mathbf{y}}. \tag{6.3.19}$$

Algorithm 6.3 can be applied as follows.

Step 1

The centralized state feedback

$$\mathbf{u} = -\mathbf{K}^*\bar{\mathbf{x}} \tag{6.3.20}$$

is determined as the solution of the minimization problem

$$\min_{\mathbf{K}} \frac{1}{2}\int_0^\infty \left[q\left(k^2 f^2 + \sum_{i=1}^{3} (p_{ti}^2 + x_{ri}^2)\right) + \sum_{i=1}^{3} u_i^2 \right] \mathrm{d}t \tag{6.3.21}$$

where $k = k_1 + k_2 + k_3 = 3769$ is the 'stiffness' of the overall plant. In this performance criterion only a scalar q can be used to shape the I/O behaviour of the resulting centralized closed-loop system.

The problem (6.3.21) is obviously an **LQ** problem and can be solved as described in Section 2.4. The response of the closed-loop system (6.3.18) and (6.3.20) to a load step of 40 MW ($d = 40\sigma(t)$) is shown in

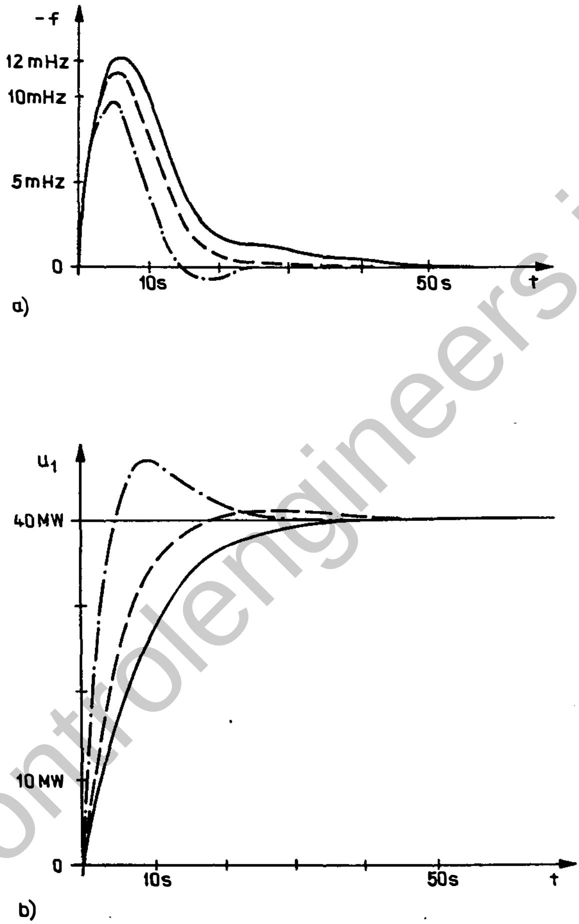


Figure 6.2 Performance of the centralized closed-loop system after load step of 40 MW in area 1: ———, $q = 0.01$; – – – –, $q = 0.02$; – · – · –, $q = 0.1$

Figure 6.2 for different values of the scalar q. The controller

$$\mathbf{K}^* = \begin{pmatrix} 0.569 & 0.679 & 0.437 \\ 0.667 & 0.039 & 0.037 \\ 0.372 & 0.016 & 0.015 \\ 0.025 & 0.003 & 0.003 \\ 0.218 & 0.026 & 0.025 \\ 0.891 & 0.041 & 0.040 \\ 1.516 & 0.034 & 0.032 \\ 8.651 & -0.157 & -0.153 \\ 0.032 & 0.597 & 0.035 \\ 0.029 & 0.035 & 0.593 \\ 0.141 & -0.001 & -0.001 \\ 0.001 & 0.141 & 0.001 \\ 0.001 & 0.001 & 0.141 \end{pmatrix}$$

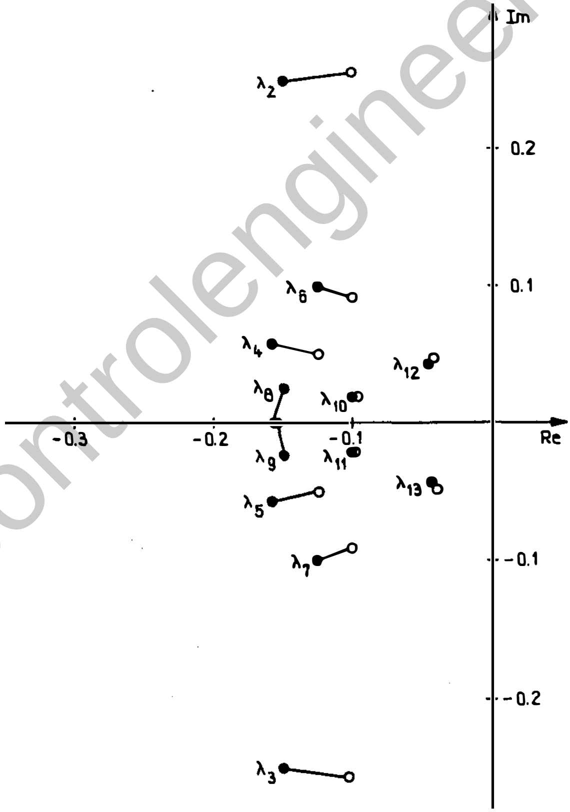


Figure 6.3 Eigenvalues of the centralized (•) and the decentralized (○) closed-loop system (without the eigenvalue at -1.7)

obtained for $q = 0.02$ leads to a reasonable loop behaviour. In particular, $u_1(t)$ has nearly no overshoot. Its static value of 40 MW is identical to the disturbance. Thus, the disturbance is attenuated by area 1 alone ($u_2(\infty) = u_3(\infty) = 0$). The resulting set σ_0 of eigenvalues is shown in Figure 6.3.

Step 2

Several weighting matrices have been used in order to get a reasonable approximation of σ_0 by the spectrum of the decentralized closed-loop system. Finally,

$$\mathbf{W} = \text{diag } w_{ii} \text{ with } \begin{aligned} &w_1 = w_{12} = w_{13} = 0 \\ &w_2 = w_3 = w_8 = w_9 = 0.9 \\ &w_4 = w_5 = 1 \\ &w_6 = w_7 = 0.6 \\ &w_{10} = w_{11} = 0.5 \end{aligned}$$

has been used.

Step 3

Equation (6.3.13) yields

$$\bar{\mathbf{K}}_{y1} = (0.363 \;\; 0.156 \;\; 0.097)$$
$$\bar{\mathbf{K}}_{y2} = (0.290 \;\; 0.190 \;\; 0.081)$$
$$\bar{\mathbf{K}}_{y3} = (0.239 \;\; 0.298 \;\; 0.096).$$

As shown in Figure 6.3, the spectrum of the decentralized closed-loop system approximates σ_0 reasonably. The same is true for the I/O behaviour of the closed-loop system (Figure 6.4). The decentralized controller (6.3.19) yields nearly the same step responses as the centralized controller (6.3.20). □

In comparison with the design methods presented in Sections 6.1 and 6.2, this design procedure has the advantage that it aims at determining a static decentralized output feedback (6.3.4). No dynamical parts are used. The price for this is that the closed-loop system will not exactly satisfy the design aim, which is given as the set σ_0 of closed-loop eigenvalues. This, however, is no disadvantage since the set σ_0 itself is merely an approximate representation of the primary control aims which, for example, are formulated in terms of the step response as in Figure 1.5.

The main difficulty with this design procedure is that there is no systematic way of choosing the weighting matrix **W** in order to come up with a reasonable closed-loop spectrum σ_d. There is even no condition which makes it possible to test whether controller parameters do exist such that the closed-loop system (6.3.1) and (6.3.4) is stable. Therefore, it is reasonable to begin the design with the static controller (6.3.4) but

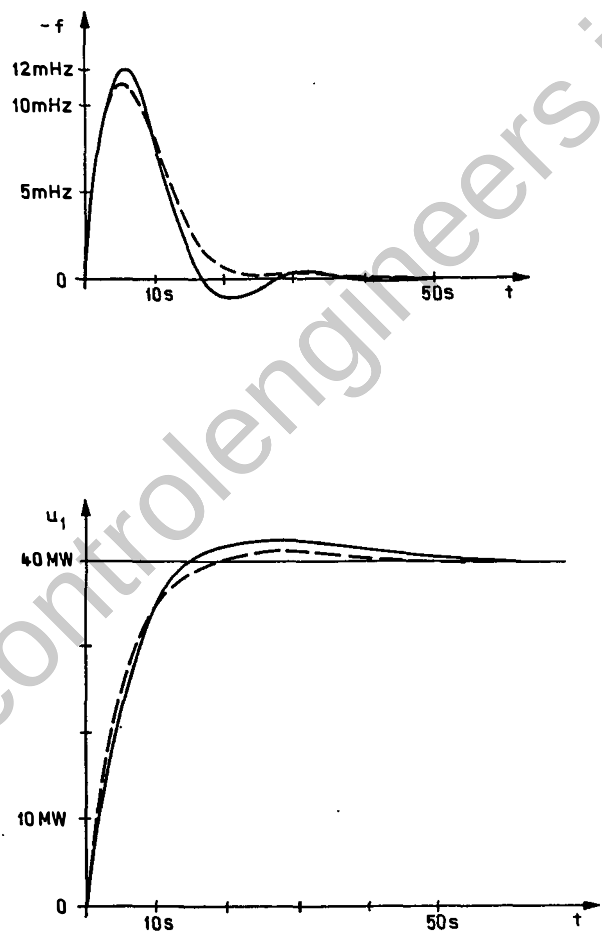


Figure 6.4 Performance of the closed-loop system after load step of 40 MW in area 1: ———, decentralized control; – – –, centralized control as in Figure 6.2

to extend the plant with $1, 2, \ldots, n_r$ integrators as in eqn (6.2.2) if no suitable design result is received. Then the state feedback has to be determined for the extended plant (6.2.2) and steps 2 and 3 of Algorithm 6.3 have to be carried out with suitably partitioned vectors $\bar{\mathbf{u}}$ and $\bar{\mathbf{y}}$. This method is guaranteed to have a solution since Theorem 6.2 shows that for a decentralized control with dynamical order n_r (cf. eqn (6.2.4)) the design aim $\sigma_0 = \sigma_d$ can be reached.

BIBLIOGRAPHICAL NOTES

Stabilization and pole assignment through a specific channel were proposed by Corfmat and Morse (1976a,b) as a result of their investigations concerning decentralized stabilizability. Although it allows an interesting insight into the mechanisms of stabilization by decentralized control it also has the disadvantage of concentrating the controller complexity at a single channel. A contrasting method, which aims at reducing the maximum dynamical order of the control stations by distributing the dynamical parts over all channels, was given by Anderson and Linnemann (1984). An extension of the method explained in Section 6.1 to dynamic control stations was made by Özgüler (1990). He used a fractional representation of a two-channel system in order to derive weaker conditions under which the overall system can be made stabilizable through the second channel by applying a dynamic compensator to the first channel.

Dynamic compensation was proposed for centralized systems by Brasch and Pearson (1970). Its extension to decentralized control was first tackled by Wang and Davison (1973) in their work on stabilizability by decentralized control. However, these authors used the idea of compensation only to prove the existence of some stabilizing decentralized controller with restricted dynamical order and gave no constructive design method. Although in the centralized case several powerful design algorithms have been derived from the principle of dynamic compensation, procedures for decentralized compensators have to overcome severe difficulties concerning the distribution of the dynamical parts among the control stations in order to get constructive and straightforward design steps. Therefore, the results of Section 6.2 are mainly used for bounding the order of the dynamical part of the controller rather than for developing computationally effective algorithms.

There are many design methods that aim at replacing a given centralized controller by some decentralized feedback (see Košut 1970).

They differ with respect to the criterion used to describe the accuracy of the approximation. Algorithm 6.3 is due to Bengtsson and Lindahl (1974). The power system example is a summary of the detailed investigations of decentralized load–frequency controllers carried out for the East German electric power system by Billerbeck *et al.* (1979). The structure of the controller is based on the area control error $k_i f + p_{ti}$, which has been used in electrical engineering practice for a long time, although Elgerd and Fosha (1970) gave a critical account of its use in large decentralized control systems.

7

Optimal Decentralized Control

7.1 THE OPTIMIZATION PROBLEM

The use of optimization methods as a means of designing linear feedback controllers is well known from multivariable systems. For a given model of the plant, controller parameters have to be found for which the closed-loop system minimizes a given objective function.

There are several reasons for reformulating and solving the design task as an optimization problem. First, in several applications the design specifications can be formulated as an objective function, which is to be minimized. Second, there are powerful methods for solving the optimization problem. Necessary or sufficient conditions for the optimality of the solution can be derived, which lead to efficient optimization algorithms and which yield characteristic properties that all optimal controllers are known to possess. Therefore, optimal control is used even if the minimization of an objective function is not the ultimate design aim.

This chapter is devoted to an extension of optimal control to design problems where the control law is subjected to structural constraints. In order to explain the severe difficulties that arise from the structural constraints, the optimization problem encountered in centralized control has to be briefly revisited (cf. Section 2.4).

In **LQ** regulation a control $\mathbf{u}(t)$ $(0 \leqslant t < \infty)$ is to be found for a linear plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{7.1.1}$$

such that the performance index (2.4.7)

$$I = \frac{1}{2} \int_0^\infty (\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u})\, dt \tag{7.1.2}$$

is minimized,

$$\min_{\mathbf{u}(t)} I \quad \text{subject to eqn (7.1.1).} \tag{7.1.3}$$

The matrices **Q** and **R** are symmetric non-negative definite or symmetric positive definite, respectively. The problem (7.1.3) is a functional optimization problem.

It is important that the solution of problem (7.1.3) can be represented as a linear feedback

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t). \tag{7.1.4}$$

That is, the functional optimization problem (7.1.3) is equivalent to the parameter optimization problem

$$\min_{\mathbf{K}} I \quad \text{subject to eqns (7.1.1) and (7.1.4).} \tag{7.1.5}$$

A second important property of the optimization problem is that the solution $\mathbf{K}$ is independent of the initial state $\mathbf{x}_0$ of the system (7.1.1). This becomes clear from the representation of the optimal feedback gain matrix by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}'\mathbf{P} \tag{7.1.6}$$

where $\mathbf{P}$ denotes the symmetric positive definite solution of the algebraic matrix Riccati equation

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P} + \mathbf{Q} = \mathbf{0}. \tag{7.1.7}$$

The same optimization problems will now be considered with structural constraints imposed on the control $\mathbf{u}(t)$. The plant model has the I/O-oriented form (3.1.2)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{7.1.8}$$

$$\mathbf{y}_i(t) = \mathbf{C}_{si}\mathbf{x}(t) \qquad (i = 1, \ldots, N).$$

According to the structural constraints, the accessible subsystem controls $\mathbf{u}_i(t)$ are restricted to those functions which can be determined from knowledge of the subsystem output $\mathbf{y}_i(t)$ only. The dependence of the input $\mathbf{u}_i(\bar{t})$ at the time instant $\bar{t}$ upon the output $\mathbf{y}_i(t)$ $(0 \leqslant t \leqslant \bar{t})$ measured until time $\bar{t}$ is expressed as $\mathbf{u}_i = \mathbf{u}_i(\mathbf{y}_i)$.

Problem 7.1

Solve the optimization problem

$$\min_{\mathbf{u}_i = \mathbf{u}_i(\mathbf{y}_i)} I \quad \text{subject to eqn (7.1.8).} \tag{7.1.9}$$

Contrary to the optimization problem (7.1.3), the solution of Problem 7.1 may be a non-linear function $\mathbf{u}_i(\mathbf{y}_i)$. This fact became evident in 1968 from an example given by Witsenhausen for a similar stochastic optimization problem.

The fact that non-linear control laws may be superior to linear controllers has already been mentioned in connection with the stabilization problem (cf. Bibliographical Notes of Chapter 4). There are systems which can be stabilized by some non-linear decentralized controller, although stabilization is impossible if the control laws $\mathbf{u}_i(\mathbf{y}_i)$ are restricted to be linear if unstable fixed modes exist.

In a more general setting, Problem 7.1 involves N decision makers which have to choose $\mathbf{u}_i(t)$ independently of each other in order to satisfy a common goal. This setting of the problem has already been mentioned in Chapter 1 (Figure 1.3), where these decision makers were characterized as local units, which act independently of each other. The desire to minimize the performance criterion (7.1.2) may motivate the decision makers to use the plant as a communication channel in order to inform each other about their measurement data $\mathbf{y}_i(t)$ or their decisions $\mathbf{u}_i(t)$. Then, the decisions can be made on the basis of more information and may be better than any decisions which refer only to locally available information $\mathbf{y}_i$. This phenomenon is called *signalling*. It leads to non-linear decision rules $\mathbf{u}_i(\mathbf{y}_i)$.

In order to make the control applicable as linear feedback it is necessary to restrict the accessible inputs explicitly to functions of the form

$$\mathbf{u}(t) = -\mathbf{K}_y\mathbf{y}(t) \quad \text{with } \mathbf{K}_y = \text{diag } \mathbf{K}_{yi} \tag{7.1.10}$$

or

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \quad \text{with } \mathbf{K} = \text{diag } \mathbf{K}_i. \tag{7.1.11}$$

Then, all signalling phenomena are excluded.

Problem 7.2

Solve the optimization problem

$$\min_{\mathbf{u} = -\text{diag } \mathbf{K}_{yi}\mathbf{y}} I \quad \text{subject to eqn (7.1.8).} \tag{7.1.12}$$

This is a parameter optimization problem in which controller parameters $\mathbf{K}_{yi}$ have to be found such that the closed-loop system (7.1.8) and (7.1.10) minimizes the objective function (7.1.2).

A further problem occurs in optimal decentralized control because the controller matrices $\mathbf{K}_{yi}$ turn out to depend on the initial state $\mathbf{x}_0$. Therefore, the parameter optimization problem (7.1.12) has to be solved for every initial state $\mathbf{x}_0$ separately. This is the reason for a further reformulation of the optimization problem. As the basis for this the closed-loop system (7.1.8) and (7.1.10) is represented as

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{7.1.13}$$

with

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C} \tag{7.1.14}$$

where $\mathbf{B}$ and $\mathbf{C}$ are composed of $\mathbf{B}_{si}$ or $\mathbf{C}_{si}$, respectively, as in eqn (3.1.3). By using the transition matrix

$$\mathbf{\Phi}(t) = \exp(\bar{\mathbf{A}}t) \tag{7.1.15}$$

the state trajectory of the closed-loop system (7.1.13) can be represented by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}_0. \tag{7.1.16}$$

Therefore, the performance index (7.1.2) can be rewritten as

$$I = \tfrac{1}{2}\mathbf{x}_0' \int_0^\infty \mathbf{\Phi}'(t)(\mathbf{Q} + \mathbf{C}'\mathbf{K}_y'\mathbf{R}\mathbf{K}_y\mathbf{C})\mathbf{\Phi}(t)\, \mathrm{d}t\mathbf{x}_0. \tag{7.1.17}$$

It is known that for every symmetric matrix $\mathbf{Q}$ the identity

$$\mathbf{x}'\mathbf{Q}\mathbf{x} = \operatorname{trace}(\mathbf{Q}\mathbf{x}\mathbf{x}')$$

holds. Hence, eqn (7.1.17) leads to

$$I = \tfrac{1}{2}\operatorname{trace}(\mathbf{P}\mathbf{x}_0\mathbf{x}_0') \tag{7.1.18}$$

with

$$\mathbf{P} = \int_0^\infty \mathbf{\Phi}'(t)(\mathbf{Q} + \mathbf{C}'\mathbf{K}_y'\mathbf{R}\mathbf{K}_y\mathbf{C})\mathbf{\Phi}(t)\, \mathrm{d}t. \tag{7.1.19}$$

It can be shown that $\mathbf{P}$ is the solution of the Lyapunov equation

$$(\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C})'\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C}) + \mathbf{C}'\mathbf{K}_y'\mathbf{R}\mathbf{K}_y\mathbf{C} + \mathbf{Q} = \mathbf{0}. \tag{7.1.20}$$

In order to make the optimization problem independent of the initial state $\mathbf{x}_0$, the design aim is altered to the new aim of finding a controller (7.1.12) for which the average cost I is minimal. The initial state is considered as a random variable and the average value of I is chosen as the new objective function

$$\tilde{I} = E[I] = E[\operatorname{trace}(\mathbf{P}\mathbf{x}_0\mathbf{x}_0')]. \tag{7.1.21}$$

$E[.]$ denotes the expected value with respect to $\mathbf{x}_0$. If $\mathbf{x}_0$ is assumed to be uniformly distributed over the n-dimensional unit sphere with

$$E[\mathbf{x}_0\ \mathbf{x}_0'] = \frac{1}{n}\, I$$

the expression

$$\tilde{I} = \frac{1}{n}\operatorname{trace}\mathbf{P}$$

follows. Obviously, $\bar{I}$ has the same optimum as

$$\bar{I} = \text{trace } \mathbf{P}. \tag{7.1.22}$$

$\bar{I}$ does not depend on $\mathbf{x}_0$.

Problem 7.3

Solve the optimization problem

$$\min_{\mathbf{u} = -\text{diag } \mathbf{K}_{yi}\mathbf{y}} \bar{I} \quad \text{subject to eqn (7.1.8).} \tag{7.1.23}$$

Contrary to Problems 7.1 and 7.2, this problem yields a linear decentralized feedback which is independent of $\mathbf{x}_0$. Therefore, the remainder of this chapter will be devoted to Problem 7.3. Section 7.2 describes an algorithm for solving this problem.

7.2 AN ALGORITHM FOR SOLVING THE OPTIMIZATION PROBLEM

A Necessary Optimality Condition

The solution to Problem 7.3 is characterized by an optimality condition which can be derived from the gradient of the objective function $\bar{I}$ with respect to the non-vanishing controller parameters entering the matrix $\mathbf{K}_y$ in eqn (7.1.10). Starting with

$$\frac{\mathrm{d}\bar{I}}{\mathrm{d}\mathbf{K}_y} = \lim_{\varepsilon \to 0} \frac{\bar{I}(\mathbf{K}_y + \varepsilon\delta\mathbf{K}_y) - \bar{I}(\mathbf{K}_y)}{\varepsilon}$$

with $\bar{I}(\mathbf{K}_y)$ from eqn (7.1.22) and $\mathbf{P}$ from (7.1.20), a rather lengthy manipulation yields

$$\frac{\mathrm{d}\bar{I}}{\mathrm{d}\mathbf{K}_y} = 2(\mathbf{R}\mathbf{K}_y\mathbf{C} - \mathbf{B}'\mathbf{P})\mathbf{L}\mathbf{C}' \tag{7.2.1}$$

where $\mathbf{L}$ is the positive symmetric solution of the equation

$$(\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C})\mathbf{L} + \mathbf{L}(\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C})' + \mathbf{I} = \mathbf{0}. \tag{7.2.2}$$

Such a matrix $\mathbf{L}$ exists if and only if the closed-loop system matrix $\mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C}$ is stable. From

$$\frac{\mathrm{d}\bar{I}}{\mathrm{d}\mathbf{K}_y} = \mathbf{0}$$

the necessary optimality condition

$$\mathbf{K}_y = \mathbf{R}^{-1}\mathbf{B}'\mathbf{PLC}'(\mathbf{CLC}')^{-1} \tag{7.2.3}$$

follows.

Theorem 7.1

The decentralized feedback controller (7.1.10) that solves Problem 7.3 satisfies the following optimality conditions

$$\mathbf{K}_y = \mathbf{R}^{-1}\mathbf{B}'\mathbf{PLC}'(\mathbf{CLC}')^{-1}$$
$$(\mathbf{A} - \mathbf{BK}_y\mathbf{C})'\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK}_y\mathbf{C}) + \mathbf{C}'\mathbf{K}_y'\mathbf{RK}_y\mathbf{C} + \mathbf{Q} = \mathbf{0}$$
$$(\mathbf{A} - \mathbf{BK}_y\mathbf{C})\mathbf{L} + \mathbf{L}(\mathbf{A} - \mathbf{BK}_y\mathbf{C})' + \mathbf{I} = \mathbf{0}.$$

Note that eqn (7.2.3) coincides with (7.1.6) if the number of subsystems is restricted to $N = 1$ and if $\mathbf{C}_{s1} = \mathbf{C} = \mathbf{I}$ holds. That is, the performance indices $\bar{I}$ and I yield the same controller matrix $\mathbf{K}$ if they are both applied to the centralized control problem (7.1.3).

Solution of the Optimization Problem

Optimal controller gains can be obtained as solutions of the three matrix equations given in Theorem 7.1. Different iterative procedures have been proposed, but only a few of them are proved to converge to the optimal solution. The algorithm, which will now be explained, is known to improve the solution in every iteration step if it is initialized by a stabilizing feedback $\mathbf{K}_y^0$.

The algorithm has been elaborated under the assumption that in the overall system (7.1.8), which can be written in the unstructured form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu}\\ \mathbf{y} &= \mathbf{Cx}\end{aligned} \tag{7.2.4}$$

the matrix $\mathbf{B}$ is block diagonal, $\mathbf{B} = \text{diag}\,\mathbf{B}_{ii}$. Furthermore, a decentralized state feedback

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i. \tag{7.2.5}$$

is used (i.e. $\mathbf{y}_i = \mathbf{x}_i$). Then, eqns (7.1.20) and (7.2.2) read as

$$(\mathbf{A} - \mathbf{BK})'\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK}) + \mathbf{K}'\mathbf{RK} + \mathbf{Q} = \mathbf{0} \tag{7.2.6}$$

$$(\mathbf{A} - \mathbf{BK})\mathbf{L} + \mathbf{L}(\mathbf{A} - \mathbf{BK})' + \mathbf{I} = \mathbf{0} \tag{7.2.7}$$

with $\mathbf{K} = \text{diag}\ \mathbf{K}_i$. A search procedure with the structure of the Davidon–Fletcher–Powell variable metric method can be used to find the optimal feedback matrix $\mathbf{K} = \text{diag}\ \mathbf{K}_i$.

Algorithm 7.1 (*Optimal decentralized control*)

Given: Plant model (7.2.4), controller structure constraints, initial feedback matrix $\mathbf{K}^0 = \text{diag}\ \mathbf{K}_i^0$ for which the closed-loop system matrix $\bar{\mathbf{A}}^0 = \mathbf{A} - \mathbf{B}\mathbf{K}^0$ is stable; $k = 1$.

1. Determine $\mathbf{P}^k$ and $\mathbf{L}^k$ as solutions of eqns (7.2.6) and (7.2.7) with $\mathbf{K} = \mathbf{K}^{k-1}$.
2. Determine $\mathrm{d}\bar{I}/\mathrm{d}\mathbf{K}$ from eqn (7.2.1) with $\mathbf{K}^{k-1}$, $\mathbf{P}^k$ and $\mathbf{L}^k$. Let $\mathbf{D}^k = \text{diag}\ \mathbf{D}_i^k$ where $\mathbf{D}_i^k$ denotes the diagonal blocks of $\mathrm{d}\bar{I}/\mathrm{d}\mathbf{K}$.
3. Determine a step size s^k for which $\bar{I}(\mathbf{K}^{k-1} - s^k\mathbf{D}^k) < \bar{I}(\mathbf{K}^k)$ holds. Let $\mathbf{K}^k = \mathbf{K}^{k-1} - s^k\mathbf{D}^k$.
4. If $s^k \|\mathbf{D}^k\| < \varepsilon$ holds for a given threshold ε, stop; otherwise let $k = k + 1$ and proceed with step 1.

Result: Near-optimal gain matrix $\mathbf{K}^k = \text{diag}\ \mathbf{K}_i^k$ of the decentralized controller (7.2.5).

Step 3 includes a one-dimensional search concerning the step size s^k. Well-known methods from static optimization can be used for it. The properties of the algorithm are described by the following Lemma.

Lemma 7.1

Assume that the pair $(\mathbf{A}, \bar{\mathbf{Q}})$ with $\bar{\mathbf{Q}}'\bar{\mathbf{Q}} = \mathbf{Q}$ is observable and that the decentralized controller $\mathbf{K}^k$ leads to a stable closed-loop system. Then

(i) there exists a step size s^k such that

$$\bar{I}(\mathbf{K}^{k-1} - s^k\mathbf{D}^k) < \bar{I}(\mathbf{K}^k)$$

holds and

(ii) $\mathbf{K}^k = \mathbf{K}^{k-1} - s^k\mathbf{D}^k$ is a stabilizing feedback, that is the closed-loop system matrix $\bar{\mathbf{A}}^k = \mathbf{A} - \mathbf{B}\mathbf{K}^k$ is stable.

That is, the algorithm does not produce instability within the closed-loop system if it is initialised by a stabilising feedback. In each iteration step the feedback gain is improved.

Initialization of the algorithm

Since a stabilizing decentralized controller has to be given, a major problem of decentralized control has to be solved before Algorithm 7.1 can be started. The following algorithm for finding such controller gains is based on sensitivity considerations of the dominant eigenvalues of the closed-loop system. The sensitivity of an eigenvalue λ of the matrix $\mathbf{A}(p)$ with respect to the parameter p is described by eqn (A1.6). It will be used here with respect to the dominant eigenvalue

$$\lambda_{\mathrm{d}}^{k} = \max_{i} \lambda_i[\bar{\mathbf{A}}^{k-1}] \tag{7.2.8}$$

of the matrix

$$\bar{\mathbf{A}}^{k-1} = \mathbf{A} - \mathbf{B}\mathbf{K}^{k-1} \tag{7.2.9}$$

and the non-zero elements k_{ij}^k of the controller matrix $\mathbf{K}^k$: $\mathbf{G}^k = (g_{ij}^k)$ with

$$\begin{aligned} g_{ij}^k &= \frac{\mathrm{d}\lambda_{\mathrm{d}}^k}{\mathrm{d}k_{ij}} = \frac{\mathbf{v}^{k\prime}\mathbf{b}_i u_j^k}{\mathbf{v}^{k\prime}\mathbf{u}^k} \quad \text{for } k_{ij}^k \neq 0 \\ g_{ij}^k &= \frac{\mathrm{d}\lambda_{\mathrm{d}}^k}{\mathrm{d}k_{ij}} = 0 \quad \text{for } k_{ij}^k = 0. \end{aligned} \tag{7.2.10}$$

$\mathbf{b}_i$ is the ith column of the matrix $\mathbf{B}$ in eqn (7.2.4); $\mathbf{u}^k = (u_1^k \dots u_n^k)'$ and $\mathbf{v}^{k\prime}$ are the right and left eigenvectors, respectively, or $\mathbf{A}^{k-1}$, which belong to λ_{d}^k.

Algorithm 7.2

Given: Plant model (7.2.4), controller structure constraints, an arbitrary initial controller matrix $\mathbf{K}^0$; $k = 1$.

1. Determine λ_{d}^k from eqns (7.2.8) and (7.2.9).
2. If $\mathrm{Re}[\lambda_{\mathrm{d}}^k] < 0$ holds, stop; otherwise proceed with step 3.
3. Determine $u_i^k, v_i^{k\prime}$ from eqns (A1.1) and (A1.2), and the gradient $\mathbf{G}^k$ according to eqn (7.2.10).
4. Search for a step size s^k such that the dominant eigenvalue of $\bar{\mathbf{A}}^k = \bar{\mathbf{A}}^{k-1} - s^k\mathbf{B}\mathbf{G}^k$ is minimal. Let $\mathbf{K}^k = \mathbf{K}^{k-1} - s^k\mathbf{G}^k$. Proceed with step 2.

Result: Decentralized control for which the closed-loop system is stable.

This algorithm can be used only as long as the dominant eigenvalue is single. However, a multiple eigenvalue λ_{d}^k can be made single by changing the controller gain $\mathbf{K}^k$ slightly. If λ_d^k converges to some value with positive real part, restart the algorithm with new initial controller gains. Although encouraging results have been reported with this algorithm, for particular controllers such as decentralized PI controllers better methods are available and should be used for initializing Algorithm 7.1 (cf. Chapter 11).

7.3 GLOBAL OPTIMALITY OF THE OPTIMAL CONTROLLERS OF ISOLATED SUBSYSTEMS

In this section an answer is given to the question of under what conditions the control stations of the optimal decentralized controller can be designed as optimal controllers of isolated subsystems. A condition will be derived under which the optimal centralized controller of the overall system represents a decentralized controller. This condition depends on the system properties as well as the performance index. In Chapters 9–12 the method of designing the control stations independently of each other will be extended to design problems in which the condition derived below is not satisfied.

The overall system is described by the model (3.1.16)

$$\begin{aligned}
&\dot{\mathbf{x}}_i(t) = \mathbf{A}_{ii}\mathbf{x}_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\
&\mathbf{y}_i(t) = \mathbf{C}_i\mathbf{x}_i(t) \qquad (i = 1, \ldots, N)
\end{aligned} \tag{7.3.1}$$

or, equivalently, by

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x}
\end{aligned}$$

with $\mathbf{B} = \text{diag}\,\mathbf{B}_i$ and $\mathbf{C} = \text{diag}\,\mathbf{C}_i$. Each subsystem is associated with a performance index

$$I_i = \frac{1}{2}\int_0^\infty (\mathbf{x}_i'\mathbf{Q}_i\mathbf{x}_i + \mathbf{u}_i'\mathbf{R}_i\mathbf{u}_i)\,\mathrm{d}t \tag{7.3.2}$$

where the weighting matrices $\mathbf{Q}_i$ and $\mathbf{R}_i$ are symmetric non-negative definite or positive definite, respectively. Each isolated subsystem

$$\begin{aligned}
&\dot{\mathbf{x}}_i(t) = \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\
&\mathbf{y}_i(t) = \mathbf{C}_i\mathbf{x}_i(t)
\end{aligned} \tag{7.3.3}$$

with $\mathbf{A}_i = \mathbf{A}_{ii}$ is assumed to be controllable. Furthermore, the goals of the subsystems are assumed to be in harmony. That is, the overall system performance index equals the sum of the subsystem indices so that

$$I = \sum_{i=1}^{N} I_i = \frac{1}{2} \int_0^\infty (\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u})\, \mathrm{d}t \tag{7.3.4}$$

holds with $\mathbf{Q} = \operatorname{diag} \mathbf{Q}_i$ and $\mathbf{R} = \operatorname{diag} \mathbf{R}_i$.

Three Alternative Design Problems

Optimal control of isolated subsystems

If the control stations

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i \tag{7.3.5}$$

are designed as optimal controllers of the isolated subsystems (7.3.3), the controller matrices $\mathbf{K}_i$ are obtained as the solution of the optimization problems

$$\min_{\mathbf{K}_i} I_i \quad \text{subject to eqns (7.3.3) and (7.3.5).} \tag{7.3.6}$$

These optimization problems can be solved independently of each other. Their solutions are labelled with a superscript 'o':

$$\mathbf{K}_i^{\mathrm{o}} = \mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i^{\mathrm{o}} \tag{7.3.7}$$

holds where $\mathbf{P}_i^{\mathrm{o}}$ satisfies the algebraic Riccati equation of the ith subsystem

$$\mathbf{A}_i'\mathbf{P}_i^{\mathrm{o}} + \mathbf{P}_i^{\mathrm{o}}\mathbf{A}_i - \mathbf{P}_i^{\mathrm{o}}\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i^{\mathrm{o}} + \mathbf{Q}_i = \mathbf{0}. \tag{7.3.8}$$

The minima of the performance indices of the isolated subsystems are given by

$$I_i^{\mathrm{o}} = \mathbf{x}_{i0}'\mathbf{P}_i^{\mathrm{o}}\mathbf{x}_{i0}. \tag{7.3.9}$$

If no interactions between the subsystems are effective and, thus, eqn (7.3.1) holds with $\mathbf{A}_{ij} = \mathbf{0}$ $(i \neq j)$, the decentralized controller (7.3.5) and (7.3.7) is globally optimal, that is it is the solution of the problem

$$\min_{\mathbf{K} = \operatorname{diag} \mathbf{K}_i} I \quad \text{subject to eqns (7.3.3) and (7.3.5).}$$

The overall system performance index is given by

$$I^{\mathrm{o}} = \sum_{i=1}^{N} I_i^{\mathrm{o}} = \sum_{i=1}^{N} \mathbf{x}_0' \operatorname{diag} \mathbf{P}_i^{\mathrm{o}}\mathbf{x}_0. \tag{7.3.10}$$

Under the influence of the interactions the value of the global index I

can be represented as

$$I^g = \mathbf{x}_0'\mathbf{P}^g\mathbf{x}_0. \tag{7.3.11}$$

$\mathbf{P}^g$ is the positive definite solution of the Lyapunov equation

$$(\mathbf{A}-\mathbf{B}\mathbf{K}^o)'\mathbf{P}^g + \mathbf{P}^g(\mathbf{A}-\mathbf{B}\mathbf{K}^o) + \mathbf{K}^{o\prime}\ \mathrm{diag}\ \mathbf{R}_i\mathbf{K}^o + \mathrm{diag}\ \mathbf{Q}_i = \mathbf{0} \tag{7.3.12}$$

where $\mathbf{K}^o = \mathrm{diag}\ \mathbf{K}_i^o$ holds (cf. eqns (7.1.17) and (7.1.20)). Depending upon the matrices $\mathbf{A}_{ij}$ the value I^g can be less than, equal to or greater than I^o

$$I^g \gtreqless I^o. \tag{7.3.13}$$

Optimal decentralized control

If the decentralized controller (7.3.5) is determined as a solution of the optimization problem

$$\min_{\mathbf{u}=-\mathrm{diag}\ \mathbf{K}_i\mathbf{x}} I \quad \text{subject to eqn (7.3.1)} \tag{7.3.14}$$

the performance index I assumes its minimum I^d with respect to the structural constraints given by the decentralization of the control law. The solution is denoted by $\mathbf{K}^d = \mathrm{diag}\ \mathbf{K}_i^d$. Similarly to I in (7.3.11) and (7.3.12),

$$I^d = \mathbf{x}_0'\mathbf{P}^d\mathbf{x}_0 \tag{7.3.15}$$

holds with $\mathbf{P}^d$ a positive definite solution of the Lyapunov equation

$$(\mathbf{A}-\mathbf{B}\mathbf{K}^d)'\mathbf{P}^d + \mathbf{P}^d(\mathbf{A}-\mathbf{B}\mathbf{K}^d) + \mathbf{K}^d\ \mathrm{diag}\ \mathbf{R}_i\mathbf{K}^d + \mathrm{diag}\ \mathbf{Q}_i = \mathbf{0}. \tag{7.3.16}$$

Obviously,

$$I^d \leqslant I^g \tag{7.3.17}$$

holds since the problem (7.3.14), which yields I^d, concerns the subsystem interactions. As known from Section 7.1, the optimal solution $\mathbf{K}^d$ depends upon $\mathbf{x}_0$ and so does the matrix $\mathbf{P}^d$.

Optimal control of the overall system

The third alternative controller can be obtained as the solution of the optimization problem

$$\min_{\mathbf{u}=-\mathbf{K}\mathbf{x}} I \quad \text{subject to eqn (7.3.1)} \tag{7.3.18}$$

which yields the optimal centralized controller

$$\mathbf{u}(t) = -\mathbf{K}^*\mathbf{x}(t) \tag{7.3.19}$$

$$\mathbf{K}^* = \mathbf{R}^{-1}\mathbf{B}'\mathbf{P}^* \tag{7.3.20}$$

with $\mathbf{P}^*$ the positive definite solution of the algebraic Riccati equation of the overall system

$$\mathbf{A}'\mathbf{P}^* + \mathbf{P}^*\mathbf{A} - \mathbf{P}^*\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P}^* + \mathbf{Q} = \mathbf{0}. \tag{7.3.21}$$

The performance index has the value

$$I^* = \mathbf{x}_0'\mathbf{P}^*\mathbf{x}_0. \tag{7.3.22}$$

A comparison of all three controllers yields the relations

$$I^* \leqslant I^d \leqslant I^g \quad \text{and} \quad I^* \leqslant I^o. \tag{7.3.23}$$

Systems With Neutral Interactions

Now, the question will be answered under what conditions eqn (7.3.23) holds with the equality signs. The system matrix $\mathbf{A}$ of the overall system (7.3.1) can be decomposed into

$$\mathbf{A} = \mathbf{A}_D + \mathbf{A}_C \quad \text{with } \mathbf{A}_D = \text{diag } \mathbf{A}_{ii} \tag{7.3.24}$$

(cf. eqn (3.1.18)). The matrix $\mathbf{A}_C$ describes the interactions between the subsystems. If the subsystems are not interconnected ($\mathbf{A}_C = \mathbf{0}$) all design problems yield the same solution, that is

$$\mathbf{K}^* = \text{diag } \mathbf{K}_i^o \tag{7.3.25}$$

$$I^* = I^d = I^g = I^o \tag{7.3.26}$$

hold. The question arises for which non-trivial interactions ($\mathbf{A}_C \neq \mathbf{0}$) these relations are also true.

A classification of the subsystem interactions can be defined according to the relation between I^g and I^o since these values describe the performance of the decentralized controller (7.3.5) and (7.3.7) in connection with the interaction-free overall system and the subsystems under the influence of the interactions, respectively.

Definition 7.1

For given performance indices (7.3.2) the subsystem interactions are called

(a) *neutral* if

$$I^g = I^o \tag{7.3.27}$$

holds,

(b) *beneficial* (or *cooperative*) if $I^g < I^o$ holds,

(c) *non-beneficial* (or *competitive*) if $I^g > I^o$ holds.

That is, neutral interactions do not change the value of the performance index of the decentralized controller (7.3.5) and (7.3.7) whereas beneficial interactions make the performance of the overall system better than the performances of the isolated subsystems. Note that this definition refers to the closed-loop system for a given performance index. Hence, the three characterizations describe properties of the interactions within the closed-loop system rather than plant properties.

Although Definition 7.1 does not refer to the optimal centralized controller, the following theorems show that both the equations (7.3.25) and (7.3.26) are valid if the interactions are neutral.

Theorem 7.2

The subsystem interactions are neutral with respect to the performance index (7.3.2) if and only if the interaction matrix $\mathbf{A}_c$ can be factorized as

$$\mathbf{A}_C = \mathbf{S} \text{ diag } \mathbf{P}_i^o \tag{7.3.28}$$

where the matrices $\mathbf{P}_i^o$ $(i = 1, \ldots, N)$ are the solutions of eqn (7.3.8) and $\mathbf{S}$ is some skew-symmetric matrix (i.e. $\mathbf{S} = -\mathbf{S}'$ holds).

Proof

To prove the necessity of eqn (7.3.28) assume that $I^g = I^o$ and, thus, $\mathbf{P}^g = \text{diag } \mathbf{P}_i^o$ hold. Then eqns (7.3.7), (7.3.12) and (7.3.24) yield

$$\mathbf{A}_D' \text{ diag } \mathbf{P}_i^o + \text{diag } \mathbf{P}_i^o\mathbf{A}_D - \text{diag } \mathbf{P}_i^o\mathbf{B} \text{ diag } \mathbf{R}_i^{-1}\mathbf{B}' \text{ diag } \mathbf{P}_i^o$$
$$+ \text{diag } \mathbf{Q}_i + \mathbf{A}_C' \text{ diag } \mathbf{P}_i^o + \text{diag } \mathbf{P}_i^o\mathbf{A}_C = \mathbf{0}. \tag{7.3.29}$$

Since the first four terms coincide with the left-hand side of eqn (7.3.8) for $i = 1, \ldots, N$, eqn (7.3.29) reduces to

$$\text{diag } \mathbf{P}_i^o\mathbf{A}_C + \mathbf{A}_C' \text{ diag } \mathbf{P}_i^o = \mathbf{0}.$$

Therefore,

$$\mathbf{A}_C \text{ diag } \mathbf{P}_i^o = -\text{diag } \mathbf{P}_i^o\mathbf{A}_C' = \tilde{\mathbf{S}}$$

represents some matrix $\tilde{\mathbf{S}}$, which is skew-symmetric. If $\tilde{\mathbf{S}}$ is chosen to be $\tilde{\mathbf{S}} = \text{diag}\ \mathbf{P}_i^{\circ}\mathbf{S}\ \text{diag}\ \mathbf{P}_i^{\circ}$, then eqn (7.3.28) results.

To show the sufficiency of eqn (7.3.28), eqn (7.3.12) is transformed into

$$\begin{aligned}\mathbf{A}_D'\mathbf{P}^g + \mathbf{P}^g\mathbf{A}_D - \mathbf{P}^g\mathbf{B}\ \text{diag}\ \mathbf{R}_i^{-1}\mathbf{B}'\mathbf{P}^g + \text{diag}\ \mathbf{Q}_i \\ + (\mathbf{P}^g - \text{diag}\ \mathbf{P}_i^{\circ})\mathbf{B}\ \text{diag}\ \mathbf{R}_i^{-1}\mathbf{B}'(\mathbf{P}^g - \text{diag}\ \mathbf{P}_i^{\circ}) \\ - \text{diag}\ \mathbf{P}_i^{\circ}\mathbf{S}\mathbf{P}^g + \mathbf{P}^g\mathbf{S}\ \text{diag}\ \mathbf{P}_i^{\circ} = \mathbf{0}\end{aligned} \qquad (7.3.30)$$

where eqns (7.3.7), (7.3.8) and (7.3.28) have been used. It follows from eqn (7.3.8) that the matrix diag $\mathbf{P}_i^{\circ}$ is positive definite and satisfies

$$\mathbf{A}_D'\ \text{diag}\ \mathbf{P}_i^{\circ} + \text{diag}\ \mathbf{P}_i^{\circ}\mathbf{A}_D - \text{diag}\ \mathbf{P}_i^{\circ}\mathbf{B}\ \text{diag}\ \mathbf{R}_i^{-1}\mathbf{B}'\ \text{diag}\ \mathbf{P}_i^{\circ} + \text{diag}\ \mathbf{Q}_i = \mathbf{0}.$$

Therefore, $\mathbf{P}^g = \text{diag}\ \mathbf{P}_i^{\circ}$ is positive definite and satisfies eqn (7.3.30). Hence, $I^g = I^{\circ}$ holds. □

Similarly, it can be proved that the solution of the optimization problem (7.3.18) is a decentralized controller if the interactions are neutral.

Theorem 7.3

The solution of the optimization problem (7.3.18) is a decentralized controller (7.3.5) and (7.3.7) if and only if the subsystem interactions are neutral.

As a result of eqn (7.3.23), Theorems 7.2 and 7.3 lead to the following corollary.

Corollary 7.1

Eqns (7.3.25) and (7.3.26) hold if and only if the subsystem interactions are neutral with respect to the given performance index (7.3.2).

This result has several interesting consequences. First, eqn (7.3.28) represents a necessary and sufficient condition under which the locally optimal controllers optimize and, hence, stabilize the overall system. The stabilization and optimization problems can be exactly represented as N independent design problems (7.3.6). All calculations that are necessary for the solution of these design problems refer merely to the isolated subsystems and involve manipulations with matrices of low order.

Second, eqn (7.3.28) clearly shows that these results hold true only for a restricted class of overall systems. The matrix $\mathbf{A}_C$ has to possess a specific property which for $N = 2$ is represented by

$$\mathbf{A}_C = \begin{pmatrix} \mathbf{0} & \mathbf{S}_{12}\mathbf{P}_2^{\circ} \\ -\mathbf{S}_{12}'\mathbf{P}_1^{\circ} & \mathbf{0} \end{pmatrix}.$$

where the matrices $\mathbf{P}_i^{\circ}$ $(i = 1, 2)$ are the solutions of eqn (7.3.8) and $\mathbf{S}_{12}$ is an arbitrary matrix. This property depends not only on the plant but also on the given performance index. In Section 9.2 methods will be presented that likewise use the optimal controllers of the isolated subsystems but do not restrict the systems to those with neutral interactions.

7.4 DECENTRALIZED OBSERVERS

A short note is now in order concerning the application of controllers that represent decentralized feedbacks of the overall system state $\mathbf{x}$

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x} \tag{7.4.1}$$

or the subsystem state $\mathbf{x}_i$

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i \tag{7.4.2}$$

(compare, for example, eqns (6.3.2), (6.3.7) or (7.3.5)). In practical situations the complete state is not accessible and only some outputs $\mathbf{y}_i$ can be measured. For the controllers (7.4.1) or (7.4.2) the state $\mathbf{x}$ or $\mathbf{x}_i$, respectively, has to be reconstructed. It will now be seen that the state reconstruction gives rise to considerable difficulties since the decentralized structure of the feedback must not be destroyed by the observer.

Reconstruction of the Overall System State x

The first observation problem occurs if the controller (7.4.1) is to be applied. For each control station an estimate $\hat{\mathbf{x}}$ of the whole state vector $\mathbf{x}$ must be available. If the classical observer theory was applied (Section 2.2), the observer would consist of the plant model with a feedback of the estimation error $\mathbf{y} - \hat{\mathbf{y}}$ (Figure 2.3). Even if the feedback matrix $\mathbf{E}_b$ is block diagonal, the controller, which consists of the observer and the decentralized feedback (7.4.1), is not decentralized but centralized, because $\mathbf{u}_i$ is related to $\mathbf{y}_j$ $(i \neq j)$ via the plant model within the observer.

An alternative way is to build separate observers O_i at each channel $(\mathbf{u}_i, \mathbf{y}_i)$ (Figure 7.1). Here, severe difficulties arise owing to the fact that

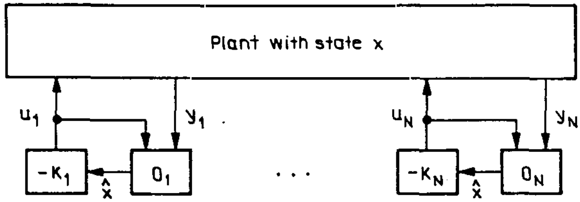


Figure 7.1 Decentralized observation of the overall system state $\mathbf{x}(t)$

any observer can be designed only if the dynamics of the whole system to be observed is known. For each observer in the structure of Figure 7.1 the system under consideration consists of the plant as well as the observer at all the other channels. In principle, each observer has to reconstruct not only the plant state **x**, but the states of all other observers O_j ($j \neq i$) too. This phenomenon is called *second guessing*. As a consequence, the dynamical order of O_i has to be made sufficiently large. If the point of view is moved now from O_i to another observer O_k it becomes clear that the observation problem for O_k is related to the large state vector of the observer O_i. The order of the observer O_k has to be extended accordingly. It this process proceeds, no upper bound on the dynamical order of the observers can be found. Furthermore, the separation theorem (Theorem 2.8) no longer holds.

A way out of this situation is the sequential design of the observers. Then, the system to be observed consists of the plant in connection with the control stations that have already been applied. In each step a classical observation problem has to be solved. The separation theorem holds for each step. However, this design method is, for really large systems, only of theoretical importance since the dynamical order of the system to be observed and, thus, the observer to be built increases rapidly from one step to the next. It can be used practically only if the observation problem can be reasonably solved for low-order approximate models of the system to be observed. This is possible if the interactions within the system are weak.

Reconstruction of the Subsystem State $\mathbf{x}_i$

The second observation problem has to be solved if the controller (7.4.2) were to be applied. At each channel $(\mathbf{u}_i, \mathbf{y}_i)$ only an estimate $\hat{\mathbf{x}}_i$ of the state $\mathbf{x}_i$ of the subsystem has to be made available by the observer O_i (Figure 7.2).

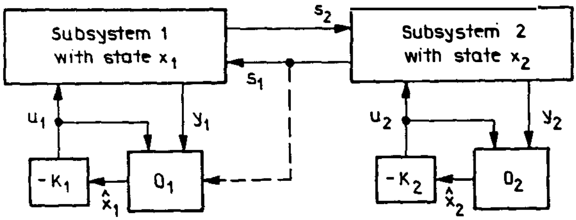


Figure 7.2 Decentralized observation of the subsystem state $\mathbf{x}_i(t)$

A solution to the problem of reconstructing the subsystem state $\mathbf{x}_i$ from knowledge of the subsystem input $\mathbf{u}_i$ and output $\mathbf{y}_i$ exists only under restrictive conditions. Note that the subsystem to be observed has the interconnection input $\mathbf{s}_i$ which is unknown for the observer O_i. Therefore, the reconstruction of $\mathbf{x}_i$ includes the reconstruction of $\mathbf{s}_i$. Roughly speaking, this is possible only if the available information $\mathbf{u}_i$, $\mathbf{y}_i$ is 'large enough'.

The problem becomes simple if in addition to $\mathbf{y}_i$ the interconnection input $\mathbf{s}_i$ can be measured as well (dashed line in Figure 7.2). This is possible, for example, in power systems (Example 3.5) where the interconnection input $s_i = f$ is measurable in all parts of the net. Then, the observation problem for each channel is the classical one. It can be solved by means of Theorem 2.7. Moreover, it can be shown that the separation Theorem 2.8 does hold for the overall decentralized control system. That is, the eigenvalues of the closed-loop system consist of those of the system with state feedback (7.4.2) and of the observers.

A second method of solving this observation problem will be outlined in Chapter 9, where the controller design and, in extension to this, the observer design are carried out for isolated subsystems. Then the separate problems of observing the subsystem states are classical even if $\mathbf{s}_i$ is not measurable.

BIBLIOGRAPHICAL NOTES

The severe difference between Problem 7.1 and the similar problem for centralized control was highlighted by Witsenhausen (1968) who gave a simple example showing that the solution to a problem similar to Problem 7.1 is represented by a non-linear function $\mathbf{u}_i(\mathbf{y}_i)$. Methods for solving Problem 7.2 have been given by Hassan and Singh (1978) and surveyed by Singh and Titli (1978). They derived the two-point-boundary-value problem and developed a three-level algorithm for determining the solution to this problem.

Problem 7.3 was derived and solved for optimal output feedback by Levine and Athans (1970) and extended to decentralized control by Košut (1970). Levine and Athans also derived eqn (7.2.1), which has been given here without proof. A more general consideration of derivatives of the form (7.2.1) can be found in the work by Berger (1976).

Several search procedures have been developed for the determination of the optimal control law from the optimality conditions presented in Theorem 7.1, for example by Levine and Athans (1970) or Naeije *et al.* (1973). Algorithm 7.1 is due to Geromel and Bernussou (1979a) who also gave the initialization procedure summarized in Algorithm 7.2. An optimal decentralized frequency–power controller for a power system of order 119 has been designed by Davison and Tripathi (1978).

Davison and Gesing (1979) proposed a sequential optimization procedure to solve Problem 7.3. One control station at a time is designed by means of the **LQ** principle as known from centralized control (Section 2.4) while the other control stations remain unchanged. In the first N steps the control stations are designed and applied sequentially, whereas all further steps serve to improve the ith control station for the plant with all other $N-1$ controllers attached. The procedure stops if the difference between the old and the new controller is small.

The relation between the optimality of the decentralized controller of the overall system and the optimality of the control stations for the isolated subsystems has been intensively investigated. For example, Özgüner (1975a) proved that for each stabilizing decentralized controller there exist weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ such that the decentralized controller is the optimal controller of the overall system. Unlike the results presented in Section 7.3 Özgüner used different performance criteria for the subsystems and the overall system. Theorems 7.2 and 7.3 have been derived from results presented by Sundareshan (1977a).

All these investigations concern the evaluation of the system behaviour in terms of a scalar performance index I. The more general problem of deriving conditions on the subsystem behaviour which ensure the practically relevant requirements of disturbance rejection, command following and stability within the overall closed-loop system is lacking a solution. Looze *et al.* (1982) and Medanic *et al.* (1989), however, have carried out some preliminary steps.

All these results are based on considerations and manipulations with the overall system model. They provide restrictions on the numerical values of the overall system and, thus, are difficult to check because of the dimensionality and uncertainties of large-scale systems. To circumvent these difficulties, two alternative approaches have been developed.

The aim of the first is to characterize larger classes of systems, for which stabilizing decentralized controllers are known to be globally optimal. Yasuda and Hirai (1980) and Ikeda *et al.* (1983) derived restrictions that concern the positions of non-zero entries of the interconnection matrix $\mathbf{A}_C$. Results of this kind lead to structural constraints on the plant for which completely decentralized design schemes as discussed in Chapters 9–12 can be used. The second approach utilizes robustness properties of the closed-loop system for proving the applicability of control stations which have been designed independently for the isolated subsystems. This approach will be presented in Chapter 9.

The decentralized observation problem was considered very early on. Aoki and Li (1973) described a manifold $\mathbf{X}(t)$ in which the overall system state $\mathbf{x}$ lies at time t. This manifold can be determined from the locally available information $\mathbf{y}_i(\bar{t})$, $\mathbf{u}_i(\bar{t})$ $(0 < \bar{t} < t)$ at time t. Fujita (1974) and Yoshikawa and Kobayashi (1975) derived necessary and sufficient conditions on the subsystems under which observers can be designed, which reconstruct $\mathbf{x}_i$ (and $\mathbf{s}_i$) from $\mathbf{y}_i$ and $\mathbf{u}_i$ for unknown interconnection input $\mathbf{s}_i(t)$. Sanders *et al.* (1976) proved that the separation theorem holds for the scheme of Figure 7.2 if the interconnection inputs are measurable. They point to the fact that if no complete measurement of $\mathbf{s}_i$ is possible the observers have to exchange information (Sanders *et al.* 1977). So-called 'decentralized estimation schemes' were proposed by Šiljak (1978) and Mahmoud and Singh (1981b), but these are classical observers for which only the error feedback matrix $\mathbf{E}_b$ is block diagonal (Figure 2.3) or where the plant model has been decomposed into the subsystems and the interaction relation (estimator with 'multilevel structure'). The use of decentralized controllers which include observers in each control station have been investigated, for example, by Bachmann and Konik (1984) and Kuhn (1985). A detailed discussion of decentralized observation and some approximate solutions, which lead to completely decentralized controllers, can be found in the monograph by Litz (1983).

8

Stability Analysis of Interconnected Systems

8.1 THE COMPOSITE-SYSTEM METHOD FOR STABILITY ANALYSIS

This and the succeeding chapters are devoted to the decentralization of the analytical and design tasks. Unlike the methods presented so far, the methods that will now be derived do not use a description of the overall system. They do not even assume that such a model is available. Although the development of these methods may start with manipulations including the overall system and the whole decentralized controller, all the analytical and design steps refer merely to precise or approximate models of the subsystems or to considerably simplified overall system descriptions.

In the field of stability analysis, the 'composite-system method' has been developed as a general framework for deriving stability criteria (Figure 8.1). The method utilizes the natural or heuristically introduced

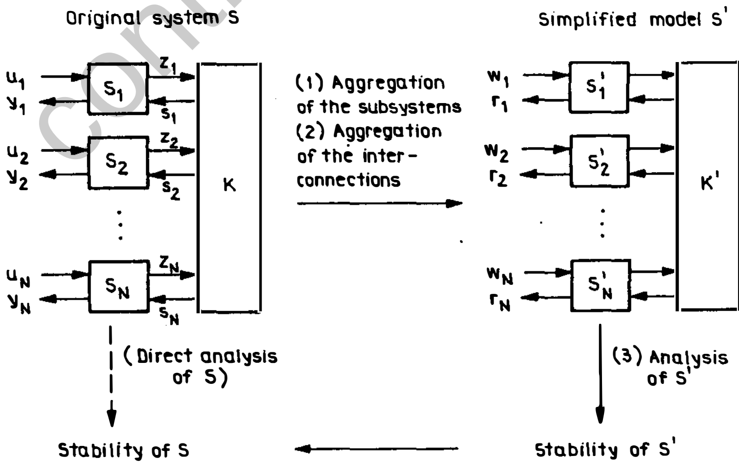


Figure 8.1 'Composite-system method' for stability analysis

decomposition of the given system S into different subsystems S_i (3.1.4) and an interconnection relation (3.1.6) denoted by K.

Two ideas provide the motivation for this method. The first is given by the fact that if all subsystems S_i are stable then the overall system S is stable for vanishing interactions ($\mathbf{L} = \mathbf{0}$). Owing to the continuity of the relation between the overall system eigenvalues and the entries of the interconnection matrix $\mathbf{L}$, the system S can be expected to be stable also for 'small' interactions. That is, the composite-system method has been elaborated to tackle the following problem.

Problem 8.1

Suppose that the isolated subsystems S_i are stable. Under what conditions on the interaction relation K does the stability of the subsystems guarantee the stability of the overall system S?

The second idea refers to information about the system, which is necessary for answering this question. Since the dimensionality of the system equations and the uncertainties of the model make the use of a complete model questionable, the stability analysis should be carried out by means of aggregate models S_i' of S_i and K' of K (Figure 8.1).

Both ideas together bring about conceptual simplifications because the stability analysis of the large-scale system S is replaced by the analysis of the low-order linear model S', which is very simple indeed. Moreover, the subsystems and interactions have to be known only to the extent that aggregate models S_i' and K' can be set up.

The following algorithm provides the framework for the stability tests which will be derived in the next sections.

Algorithm 8.1 (*Stability analysis of interconnected systems*)

Given: System composed of subsystems S_i ($i = 1, 2, \ldots, N$) and an interaction relation K.

1. Check the stability of the subsystems S_i and determine approximate models S_i' of all subsystems S_i. If not all subsystems are stable, stop (the method is not applicable).
2. Determine an aggregate interconnection relation K'.
3. Combine the subsystems S' via the interconnection relation K' to get the aggregate model S'.
4. Prove the stability of S'.

Result: If S' is stable then the original system S is known to be stable.

In order to refine this algorithm the following questions arise:

- Which subsystem properties are important for testing the overall system stability and by what approximate models S_i' are they represented?
- In what way can the interaction relation be aggregated so that S_i' can be coupled via K' to an aggregate overall system S'?
- Under what conditions is the aggregate system S' stable?
- Under what conditions does the stability of S' imply the stability of S?

The following sections provide different answers to these questions.

8.2 STABILITY ANALYSIS WITH SCALAR LYAPUNOV FUNCTIONS

The first method for analysing the stability of interconnected systems uses Lyapunov functions $v_i(\mathbf{x}_i)$ of the subsystems as a means to determine the simplified subsystem models S_i' (Figure 8.1). Consider the autonomous system ($\mathbf{u} = \mathbf{0}$) which is described by the interconnection-oriented model (3.1.4), (3.1.5):

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}_i\mathbf{x}_i + \mathbf{E}_i\mathbf{s}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{z}_i &= \mathbf{C}_{zi}\mathbf{x}_i \end{aligned} \tag{8.2.1}$$

and

$$\mathbf{s} = \mathbf{L}\mathbf{z}. \tag{8.2.2}$$

The system has a unique equilibrium point $\mathbf{x} = \mathbf{0}$ whose stability should be investigated. The isolated subsystems

$$\dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \tag{8.2.3}$$

are assumed to be stable, that is all eigenvalues of the matrices $\mathbf{A}_i$ ($i = 1, 2, \ldots, N$) have negative real parts. Then there exist Lyapunov functions

$$v_i(\mathbf{x}_i) = \mathbf{x}_i'\mathbf{P}_i\mathbf{x}_i \tag{8.2.4}$$

for all subsystems with $\mathbf{P}_i$ a symmetric positive definite matrix. Such a matrix can be found by solving the Lyapunov equation

$$\mathbf{A}_i'\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_i = -\mathbf{Q}_i$$

for some symmetric positive definite matrix $\mathbf{Q}_i$. The functions $v_i(\mathbf{x}_i)$ are positive definite

$$v_i > 0 \quad \text{for } \mathbf{x}_i \neq \mathbf{0} \tag{8.2.5}$$

and have a non-positive definite derivative along the trajectory of the isolated subsystem (8.2.3)

$$\dot{v}_i|_{SS} = \frac{\mathrm{d}v_i}{\mathrm{d}t} = \left(\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right)' \dot{\mathbf{x}}_i \leqslant 0. \tag{8.2.6}$$

The Lyapunov functions $v_i(\mathbf{x}_i)$ describe the stability property of the subsystems and should be used to derive the global model S_i'. To this end, four constants $\tilde{c}_{ij}$ $(i = 1, 2, \ldots, N;\ j = 1, \ldots, 4)$ are determined for each subsystem such that the following inequalities hold

$$\tilde{c}_{i1}\|\mathbf{x}_i\|^2 \leqslant v_i(\mathbf{x}_i) \leqslant \tilde{c}_{i2}\|\mathbf{x}_i\|^2 \tag{8.2.7}$$

$$\dot{v}_i|_{SS} \leqslant -\tilde{c}_{i3}\|\mathbf{x}_i\|^2 \tag{8.2.8}$$

$$\left\|\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right\| \leqslant \tilde{c}_{i4}\|\mathbf{x}_i\|. \tag{8.2.9}$$

Appropriate values for $\tilde{c}_{ij}$ can be derived from eqns (8.2.4)–(8.2.6) as follows. From

$$\lambda_{\min}[\mathbf{P}_i]\ \|\mathbf{x}_i\|^2 \leqslant v_i \leqslant \lambda_{\max}[\mathbf{P}_i]\ \|\mathbf{x}_i\|^2$$

and

$$\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i} = 2\mathbf{P}_i\mathbf{x}_i$$

the constants $\tilde{c}_{i1}$, $\tilde{c}_{i2}$ and $\tilde{c}_{i4}$ follow:

$$\tilde{c}_{i1} = \lambda_{\min}[\mathbf{P}_i] \qquad \tilde{c}_{i2} = \lambda_{\max}[\mathbf{P}_i] \qquad \tilde{c}_{i4} = 2\lambda_{\max}[\mathbf{P}_i] \tag{8.2.10}$$

(cf. conditions (A1.28) and (A1.29)). The derivative (8.2.6) of v_i can be written as

$$\begin{aligned}\dot{v}_i|_{SS} = \left(\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right)' \dot{\mathbf{x}}_i &= \mathbf{x}_i'(\mathbf{P}_i\mathbf{A}_i + \mathbf{A}_i'\mathbf{P}_i)\mathbf{x}_i = -\mathbf{x}_i'\mathbf{Q}_i\mathbf{x}_i \\ &\leqslant -\lambda_{\min}[\mathbf{Q}_i]\ \|\mathbf{x}_i\|^2.\end{aligned}$$

Hence, eqn (8.2.8) holds with

$$\tilde{c}_{i3} = \lambda_{\min}[\mathbf{Q}_i]. \tag{8.2.11}$$

Two further constants b_{i1} and b_{i2} have to be determined for all subsystems (8.2.1) so that

$$\begin{aligned}\|\mathbf{E}_i\mathbf{s}_i\| &\leqslant b_{i1}\|\mathbf{s}_i\| \\ \|\mathbf{z}_i\| &\leqslant b_{i2}\|\mathbf{x}_i\|\end{aligned} \tag{8.2.12}$$

hold. Possible values are

$$b_{i1} = \| \mathbf{E}_i \| \qquad b_{i2} = \| \mathbf{C}_{zi} \|. \tag{8.2.13}$$

The interconnection relation (8.2.2) is aggregated as

$$\| \mathbf{s}_i \| \leqslant \sum_{j=1}^{N} l_{ij} \| \mathbf{z}_j \| \tag{8.2.14}$$

with

$$l_{ij} = \| \mathbf{L}_{ij} \|. \tag{8.2.15}$$

The stability of the interconnected system can be analysed with $\tilde{c}_{ij}$, b_{i1} and b_{i2} as information about the subsystems and l_{ij} about the interactions. This will be done by investigating the question of under what conditions there are positive constants $a_i > 0$ such that the sum

$$v(\mathbf{x}) = \sum_{i=1}^{N} a_i v_i(\mathbf{x}_i) \tag{8.2.16}$$

is a Lyapunov function of the overall system (8.2.1) and (8.2.2). Obviously, the function $v(\mathbf{x})$ is positive for $\mathbf{x} \neq \mathbf{0}$ and, thus, can be used as a tentative Lyapunov function, that is as a function for which the property

$$\dot{v}\,|_{\mathrm{OS}} = \left(\frac{\mathrm{d}v}{\mathrm{d}\mathbf{x}}\right)' \dot{\mathbf{x}} \leqslant 0$$

remains to be proved. If this proof is successful, $v(\mathbf{x})$ defined in eqn (8.2.16) is a Lyapunov function of the overall system (8.2.1) and (8.2.2) and thus the overall system is stable.

The time derivative of $v(\mathbf{x})$ along the trajectory of the overall system (8.2.1) and (8.2.2) can be written as

$$\dot{v}(\mathbf{x})\,|_{\mathrm{OS}} = \sum_{i=1}^{N} a_i \left(\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right)' \dot{\mathbf{x}}_i = \sum_{i=1}^{N} a_i \left[\left(\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right)' \mathbf{A}_i \mathbf{x}_i + \left(\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right)' \mathbf{E}_i \mathbf{s}_i \right].$$

The first term within the parentheses represent $\dot{v}_i\,|_{\mathrm{SS}}$. Because of the inequalities (A1.28) and (A1.29) the expression above yields

$$\begin{aligned}
\dot{v}(\mathbf{x})\,|_{\mathrm{OS}} &\leqslant \sum_{i=1}^{N} a_i(-\tilde{c}_{i3} \| \mathbf{x}_i \|^2 + \tilde{c}_{i4} \| \mathbf{x}_i \| \, \| \mathbf{E}_i \mathbf{s}_i \|) \\
&\leqslant \sum_{i=1}^{N} a_i \left(-\tilde{c}_{i3} \| \mathbf{x}_i \|^2 + \tilde{c}_{i4} b_{i1} \sum_{j=1}^{N} l_{ij} b_{j2} \| \mathbf{x}_i \| \, \| \mathbf{x}_j \| \right) \\
&= -\bar{\mathbf{x}}' \operatorname{diag} a_i \, \tilde{\mathbf{S}} \, \bar{\mathbf{x}} = -\bar{\mathbf{x}}' \left(\frac{\operatorname{diag} a_i \, \tilde{\mathbf{S}} + \tilde{\mathbf{S}}' \operatorname{diag} a_i}{2} \right) \bar{\mathbf{x}}
\end{aligned} \tag{8.2.17}$$

where

$$\bar{\mathbf{x}} = (\|\mathbf{x}_1\| \ \ \|\mathbf{x}_2\| \ \ \dots \ \ \|\mathbf{x}_N\|)'$$

and

$$\begin{aligned} \tilde{\mathbf{S}} &= (\tilde{s}_{ij}) \\ \tilde{s}_{ii} &= \bar{c}_{i3} - \bar{c}_{i4} b_{i1} l_{ii} b_{i2} \\ \tilde{s}_{ij} &= -\bar{c}_{i4} b_{i1} l_{ij} b_{j2} \quad \text{for } i \neq j \end{aligned} \tag{8.2.18}$$

hold; $\dot{v}$ is negative for $\mathbf{x} \neq \mathbf{0}$ if the symmetric matrix

$$\tilde{\mathbf{S}} = (\text{diag } a_i \tilde{\mathbf{S}} + \tilde{\mathbf{S}}' \text{ diag } a_i)$$

is positive definite.

Lemma 8.1

Consider a composite system (8.2.1) and (8.2.2) with stable subsystems (8.2.3). Assume that bounds $\bar{c}_{ij}$, b_{ij} and l_{ij} of the Lyapunov functions of the isolated subsystems and the interaction relation, respectively, are known (cf. eqns (8.2.7)–(8.2.9), (8.2.12), (8.2.14)). If there exist constants a_i such that the matrix (diag a_i $\tilde{\mathbf{S}} + \tilde{\mathbf{S}}'$ diag a_i) with $\tilde{\mathbf{S}}$ from eqn (8.2.18) is positive definite then the overall system (8.2.1) and (8.2.2) is stable.

This result can be reformulated as follows. Since $\tilde{\mathbf{S}}$ is a matrix with non-positive non-diagonal elements ($\tilde{s}_{ij} \leqslant 0$), and because of the property of M-matrices stated in Theorem A1.4(iii) in Appendix 1, the condition stated in Lemma 8.1 can be replaced by an equivalent condition claiming $\tilde{\mathbf{S}}$ to be an M-matrix.

Theorem 8.1

Consider a composite system (8.2.1) and (8.2.2) with stable subsystems (8.2.3). Assume that bounds $\bar{c}_{ij}$, b_{ij} and l_{ij} of the Lyapunov functions of the isolated subsystems and the interaction relation, respectively, are known (cf. eqns (8.2.7)–(8.2.9), (8.2.12), (8.2.14)). If the matrix $\tilde{\mathbf{S}}$ in eqn (8.2.18) is an M-matrix the overall system (8.2.1) and (8.2.2) is stable.

This theorem leads to the following stability test, the main steps of which correspond to those of Algorithm 8.1.

Algorithm 8.2 (*Stability test with scalar Lyapunov functions*)

Given: Interconnected system (8.2.1) and (8.2.2).

1. Check the stability of the isolated subsystems (8.2.3) and determine Lyapunov functions (8.2.4). If one or more subsystems are unstable, stop (the method is not applicable). Determine the constants $\tilde{c}_{ij}$, b_{ij} according to eqns (8.2.10), (8.2.11), (8.2.13) for a pair $\mathbf{P}_i$, $\mathbf{Q}_i$ which satisfies the subsystem Lyapunov equation.
2. Determine l_{ij} in eqn (8.2.15).
3. Determine the matrix $\tilde{\mathbf{S}}$ from eqn (8.2.18).
4. Check whether $\tilde{\mathbf{S}}$ is an M-matrix.

Result: If $\tilde{\mathbf{S}}$ is an M-matrix then the system (8.2.1) and (8.2.2) is stable.

If instead of eqns (8.2.1) and (8.2.2) the model (3.1.16) is used the matrix $\tilde{\mathbf{S}}$ can be determined with $\| \mathbf{A}_{ij} \|$ replacing the term $b_{i1} l_{ij} b_{j2}$.

In this stability test, the function $v(\mathbf{x})$ defined in eqn (8.2.16) is used to characterize the stability properties of the original system S. Since $\dot{v}$ has to satisfy merely an inequality, the description (8.2.17), which represents an autonomous aggregate system S' on the right-hand side of Figure 8.1, is by no means a complete model of the original system. The advantage of applying the composite-system method is the simplicity of the stability test. Merely an (N, N) matrix $\tilde{\mathbf{S}}$ has to be checked for the n-dimensional system (8.2.1) and (8.2.2). Moreover, the test can be used if the system is not completely known. Only that information about the system S has to be available which makes the determination of the constants $\tilde{c}_{ij}$, b_{ij} and l_{ij} used in eqns (8.2.7)–(8.2.9), (8.2.12) and (8.2.14) possible.

8.3 STABILITY ANALYSIS WITH VECTOR LYAPUNOV FUNCTIONS

In this section, an alternative stability test is derived for the same kind of autonomous interconnected systems

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}_i \mathbf{x}_i + \mathbf{E}_i \mathbf{s}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{z}_i &= \mathbf{C}_{zi} \mathbf{x}_i \end{aligned} \tag{8.3.1}$$

and

$$\mathbf{s} = \mathbf{L}\mathbf{z}. \tag{8.3.2}$$

The main difference with the method derived in the preceding section is the fact that the Lyapunov functions $v_i(\mathbf{x}_i)$ of the subsystems will not be combined to a scalar Lyapunov function (8.2.16) but to a vector $\mathbf{v}(\mathbf{x})$ that can be interpreted as the state vector of the aggregate model S' (cf. eqn (8.3.13) below).

For the isolated subsystems, which are assumed to be stable, 'first-order' Lyapunov functions

$$v_i(\mathbf{x}_i) = \sqrt{\mathbf{x}_i'\mathbf{P}_i\mathbf{x}_i} \tag{8.3.3}$$

have to be constructed. Appropriate matrices $\mathbf{P}_i$ can be obtained as the solution of the Lyapunov equation

$$\mathbf{A}_i'\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_i = -\mathbf{Q}_i.$$

The functions $v_i(\mathbf{x}_i)$ are positive definite and have a non-positive definite derivative along the trajectory of the isolated subsystem

$$\dot{v}_i|_{ss} = \frac{\mathrm{d}v_i}{\mathrm{d}t} = \left(\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right)'\dot{\mathbf{x}}_i \leqslant \mathbf{0}. \tag{8.3.4}$$

Similarly to eqns (8.2.7)–(8.2.9) four constants c_{ij} ($i = 1, 2, \ldots, N$; $j = 1, \ldots, 4$) are determined for every subsystem such that the following inequalities hold

$$c_{i1}\,\|\mathbf{x}_i\| \leqslant v_i(\mathbf{x}_i) \leqslant c_{i2}\,\|\mathbf{x}_i\| \tag{8.3.5}$$

$$\dot{v}_i|_{ss} \leqslant -c_{i3}\,\|\mathbf{x}_i\| \tag{8.3.6}$$

$$\left\|\frac{\mathrm{d}v_i}{\mathrm{d}\mathbf{x}_i}\right\| \leqslant c_{i4}. \tag{8.3.7}$$

Appropriate values for c_{ij} can be determined analogously to eqns (8.2.10), (8.2.11)

$$\begin{aligned} c_{i1} &= \lambda_{\min}[\mathbf{P}_i] & c_{i2} &= \lambda_{\max}[\mathbf{P}_i] \\ c_{i3} &= \frac{\lambda_{\min}[\mathbf{Q}_i]}{2\sqrt{\lambda_{\max}[\mathbf{P}_i]}} & c_{i4} &= \frac{\lambda_{\max}[\mathbf{P}_i]}{\sqrt{\lambda_{\min}[\mathbf{P}_i]}} \end{aligned} \tag{8.3.8}$$

The constants b_{i1}, b_{i2} and l_{ij} are used as in Section 8.2:

$$\begin{aligned} \|\mathbf{E}_i\mathbf{s}_i\| &\leqslant b_{i1}\,\|\mathbf{s}_i\| \\ \|\mathbf{z}_i\| &\leqslant b_{i2}\,\|\mathbf{x}_i\| \end{aligned} \tag{8.3.9}$$

holds with

$$b_{i1} = \|\mathbf{E}_i\| \qquad b_{i2} = \|\mathbf{C}_{zi}\| \tag{8.3.10}$$

and the interconnection relation (2) is aggregated as

$$\|\mathbf{s}_i\| \leqslant \sum_{j=1}^{N} l_{ij}\,\|\mathbf{z}_j\| \tag{8.3.11}$$

with

$$l_{ij} = \| \mathbf{L}_{ij} \|. \tag{8.3.12}$$

With c_{ij} and b_{ij} as information about the subsystems and l_{ij} about the interactions, the stability of the system (8.3.1) and (8.3.2) can be analysed as follows. The time derivative of v_i along the trajectory of the overall system is

$$\begin{aligned}\dot{v}_i(\mathbf{x})\,|_{\mathrm{OS}} &= \frac{\mathrm{d}v_i'}{\mathrm{d}\mathbf{x}_i}\,\mathbf{A}_i\mathbf{x}_i + \frac{\mathrm{d}v_i'}{\mathrm{d}\mathbf{x}_i}\,\mathbf{E}_i\mathbf{s}_i \\ &\leqslant -c_{i3}\,\|\mathbf{x}_i\| + c_{i4}\,\|\mathbf{E}_i\mathbf{s}_i\| \\ &\leqslant -\frac{c_{i3}}{c_{i2}}\,v_i(\mathbf{x}_i) + c_{i4}b_{i1}\,\frac{l_{ij}b_{j2}}{c_{j1}}\,v_j(\mathbf{x}).\end{aligned}$$

Hence, $\dot{v}_i$ depends on v_j ($j = 1, \ldots, N$). These relations can be summarized in a single inequality if the vector

$$\mathbf{v} = (v_1\ \ v_2\ \ \ldots\ \ v_N)'$$

is used

$$\dot{\mathbf{v}} \leqslant -\mathbf{S}\mathbf{v} \qquad \mathbf{v}(0) = (v_1(\mathbf{x}_{10})\ \ \ldots\ \ v_N(\mathbf{x}_{N0}))' \tag{8.3.13}$$

where $\mathbf{S} = (s_{ij})$ and

$$\begin{aligned} s_{ii} &= \frac{c_{i3}}{c_{i2}} - c_{i4}b_{i1}\,\frac{l_{ii}b_{i2}}{c_{i1}} \\ s_{ij} &= -\,c_{i4}b_{i1}\,\frac{l_{ij}b_{j2}}{c_{j1}} \quad (i \neq j)\end{aligned} \tag{8.3.14}$$

hold. The N vector $\mathbf{v}$ is called the *vector Lyapunov function.* Although v_i has been defined in eqn (8.3.3) in terms of $\mathbf{x}_i$, it is considered in eqn (8.3.13) along the trajectory of the overall system (8.3.1) and (8.3.2) and can thus be interpreted as a function of time t: $\mathbf{v} = \mathbf{v}(t)$.

Equation (8.3.13) represents the aggregate model of the original overall system (8.3.1) and (8.3.2). As a result of the '$\leqslant$' sign, eqn (8.3.13) has for a given initial value $\mathbf{v}(0)$ a set of solutions $\mathbf{v}(t)$ rather than a unique solution. That is, eqn (8.3.13) describes not only the given overall system but the set of all systems (8.3.1) and (8.3.2) for which the functions v_i defined in eqn (8.3.3) satisfy this inequality. This set is called $\mathcal{S}$:

$$\mathcal{S} = \{\text{systems (8.3.1) and (8.3.2) for which } \mathbf{v} \text{ satisfies eqn (8.3.13)}\}. \tag{8.3.15}$$

In order to prove the stability of the original system (8.3.1) and (8.3.2), the stability of all systems within the set $\mathscr{S}$ has to be proved. This can be done by considering eqn (8.3.13) with the equality sign

$$\dot{\bar{\mathbf{v}}} = -\mathbf{S}\bar{\mathbf{v}} \qquad \bar{\mathbf{v}}(0) = (v_1(\mathbf{x}_{10}) \;\dots\; v_N(\mathbf{x}_{N0}))' \tag{8.3.16}$$

which represents a single aggregate model $\mathbf{S}'$, for which the stability analysis is evident (Figure 8.1). The justification for doing this is provided by the following definition and lemma.

Definition 8.1

An n-dimensional function

$$\mathbf{f}(\mathbf{z}) = (f_1(\mathbf{z}) \;\dots\; f_n(\mathbf{z}))'$$

of the n vector $\mathbf{z}$ is called *quasimonotonous increasing* if the inequalities

$$f_i(z_1 \;\dots\; z_{i-1} \; z_i \; z_{i+1} \;\dots\; z_n) \leqslant f_i(\bar{z}_1 \;\dots\; \bar{z}_{i-1} \; z_i \; \bar{z}_{i+1} \;\dots\; \bar{z}_n) \tag{8.3.17}$$

$(i = 1, 2, \dots, n)$ hold for all $z_j \leqslant \bar{z}_j$ $(j = 1, 2, \dots, n)$.

Lemma 8.2

Consider the solutions $\mathbf{z}(t)$ and $\bar{\mathbf{z}}(t)$ of

$$\dot{\mathbf{z}}(t) \leqslant \mathbf{f}(\mathbf{z}) \qquad \mathbf{z}(0) = \mathbf{z}_0 \tag{8.3.18}$$

or

$$\dot{\bar{\mathbf{z}}}(t) = \mathbf{f}(\bar{\mathbf{z}}) \qquad \bar{\mathbf{z}}(0) = \bar{\mathbf{z}}_0 \tag{8.3.19}$$

respectively, where $\mathbf{f}(\mathbf{z})$ is a quasimonotonous increasing function. Then the relation $\mathbf{z}_0 \leqslant \bar{\mathbf{z}}_0$ implies

$$\mathbf{z}(t) \leqslant \bar{\mathbf{z}}(t) \qquad t > 0. \tag{8.3.20}$$

The result described in this lemma is often referred to as the *comparison principle*. It provides the basis for analysing a set of functions $\mathbf{z}(t)$ which is known to satisfy the inequality (8.3.18) by means of another function $\bar{\mathbf{z}}(t)$. The function $\bar{\mathbf{z}}(t)$ is called the comparison function, the system (8.3.19) the comparison system.

For linear functions $\mathbf{f}(\mathbf{z}) = -\mathbf{S}\mathbf{z}$ the condition (8.3.17) is satisfied if

$$s_{ij} \leqslant 0 \quad (i \neq j) \tag{8.3.21}$$

hold. This requirement is met by the matrix $\mathbf{S}$ in eqn (8.3.14). According to Lemma 8.2 the model S' represented by eqn (8.3.16) describes the set

$\mathscr{S}$ defined in eqn (8.3.15) as

$$\mathscr{S} = \{\text{systems (8.3.1) and (8.3.2) for which } \mathbf{v} \leqslant \bar{\mathbf{v}} \text{ holds}\} \qquad (8.3.22)$$

where $\bar{\mathbf{v}}(t)$ is the trajectory of the comparison system (8.3.16). As seen from eqn (8.3.5), $v_i(t)$ vanishes for $t \to \infty$ if and only if $\mathbf{x}_i(t)$ vanishes. Therefore, all systems of the set $\mathscr{S}$ are stable if the comparison system (8.3.16) is stable.

Theorem 8.2

Consider a composite system (8.3.1) and (8.3.2) with stable subsystems. Assume that bounds c_{ij}, b_{ij} and l_{ij} of the Lyapunov functions (8.3.3) of the isolated subsystems and the interaction relation (8.3.2), respectively, are known (cf. eqns (8.3.8), (8.3.10), (8.3.12)). If all eigenvalues of the matrix **S** described in eqns (8.3.14) have positive real parts then the overall system (8.3.1) and (8.3.2) is stable.

Since the matrix **S** satisfies the relation (8.3.21), the condition on **S** formulated in Theorem 8.2 is equivalent to the claim of **S** being an M-matrix (cf. Theorem A1.3). Therefore, the simple tests summarized in Appendix 1 can be used to check the matrix **S**. For example, the (N, N) matrix **S** has only eigenvalues with positive real parts if and only if the so-called *Sevyastyanov–Kotelyanskii conditions*

$$(-1)^k \det\begin{pmatrix} s_{11} & \dots & s_{1k} \\ \vdots & & \vdots \\ s_{k1} & \dots & s_{kk} \end{pmatrix} > 0 \qquad (k = 1, 2, \dots, N) \qquad (8.3.23)$$

are satisfied. These conditions are similar to those known from the Hurwitz stability test.

Algorithm 8.3 (*Stability test with vector Lyapunov functions*)

Given: Interconnected system (8.3.1) and (8.3.2).

1. Check the stability of the isolated subsystems and determine Lyapunov functions (8.3.3). If one or more subsystems are unstable, stop (the method is not applicable). Determine the constants c_{ij}, b_{ij} according to eqns (8.3.8) and (8.3.10).
2. Determine l_{ij} in eqn (8.3.12).
3. Determine the matrix **S** from eqn (8.3.14).
4. Check whether **S** is an M-matrix.

Result: If **S** is an M-matrix then the system (8.3.1) and (8.3.2) is stable.

If instead of eqns (8.3.1) and (8.3.2) the model (3.1.16) is used the matrix $\mathbf{S}$ can be determined with $\| \mathbf{A}_{ij} \|$ replacing the term $b_{i1} l_{ij} b_{j2}$.

This and the preceding sections have shown that the Lyapunov functions of the subsystems can be used as a means to aggregate the information about the overall system which is relevant for stability analysis. As a result, the entire set of n first-order differential equations describing the overall system (8.3.1) and (8.3.2) is reduced to smaller sets of differential inequalities (8.3.13) or differential equations (8.3.16), both of which comprise only N lines. For the Nth-order system (8.3.16), the stability test is very simple indeed.

This reduction of the dimensionality is achieved at the expense of detailed information about the overall system. If the subsystem has a high order, the constants c_{ij} can provide only a very rough description. Hence, the stability test may fail although the overall system is stable. This fact will be investigated in more detail in Section 8.5.

8.4 STABILITY ANALYSIS WITH MULTIDIMENSIONAL COMPARISON SYSTEMS

In this section, a third method will be developed within the framework of the composite-system method. In contrast to those explained in the preceding sections, this method makes it possible to prove the I/O stability of the initial quiescent system rather than the internal stability. The system is given by the interconnection-oriented description

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i \mathbf{u}_i + \mathbf{E}_i \mathbf{s}_i \qquad \mathbf{x}_i(0) = \mathbf{0} \\ \mathbf{y}_i &= \mathbf{C}_i \mathbf{x}_i \\ \mathbf{z}_i &= \mathbf{C}_{zi} \mathbf{x}_i \end{aligned} \tag{8.4.1}$$

and

$$\mathbf{s} = \mathbf{L}\mathbf{z}. \tag{8.4.2}$$

As in Section 8.3, comparison functions are used as boundaries of signals within the system (8.4.1) and (8.4.2). However, since the I/O behaviour of the system (8.4.1) and (8.4.2) has to be described, the comparison functions are generated as outputs of multi-input multi-output systems

$$\mathbf{r}(t) = \mathbf{V}(t) * \mathbf{w}(t) \tag{8.4.3}$$

rather than as the free motion of autonomous systems (8.3.16). In eqn (8.4.3), the asterisk denotes the convolution operation (cf. Section 2.1). Since $\mathbf{r}(t)$ should be larger than the original system output, the notion of the comparison system came into use also for the systems (8.4.3).

Definition 8.2

Consider a linear system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \mathbf{x}(0) = \mathbf{0}\\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}.\end{aligned} \tag{8.4.4}$$

An auxiliary system (8.4.3) with $\dim \mathbf{w} = \dim \mathbf{u}$ and $\dim \mathbf{r} = \dim \mathbf{y}$ is called the *comparison system* of the original system (8.4.4) if

$$\mathbf{r}(t) = \mathbf{V}(t) * |\mathbf{u}(t)| \geqslant |\mathbf{y}(t)| \tag{8.4.5}$$

holds for arbitrary but bounded functions $\mathbf{u}(t)$.

Note that the $|\,.\,|$ and $\geqslant$ signs apply to every element of vectors or matrices. That is, eqn (8.4.5) is identical to

$$r_i(t) = \sum_{j=1}^{\dim \mathbf{u}} v_{ij} * |u_j| \geqslant |y_i| \qquad (i = 1, 2, \ldots, \dim \mathbf{y}).$$

Equation (8.4.5) implies $\mathbf{r}(t) \geqslant \mathbf{0}$. Hence, the impulse response matrix of the comparison system possesses the property

$$\mathbf{V}(t) \geqslant \mathbf{0} \quad \text{for all } t. \tag{8.4.6}$$

The question of how to determine a comparison system has the following answer, which is a direct consequence of eqn (8.4.5).

Lemma 8.3

The system (8.4.3) is a comparison system of the linear system (8.3.4) if and only if the impulse response matrix $\mathbf{V}(t)$ satisfies the relation

$$\mathbf{V}(t) \geqslant |\mathbf{G}(t)| \quad \text{for all } t \tag{8.4.7}$$

where

$$\mathbf{G}(t) = \mathbf{D}\delta(t) - \mathbf{C}\exp(\mathbf{A}t)\mathbf{B} \tag{8.4.8}$$

is the impulse response matrix of the original system (8.4.4).

In order to investigate the stability of the composite system (8.4.1) and (8.4.2), comparison systems have to be determined which provide upper bounds of the I/O behaviour of all subsystems

$$\begin{aligned}\mathbf{r}_{yi}(t) &= \mathbf{V}_{yui} * |\mathbf{u}_i| + \mathbf{V}_{ysi} * |\mathbf{s}_i| \geqslant |\mathbf{y}_i|\\ \mathbf{r}_{zi}(t) &= \mathbf{V}_{zui} * |\mathbf{u}_i| + \mathbf{V}_{zsi} * |\mathbf{s}_i| \geqslant |\mathbf{z}_i|\end{aligned} \tag{8.4.9}$$

($i = 1, \ldots, N$). Lemma 8.3 says that eqn (8.4.9) holds with

$$\begin{aligned} \mathbf{V}_{yui}(t) &\geqslant |\mathbf{C}_i \exp(\mathbf{A}_i t)\mathbf{B}_i| \\ \mathbf{V}_{ysi}(t) &\geqslant |\mathbf{C}_i \exp(\mathbf{A}_i t)\mathbf{E}_i| \\ \mathbf{V}_{zui}(t) &\geqslant |\mathbf{C}_{zi} \exp(\mathbf{A}_i t)\mathbf{B}_i| \\ \mathbf{V}_{zsi}(t) &\geqslant |\mathbf{C}_{zi} \exp(\mathbf{A}_i t)\mathbf{E}_i|. \end{aligned} \tag{8.4.10}$$

Equation (8.4.10) is used with the equality sign if the comparison systems (8.4.9) have to be determined for a given subsystem (8.4.1). The interconnection relation (8.4.2) can be estimated by a static comparison system

$$\mathbf{r}_s(t) = \bar{\mathbf{L}}\,|\mathbf{z}(t)| \geqslant |\mathbf{s}(t)| \tag{8.4.11}$$

where the constant matrix $\bar{\mathbf{L}}$ satisfies the relation

$$\bar{\mathbf{L}} \geqslant |\mathbf{L}|. \tag{8.4.12}$$

A comparison system for the overall system can be derived by combining the comparison systems described by eqns (8.4.9) and (8.4.11). Since for the vectors $\mathbf{y}$, $\mathbf{u}$, $\mathbf{s}$ and $\mathbf{z}$ the relations $|\mathbf{y}| = (|\mathbf{y}_1| \ \ldots \ |\mathbf{y}_N|)'$ etc. hold, eqns (8.4.9) and (8.4.11) yield

$$\begin{aligned} |\mathbf{y}(t)| &\leqslant \operatorname{diag} \mathbf{V}_{yui} * |\mathbf{u}| + \operatorname{diag} \mathbf{V}_{ysi} * \bar{\mathbf{L}}\,|\mathbf{z}| \\ |\mathbf{z}(t)| &\leqslant \operatorname{diag} \mathbf{V}_{zui} * |\mathbf{u}| + \operatorname{diag} \mathbf{V}_{zsi} * \bar{\mathbf{L}}\,|\mathbf{z}|. \end{aligned} \tag{8.4.13}$$

The second equation represents $|\mathbf{z}|$ in terms of $|\mathbf{u}|$ and $|\mathbf{z}|$ itself. An explicit statement of $|\mathbf{z}|$ in terms of $|\mathbf{u}|$ can be obtained by means of the following lemma.

Lemma 8.4

Consider functions $\mathbf{r}(t)$ and $\bar{\mathbf{r}}(t)$ that satisfy the relations

$$\mathbf{r}(t) \leqslant \mathbf{V}_1 * \mathbf{r} + \mathbf{V}_2 * |\mathbf{u}| \tag{8.4.14}$$

or

$$\bar{\mathbf{r}}(t) = \bar{\mathbf{V}} * |\mathbf{u}| \quad \text{with } \bar{\mathbf{V}}(t) = \mathbf{V}_2 + \mathbf{V}_1 * \bar{\mathbf{V}} \tag{8.4.15}$$

respectively, where $\mathbf{V}_i = \mathbf{K}_i\delta(t) + \tilde{\mathbf{V}}_i(t) \geqslant 0$ has the distributive part $\mathbf{K}_i\delta(t)$ and the piecewise continuous part $\tilde{\mathbf{V}}_i(t)$. Then the relation

$$\mathbf{r}(t) \leqslant \bar{\mathbf{r}}(t) \tag{8.4.16}$$

holds for arbitrary bounded functions $\mathbf{u}(t)$ if and only if the matrix $\mathbf{P} = \mathbf{I} - \mathbf{K}_1\mathbf{K}_2$ is an M-matrix.

This lemma can be called the *comparison principle for I/O systems.* Similar to Lemma 8.2 it says that $\bar{\mathbf{r}}(t)$, which is obtained from eqn (8.4.15), is not less than any solution of the inequality (8.4.14). However, whereas Lemma 8.2 referred to the free motion of autonomous systems, Lemma 8.4 deals with I/O descriptions.

Since the subsystems (8.4.1) are assumed to have no direct throughput, the impulse response matrices of the comparison systems (8.4.9) have no distributive parts. Then, according to Lemma 8.3, the signal

$$\bar{\mathbf{r}}_z(t) = \bar{\mathbf{V}}_{zu} * |\mathbf{u}| \tag{8.4.17}$$

with

$$\bar{\mathbf{V}}_{zu}(t) = \operatorname{diag} \mathbf{V}_{zui} + \operatorname{diag} \mathbf{V}_{zsi} \bar{\mathbf{L}} * \bar{\mathbf{V}}_{zu} \tag{8.4.18}$$

is known to be larger than $|\mathbf{z}|$:

$$\bar{\mathbf{r}}_z(t) \geqslant |\mathbf{z}(t)|. \tag{8.4.19}$$

Therefore,

$$\bar{\mathbf{r}}_y(t) = \bar{\mathbf{V}}(t) * |\mathbf{u}(t)| \geqslant |\mathbf{y}(t)| \tag{8.4.20}$$

with

$$\bar{\mathbf{V}}(t) = \operatorname{diag} \mathbf{V}_{yui} + \operatorname{diag} \mathbf{V}_{ysi} \bar{\mathbf{L}} * \bar{\mathbf{V}}_{zu} \tag{8.4.21}$$

is obtained from (8.3.13) and (8.3.17) as a comparison system of the overall system (8.4.1) and (8.4.2). That is, the original system is embedded in the set

$$\mathscr{S} = \{\text{system (8.4.1) and (8.4.2) with } |\mathbf{y}| \leqslant \bar{\mathbf{V}} * |\mathbf{u}|\}. \tag{8.4.22}$$

The I/O stability of the comparison system (8.4.20), which implies the I/O stability of the original system (8.4.1) and (8.4.2), can be investigated by means of matrices $\mathbf{M}$ that denote the integral of the non-negative impulse response matrices $\mathbf{V}(t)$ (with identical indices), for example

$$\mathbf{M}_{zsi} = \int_0^\infty \mathbf{V}_{zsi}(t)\, dt. \tag{8.4.23}$$

According to Theorem 2.10, the system (8.4.20) is stable if and only if

$$\bar{\mathbf{M}} = \int_0^\infty \bar{\mathbf{V}}(t)\, dt$$

is finite. Equations (8.4.18) and (8.4.21) yield

$$\begin{aligned} \bar{\mathbf{M}} &\leqslant \operatorname{diag} \mathbf{M}_{yui} + \operatorname{diag} \mathbf{M}_{ysi} \bar{\mathbf{L}} \bar{\mathbf{M}}_{zu} \\ \bar{\mathbf{M}}_{zu} &\leqslant \operatorname{diag} \mathbf{M}_{zui} + \operatorname{diag} \mathbf{M}_{zsi} \bar{\mathbf{L}} \bar{\mathbf{M}}_{zu} \end{aligned} \tag{8.4.24}$$

and

$$(\mathbf{I} - \operatorname{diag} \mathbf{M}_{zsi}\, \bar{\mathbf{L}})\bar{\mathbf{M}}_{zu} \leqslant \operatorname{diag} \mathbf{M}_{zui}. \qquad (8.4.25)$$

Assume that all comparison systems are stable, that is $\mathbf{M}_{yui}$, $\mathbf{M}_{ysi}$, $\mathbf{M}_{zui}$ and $\mathbf{M}_{zsi}$ are finite. If the matrix

$$\mathbf{S} = (\mathbf{I} - \operatorname{diag} \mathbf{M}_{zsi}\, \bar{\mathbf{L}}) \qquad (8.4.26)$$

is an M-matrix, then $\mathbf{S}^{-1}$ exists and is non-negative, and eqn (8.4.25) yields

$$\bar{\mathbf{M}}_{zu} \leqslant (\mathbf{I} - \operatorname{diag} \mathbf{M}_{zsi}\, \bar{\mathbf{L}})^{-1} \operatorname{diag} \mathbf{M}_{zui}$$

(cf. Theorem A1.3). Hence, $\bar{\mathbf{M}}$ is finite, which proves the stability of the system (8.4.20).

Theorem 8.3

Consider a composite system (8.4.1) and (8.4.2) with stable subsystems. Assume that stable comparison systems (8.4.9) or (8.4.11) of the subsystems (8.4.1) or the interactions (8.4.2), respectively, are known. If the matrix $\mathbf{S}$ defined in eqns (8.4.23) and (8.4.26) is an M-matrix then the overall system (8.4.1) and (8.4.2) is stable.

This stability criterion leads to the following test procedure.

Algorithm 8.4 (*Stability test with multidimensional comparison systems*)

Given: Interconnected system (8.4.1) and (8.4.2).

1. Check the I/O stability of the isolated subsystems and determine comparison systems (8.4.9). If one or more subsystems are unstable, stop (the method is not applicable).
2. Determine the matrix $\bar{\mathbf{L}}$ in eqn (8.4.12).
3. Determine the matrix $\mathbf{S}$ from eqns (8.4.23) and (8.4.26).
4. Check whether $\mathbf{S}$ is an M-matrix.

Result: If $\mathbf{S}$ is an M-matrix, the system (8.4.1) and (8.4.2) is I/O-stable.

Compared with the composite-system method illustrated in Figure 8.1, the comparison systems (8.4.9) and (8.4.11) of the subsystems and the interactions, respectively, represent the aggregate models S_i' and K'. They have been combined in the comparison system (8.4.20), which corresponds to the simplified model S'. The stability analysis of the aggregate model (8.4.20) is very simple since only the matrix of order dim $\mathbf{z}$ has to be proved to be an M-matrix (for such tests see Theorem A1.3).

8.5 GENERALIZATIONS AND EXTENSIONS

In the preceding sections, three different methods for analysing the stability of interconnected systems have been developed. These methods differ from one another regarding the information about the original system they use. Therefore, they are effective for different kinds of original systems. That is, if the stability of a given system can be proved by one method then another stability test may fail. This is the reason for the large number of stability tests that have been elaborated and published in the literature. The aim of this section is to discuss the composite-system method in more detail and to outline extensions of the criteria explained so far.

The majorization principle

The main problem of the composite-system method is how to find an aggregate model S' (steps 1, 2 and 3 of algorithm 8.1, cf. Figure 8.1) whose stability implies the stability of the original system **S** (step 4). The methods explained above give different answers to this question, because they use different characteristics of the subsystems ($\tilde{c}_{ij}$, c_{ij}, $\mathbf{V}_{ij}(t)$) and of the interactions (l_{ij}, $\bar{\mathbf{L}}$). However, under a formal examination of these methods, the tests are similar in that they embed the original system S in the set $\mathscr{S}$ of systems, which is described by the aggregate model S'. It will now be shown that these formal similarities have a deeper origin. All methods use models S' whose signals 'majorize' the signals of the original system S, that is these signals generated by the model are known to be larger than the original signals.

The majorization principle states that a given original system S should be replaced by some system S' whose signals provide upper bounds of the signals of S. Its use became clear in Section 8.4 in the definition of the comparison system, whose output $\mathbf{r}(t)$ is an upper bound of $|\mathbf{y}(t)|$

$$|\mathbf{y}_i(t)| \leqslant \mathbf{r}_i(t). \tag{8.5.1}$$

But similar inequalities have also been used in the first and the second method although this was not explicitly explained. From eqns (8.2.7) and (8.3.5) the relations

$$\| \mathbf{x}_i \| \leqslant \sqrt{\frac{v_i(\mathbf{x}_i)}{\tilde{c}_{i1}}} \tag{8.5.2}$$

$$\| \mathbf{x}_i \| \leqslant \frac{v_i(\mathbf{x}_i)}{c_{i1}} \tag{8.5.3}$$

can be derived. In all three inequalities, the signals of the original system ($\mathbf{y}_i$ or $\mathbf{x}_i$) are bounded from above by the signals of the aggregate models ($\mathbf{r}_i$ or v_i).

The majorization principle is used in steps 1 and 2 and in the interpretation of the result of step 4 of Algorithm 8.1. In the former, the majorization principle gives guidelines on how to find S_i' and K' from information about the original system S. In the latter, the majorization property is the reason why the stability of S' implies the stability of all systems of the set $\mathscr{S}$ and, thus, of the original system S.

The comparison principle (Lemmas 8.2 and 8.4) provide conditions under which the model S' can be obtained by applying the majorization principle to the isolated subsystems S_i and to the interconnection relation K separately and by combining the resulting models S_i' and K' to form S'.

The three methods for analysing the stability of interconnected systems which have been explained in the preceding sections are based on three different ways in which the majorization principle can be applied. In Section 8.4, multi-input multi-output systems (8.4.9) have been used as aggregate models S_i', so that S' is the comparison system (8.4.20) with the same number of inputs and outputs as the original system (8.4.1) and (8.4.2). Since the impulse response matrices $\mathbf{V}_{ij}(t)$ may have arbitrary stepwise continuous elements, their representation in the state space would lead to dynamical systems of large order. However, their property of being non-negative (cf. eqn (8.4.6)) makes the stability test of S' very simple.

In contrast, the stability test with vector Lyapunov functions uses merely first-order models S_i'

$$\begin{aligned} \dot{\bar{v}}_i &= -\frac{c_{i3}}{c_{i2}}\,\bar{v}_i + c_{i4} b_{i1} w_i \\ r_{zi} &= \frac{b_{i2}}{c_{i1}}\,\bar{v}_i \geqslant \| \mathbf{z}_i \| . \end{aligned} \tag{8.5.4}$$

Equation (8.5.4) represents a comparison system of the system (8.3.1) according to Definition 8.2. Although the aggregate model S' in eqn

(8.3.16) had been derived in a different way, this model can easily be shown to consist of the subsystem models (8.5.4). If these subsystem models (8.5.4) are coupled via the relation

$$\mathbf{w} = \hat{\mathbf{L}}\mathbf{r}_z \quad \text{with } \hat{\mathbf{L}} = (l_{ij})$$

(cf. eqns (8.3.11) and (8.3.12)), the model S' described in eqn (8.3.16) is obtained. S' is only of Nth order; it provides the upper bound (8.5.3) for the state of the original system (8.3.1) and (8.3.2).

A similar interpretation in the sense of the majorization principle can be made for the stability analysis with scalar Lyapunov functions, although the argument in Section 8.2 was to show that $v(\mathbf{x})$ is a Lyapunov function of the original system rather than an upper bound of $\mathbf{x}(t)$. This shows that the Lyapunov functions were used in Sections 8.2 and 8.3 as a means of determining majorizing aggregate models.

The majorization principle clearly shows that the aggregate models S' are not simply some approximate models of S but approximate models which have been determined so as to provide upper bounds on the motion of S. They describe sets $\mathcal{S}$ to which the original system S is known to belong. Therefore, the stability of S' implies the stability of S with certainty.

Example 8.1

The following example should illustrate the composite-system method and indicate the different fields of application of the methods described in Sections 8.2–8.4. Consider a system which consists of two subsystems (8.4.1) and (8.4.2)

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \begin{pmatrix} -0.5 & 2 \\ -2 & -0.5 \end{pmatrix}\mathbf{x}_1 + \mathbf{B}_1\mathbf{u}_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_1 \\ \mathbf{y}_1 &= \mathbf{C}_1\mathbf{x}_1 \\ z_1 &= (1 \;\; 1)\mathbf{x}_1 \end{aligned} \tag{8.5.5}$$

and

$$\begin{aligned}\dot{\mathbf{x}}_2 &= \begin{pmatrix} -0.5 & 0 \\ 0 & -0.2 \end{pmatrix}\mathbf{x}_2 + \mathbf{B}_2\mathbf{u}_2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 \\ \mathbf{y}_2 &= \mathbf{C}_2\mathbf{x}_2 \\ z_2 &= (-c \;\; 0.2)\mathbf{x}_2 \end{aligned} \tag{8.5.6}$$

which are related to one another by

$$\mathbf{s} = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix}\mathbf{z}. \tag{8.5.7}$$

First, set $c = 0.3$. Algorithm 8.2 is used for the autonomous system (8.5.5) and (8.5.6) with $v_i(\mathbf{x}_i) = \|\mathbf{x}_i\|^2$. Equations (8.2.10), (8.2.11) and (8.2.13) lead to

$$\tilde{c}_{11} = \tilde{c}_{12} = 1 \qquad \tilde{c}_{13} = -\lambda_{\min}[\mathbf{A}_1' + \mathbf{A}_1] = 1$$
$$\tilde{c}_{14} = 2 \qquad b_{11} = b_{12} = 2$$
$$\tilde{c}_{21} = \tilde{c}_{22} = 1 \qquad \tilde{c}_{23} = -\lambda_{\min}[\mathbf{A}_2' + \mathbf{A}_2] = 0.4$$
$$\tilde{c}_{24} = 2 \qquad b_{21} = 2 \qquad b_{22} = 0.36.$$

The interactions are described by eqn (8.2.14) with

$$l_{11} = l_{22} = 0 \qquad l_{12} = 1 \qquad l_{21} = |k|.$$

Hence, the test matrix $\tilde{\mathbf{S}}$ in Theorem 8.1 is

$$\tilde{\mathbf{S}} = \begin{pmatrix} 1 & -1.02 \\ -4|k| & 1 \end{pmatrix} = \mathbf{I} - \begin{pmatrix} 0 & 1.02 \\ 4|k| & 0 \end{pmatrix}. \tag{8.5.8}$$

The test for the M-matrix property can be done according to Theorem A1.5 by determining the maximum eigenvalue λ_p of the second matrix. The characteristic polynomial of this matrix leads to the eigenvalue

$$\lambda_p\left[\begin{pmatrix} 0 & 1.02 \\ 4|k| & 0 \end{pmatrix}\right] = 4.08|k|.$$

That is, the stability of the overall system is identified by the scalar Lyapunov function approach for

$$|k| < 0.245. \tag{8.5.9}$$

Algorithm 8.3 is applied to the autonomous system (8.5.5) and (8.5.6) with the Lyapunov functions $v_i(\mathbf{x}_i) = \|\mathbf{x}_i\|$, which lead to the constants

$$c_{11} = c_{12} = 1 \qquad c_{13} = \tfrac{1}{2} \qquad c_{14} = 1$$
$$c_{21} = c_{22} = 1 \qquad c_{23} = 0.2 \qquad c_{24} = 1.$$

All other constants are the same as above. The test matrix $\mathbf{S}$ is

$$\mathbf{S} = \begin{pmatrix} 0.5 & -0.51 \\ -2|k| & 0.2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix} - \begin{pmatrix} 0 & 0.51 \\ -2|k| & 0 \end{pmatrix}. \tag{8.5.10}$$

It is an M-matrix if and only if

$$\lambda_p\left[\begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0.51 \\ -2|k| & 0 \end{pmatrix}\right] < 1$$

holds. This is true for

$$|k| < 0.098. \tag{8.5.11}$$

Algorithm 8.4 necessitates the determination of comparison systems for the isolated subsystems. For the test matrix **S**, only the impulse responses $v_{zsi}(t)$ $(i = 1, 2)$ have to be determined. Subsystem 1 yields

$$v_{zs1}(t) = |2 \exp(-0.5t)\cos 2t| \tag{8.5.12}$$

and

$$v_{zs2}(t) = |-0.3 \exp(-0.5t) + 0.2 \exp(-0.2t)| \tag{8.5.13}$$

(cf. eqn (8.4.10) with the equality sign and Figure 8.2(a), (b)). The test matrix

$$\mathbf{S} = \mathbf{I} - \begin{pmatrix} 2.56 & 0 \\ 0 & 0.534 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ |k| & 0 \end{pmatrix} \tag{8.5.14}$$

is an M-matrix if

$$|k| < 0.732 \tag{8.5.15}$$

holds.

This example illustrates the method of stability analysis by means of the composite-system method. The algorithms lead to different stability conditions, which indicate the different power of these tests. The largest range of stability is indicated by the comparison system approach while the vector Lyapunov method yields the most conservative result. This comparison is, however, incomplete since the freedom in choosing the Lyapunov functions has not been exploited. Nevertheless, it shows that the different algorithms, although elaborated from the same composite-system method, yield different results.

The superiority of Algorithm 8.4 is based on the possibility of finding comparison systems that majorize the subsystems better than a first-order system (8.5.4). This can clearly be seen in Figure 8.2(a). The impulse response v_{zs1} of the comparison system (8.4.9), which is given in eqn (8.5.12), provides a smaller upper bound on the subsystem impulse response g_{zs1} than the comparison system (8.5.4), which has the impulse response

$$\bar{v}_{zs1}(t) = \frac{c_{11}b_{12}c_{14}}{c_{12}} \exp\left(-\frac{c_{13}}{c_{11}}t\right) = 4 \exp(-0.5t)$$

whose constants have been derived from the Lyapunov function $v_1(\mathbf{x}_1)$.

The application of the stability tests for uncertain original systems can be illustrated if the parameter c of subsystem 2 is assumed to be incompletely known. For $c \in [0.3, 0.4]$ the graph of the impulse response g_{zs2} lies in the region depicted in Figure 8.2(c). According to Lemma 8.3 the impulse response v_{zs2} of the comparison system (8.4.9) has to

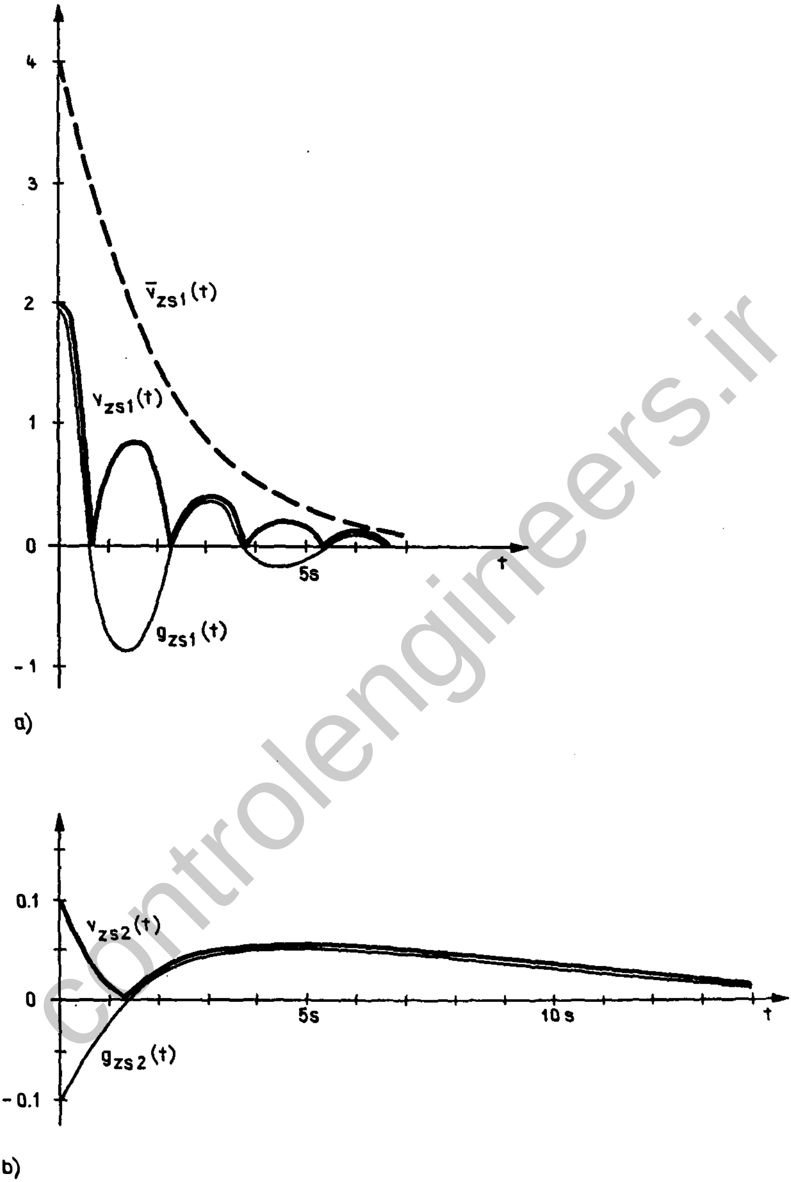


Figure 8.2 Graphical determination of comparison systems: (a) impulse response of subsystem 1 and of comparison systems; (b) impulse response of subsystem 2 for $c = 0.3$ and of the comparison system; (c) impulse responses for uncertain parameter c

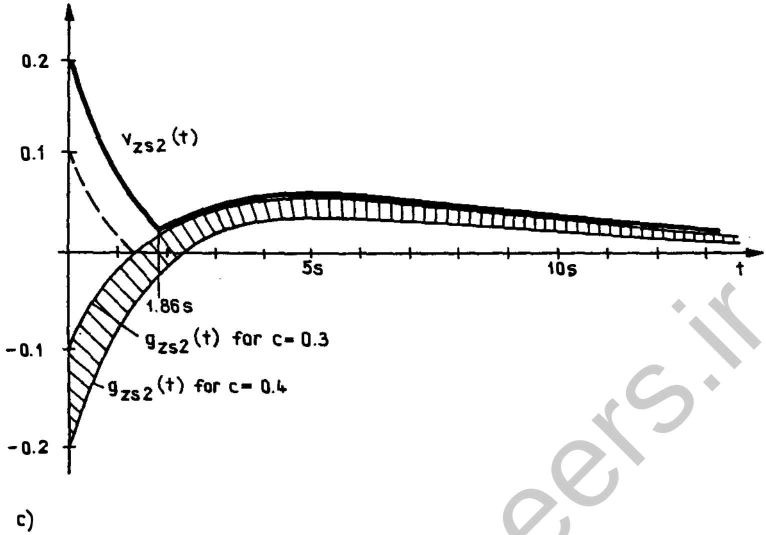


Figure 8.2 (*Continued*)

majorize g_{zs2} for all admissible c. Such a function can be found analytically as

$$v_{zs2}(t) = \begin{cases} 0.4 \exp(-0.5t) - 0.2 \exp(-0.2t) & \text{for } t \leqslant 1.86 \\ -0.3 \exp(-0.5t) + 0.2 \exp(-0.2t) & \text{for } t > 1.86. \end{cases}$$

Since only the integral M_{sz2} must be determined, the function v_{zs2} can also be graphically determined and integrated. With this new function $v_{zs2}(t)$ the matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \mathbf{I} - \begin{pmatrix} 2.56 & 0 \\ 0 & 0.62 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ |k| & 0 \end{pmatrix}.$$

The overall system is stable for all $c \in [0.3 \;\; 0.4]$ if

$$|k| < 0.63$$

holds. □

Further consequences of the majorization principle are outlined below.

Conservatism of the stability test

The conditions stated in Theorems 8.1–8.3 are sufficient but, in general, not necessary for the stability of the original system S. The conservatism of the stability criteria means that the test may fail although the system under consideration is stable.

With the notions introduced above the conservatism is based on the fact that the original system S is embedded in a set $\mathcal{S}$. The test has to prove the stability of *all* elements of $\mathcal{S}$. The larger is the set, the weaker the stability test.

Multidimensional comparison systems provide more freedom for tailoring the model S' to the given system S than the Lyapunov methods. Therefore, Algorithm 8.4 is, in general, less conservative than Algorithms 8.2 and 8.3. That is, Algorithm 8.4 may prove the stability of S although the other algorithms do not. The reason why the Lyapunov methods are, none the less, widely used in the design of decentralized controllers is given by the fact that design procedures which are based on optimization methods provide, as a byproduct, the Lyapunov functions.

The conservatism of the stability tests may be significant in practical situations. The reason for this is given by the fact that the signals of S_i' and K' are upper bounds of the norm or the absolute values of the signals of S_i and K. In particular, K' ignores the sign of the interactions. All couplings between the subsystems are considered as dangerous for system stability. The stability criteria may easily be violated by increasing the magnitude of the interconnections (cf. Example 8.5, eqns (8.5.9), (8.5.11) and (8.5.15)) or by adding further subsystems to the overall system.

The criteria claim, in principle, that all loops which occur among the subsystems have a sufficiently low gain. They are thus called *small-gain theorems*. This is clearly seen if Theorem 8.3 is used for two subsystems with $\dim \mathbf{z}_i = \dim \mathbf{s}_i = 1$. Then $\mathbf{S}$ is an M-matrix if and only if $m_{zs1} m_{zs2} l_{12} l_{21} < 1$ holds, where m_{zsi} is the static reinforcement of the comparison system of the ith subsystem between $\mathbf{s}_i$ and $\mathbf{z}_i$. This inequality states that the loop, which is formed by the two subsystems, must have a gain that is less than one.

For these theorems the field of application of the stability criteria can be outlined by means of the following classification of subsystem interactions:

- The subsystems are in *competition* if the interconnections endanger the stability of the overall system. Hence the stability of the whole system is ensured by the stability of the isolated subsystems and a properly limited magnitude of interactions.
- The subsystems are in *cooperation* if the interactions are beneficial for overall system stability. Hence, the stability of the whole system is produced by the interconnections of the possibly unstable subsystems.

Although many kinds of interactions cannot be strictly classified as purely competitive or cooperative, the discussion above shows that the stability tests consider the subsystems as competitive. They are less conservative for this class of systems. Conversely, if a given system should be decomposed prior to the application of the stability test, the whole system should be divided into competitive subsystems but with cooperative couplings left within the subsystems.

The disadvantage of the composite-system method for interpreting interactions as dangerous for stability can be overcome by tailoring the stability analysis to specific structural properties such as the symmetry of the overall system (Chapter 12).

Analysis of uncertain systems

If the majorizing auxiliary system S' is to be determined, no precise model of the original system S has to be available. The constants $\bar{c}_{ij}$, c_{ij}, b_{ij} or the matrices $\mathbf{V}_{ij}$ and $\bar{\mathbf{L}}$ have to be chosen so as to satisfy the inequalities (8.2.7)–(8.2.9), (8.2.12), (8.2.14), (8.3.5)–(8.3.7) or (8.4.9) and (8.4.11). They can be found even for incompletely known systems if eqns (8.2.10), (8.2.11), (8.2.13), (8.2.15), (8.3.8), (8.4.10) and (8.4.12) are used with '$\geq$' instead of '$=$'. These equations show which properties of S have to be assessed and how to choose the parameters of S'.

As a specific application of stability criteria to uncertain systems, the tests may be used to prove stability in spite of structural perturbations that occur in the normal operation regime of S. The set $\mathcal{S}$ has simply to be chosen so as to include all systems that result from S after such structural changes.

This extension of the stability analysis should be discussed for systems in which the disconnection of subsystems must not endanger overall system stability. The disconnection of the ith subsystem is reflected in the model by $\mathbf{z}_i = \mathbf{0}$. If corresponds to setting $l_{ji} = 0$ $(j = 1, 2, \ldots, N)$ in the interconnection matrix $\mathbf{L}$. More generally, it can be described by multiplying the elements l_{ji} by some function $e_i(t) \in \{0, 1\}$, where $e_i(t) = 1$ means that the subsystem is connected and $e_i(t) = 0$ that the subsystem is disconnected from the overall system. With

$$\mathbf{L}(e_1, \ldots, e_N) = \begin{pmatrix} e_1 l_{11} & e_2 l_{12} & \ldots & e_N l_{1N} \\ e_1 l_{21} & e_2 l_{22} & \ldots & e_N l_{2N} \\ \vdots & \vdots & & \vdots \\ e_1 l_{N1} & e_2 l_{N2} & \ldots & e_N l_{NN} \end{pmatrix} \tag{8.5.16}$$

in the interconnection relation (8.2.2) the models (8.2.1), (8.3.1) or

(8.4.1) describe the overall system with structural perturbations. The matrix $\mathbf{L}(1, \ldots, 1)$ is called the fundamental interconnection matrix since it describes all couplings between the subsystems that may be effective within the system at some time. The problem is to analyse the stability of the system (8.2.1), (8.2.2) and (8.5.16) for different combinations of $e_i \in \{0, 1\}$.

Definition 8.3

Consider the system (8.2.1) and (8.2.2) with $\mathbf{L}$ from eqn (8.5.16). The system is called *connectively stable* if it is stable for all combinations of $e_i \in \{0, 1\}$.

The extension of stability criteria derived in this chapter to tests for connective stability is very simple. If the relation (8.2.14) holds for $e_i = 1$ $(i = 1, 2, \ldots, N)$ then it holds for all e_i. Hence, the system S' (i.e. eqns (8.2.17), (8.3.16) and (8.4.20)) does majorize the system S for all e_i. All stability criteria prove the connective stability.

Corollary 8.1

If the stability of the systems (8.2.1) and (8.2.2) or (8.3.1) and (8.3.2) or (8.4.1) and (8.4.2) can be proved by means of Theorems 8.1 or 8.2 or 8.3, respectively, then these systems with $\mathbf{L}$ from eqn (8.5.16) are connectively stable.

Stability analysis of non-linear composite systems

Since the majorization principle represents the basis of the composite-system method, most of the stability criteria can be simply extended to non-linear systems. This will be outlined for a system S that consists of the subsystems

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{f}_{i1}(\mathbf{x}_i) + \mathbf{f}_{i2}(\mathbf{s}_i) \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{z}_i &= \mathbf{g}_i(\mathbf{x}_i) \end{aligned} \tag{8.5.17}$$

and interconnections

$$\mathbf{z} = \mathbf{h}(\mathbf{s}). \tag{8.5.18}$$

If Lyapunov functions $v_i(\mathbf{x}_i)$ can be constructed for the isolated subsystems (eqn (8.5.17) for $\mathbf{s}_i = \mathbf{0}$) such that constants $\bar{c}_{ij}$ or c_{ij} exist for the relations (8.2.7)–(8.2.9) or (8.3.5)–(8.3.7), respectively, and if constants b_{ij} and l_{ij} can be found so as to satisfy eqns (8.2.12) and (8.2.14), then the stability tests formulated in Theorems 8.1 and 8.2 are also valid for the non-linear system (8.5.17) and (8.5.18). Similarly, the I/O stability test given in Theorem 8.3 can be used if linear comparison systems (8.4.9) are found such that the non-linear subsystems (8.5.17) or interactions (8.5.18) are majorized.

Summary

The most important features of the stability tests for composite systems can be summarized as follows:

- The stability tests exploit the property of the overall system to consist of several subsystems. This brings about a considerable reduction in the dimensionality of the equation to be handled. This can be seen by the fact that the test matrix has only the order N or dim $\mathbf{z}$, respectively, independently of the system order.
- The stability test can be used for incompletely known systems or systems with structural perturbations. This is exemplified by the possibility of proving the connective stability of a system under structural perturbations.
- The tests are effective if the subsystems have a certain autonomy, that is if the behaviour of each subsystem depends mainly on the properties of the free subsystem and is affected by the other subsystems merely to a lower extent. The conservatism is low if the interactions are competitive.
- In the test procedures the analysis of the isolated subsystems must consider a detailed subsystem description, whereas the analysis of the whole system takes into consideration essentially an aggregate description K' of the interconnections and only the main properties of all the subsystems, which are described by S_i'. Hence, the system is analysed by means of adequate models at both levels of abstraction (Figure 8.3). The information structure of all tests that are based on Algorithm 8.1 is a multilayer structure. On the lower layer the stability of the isolated subsystems is considered and the simplified subsystem models are set up. The higher layer has the quite different job of checking overall system stability by means of the aggregate subsystem and interaction models.

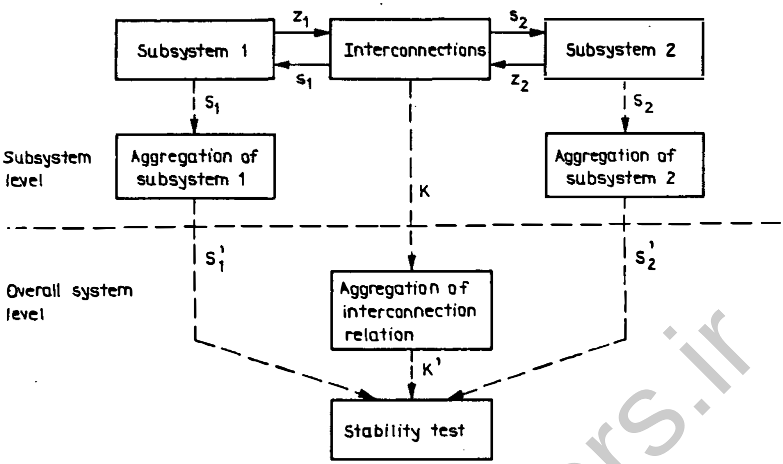


Figure 8.3 Information structure of the composite-system method

BIBLIOGRAPHICAL NOTES

There is a huge literature on the stability analysis of interconnected systems but only part of it has been exploited in the design of feedback controllers. During the last decade, methods for intrinsically non-linear systems, systems described by differential and integral equalities ('hybrid systems'), distributed parameter systems, or systems with unstable subsystems have been elaborated. Review papers and monographs have been written, for example, by Michel and Miller (1977), Bernussou and Titli (1982), and Grujić *et al.* (1987).

The composite-system method has several origins but a rapid development of stability tests for different kinds of systems began with the advent of large-scale systems in systems and control theory in the mid 1970s. Thompson (1970) and Michel and Porter (1972) were some of the first who constructed scalar Lyapunov functions for interconnected systems. Theorem 8.1 was proved for non-linear systems by Araki and Kondo (1972). Araki (1975) has produced a review of the scalar Lyapunov function approach.

Vector Lyapunov functions were introduced by Matrosov (1972) and Bellman (1962) and applied to interconnected systems by Bailey (1966). As a basis for this, the fundamentals of the theory of differential and integral inequalities were summarized by Lakshmikantham and Leela (1969) and comparison functions were investigated by Hahn (1967). Hahn's results are cited in detail in the recent monograph by Grujić *et al.* (1987), who also reviewed the current literature on the vector Lyapunov function approach. The type of aggregation introduced by using vector Lyapunov functions instead of the subsystem model for stability analysis was investigated by Grujić *et al.* (1976). A generalization of the vector Lyapunov function approach, which is based on the overlapping decomposition of the given system, was reported by Ikeda and Šiljak (1981).

Michel (1977) made a comparative study of the scalar and the vector Lyapunov function criteria. An extension of these methods to the I/O stability has been published by Willems (1976) and Araki (1978).

An alternative approach, which is mainly based on functional analytical means, interprets the subsystems and interactions as mappings of their input and output between function spaces. Upper bounds for the norm of the subsystem and interconnection operators are used as aggregate model S'. The development of such methods began with a publication by Zames (1966), which initiated different approaches by Cook (1974), Araki (1976), Lasley and Michel (1976), Moylan and Hill (1978) and Saeki *et al.* (1980). Since all these stability criteria are 'small-gain theorems', another group of researchers used the properties of dissipativeness and passivity (see Desoer and Vidyasagar 1975; Vidyasagar 1979; hill and Moylan 1980) in order to come up with less conservative tests for systems with cooperative interactions. For a common representation of small-gain and passivity criteria see the publication by Moylan and Hill (1979).

Comparison systems in the sense of Definition 8.2 were introduced by Tokumaru *et al.* (1975) as first-order systems and extended by Lunze (1980b, 1983c) to multidimensional high-order systems. Bitsoris (1984) investigated the possibility of using non-linear comparison systems for non-linear overall systems, but owing to the difficulties in determining such comparison systems and in analysing the stability of the non-linear aggregate model S' this method has not been pursued further.

The majorization principle as the common basis of Lyapunov methods and I/O methods was investigated by Lunze (1983c, 1984). The term 'connective stability' was introduced by Šiljak (1972) and has been used by many authors. Example 8.1 is due to Lunze (1979, 1980d).

Applications of the stability tests to large power systems are described, for example, by Pai (1981), Ribbens-Pavella and Evans (1985) and Grujić *et al.* (1987). These papers are also theoretically interesting, because they present extensions of the results explained in Sections 8.2 and 8.3 to systems with special classes of non-linearities.

9

Decentralized Control of Strongly Coupled Systems

9.1 MOTIVATION AND INFORMATION STRUCTURE OF DECENTRALIZED DESIGN

This and the next chapters are devoted to the decentralization of the design process. The control stations should be obtained as results of independent design problems, which are given by different models S_i and design aims A_i. Whereas the decentralization of the control law concerns on-line information about the state $\mathbf{x}$ and the command $\mathbf{v}$ and makes a completely independent implementation of the control stations possible, the decentralization of the process refers to *a priori* information (model S, design aim A) and supports a way in which the control stations are found independently as solutions of separate design tasks.

The motivation for the decentralization of the design process is manifold:

- If the subsystems are weakly coupled, it is reasonable to design the control stations independently using the methods for decentralized feedback control. The interactions can be ignored during the design process.
- If the subsystems are assigned to independent authorities, each control station has to be designed by the responsible authority alone on the basis of the information which is available to the local decision maker.
- If the subsystems have contradictory design aims, these aims cannot be summarized within a single global aim.
- If the plant is structurally perturbed during normal operation, the control station of a given subsystem should be designed so as to satisfy the requirements of subsystem performance and to tolerate the structural perturbations that affect this subsystem.
- If the original system is incompletely known, the uncertainties should be investigated for all subsystems independently.
- The high dimensionality of the overall system is a further reason for the decentralization of the design process.

All these items contribute to the attempt to divide the overall design task into subtasks that are independent or at least almost independent. Each practical application will usually refer not to all but to some of these motivations.

Decentralized Design as a Problem of Decision Making

In the following, the possibilities of imposing structural constraints on the information structure of the design process and the problems that arise from these constraints will be reviewed. As in Chapter 1, the control task is considered as a problem of decision making, which is posed by the plant model S and by the decision aim A. The aim A includes general properties to be reached such as the closed-loop stability as well as time-dependent aims, which are described in terms of the current values of the command signal $\mathbf{v}$ and the system state $\mathbf{x}$. S and A together comprise the *a priori* information, $\mathbf{v}$ and $\mathbf{x}$ the *a posteriori* information. The problem (S, A) has to be solved by the common effort of several relatively independent decision makers (control agents).

As illustrated by Figure 1.6(a) every kind of decentralization of the design process yields uncertainties in the separate design problems and conflicts among the solutions received by the different decision makers. For example, for the decision maker which is in charge of subsystem i and has to select the control station $\mathbf{u}_i = -\mathbf{K}_{yi}\mathbf{y}_i$, the other subsystems and their control stations are incompletely known. Note that these parts of the overall system belong to the system to be controlled by control station i. Therefore, coordination of the design activities of the decision makers or at least a test concerning the compatibility of the resulting control stations is necessary.

The question arises of which way multiperson decision theory may provide starting points for the solution of decentralized design problems. This requires the replacement of the control aim A, which is a collection of several specifications on the closed-loop system, by a scalar performance index I.

Two basic situations are known from decision theory, as follows.

Decentralized control as a dynamic team problem

If the control problem can be formulated in terms of a single global performance index $I(\mathbf{y}_1, \ldots, \mathbf{y}_N, \mathbf{u}_1, \ldots, \mathbf{u}_N)$ or if the subsystem aims I_i are in harmony and can be summarized within such a single global aim, decentralized control can be considered as a dynamic team problem. N

independent control agents have to decide which control law they want to use for their subsystems in order to minimize I. Stated in this way, the decentralized design of decentralized controllers is relevant to the theory of teams, which investigates how the team members might best achieve their common goal.

Since the effect of the control input $\mathbf{u}_i(t)$ on the performance of the overall plant depends on the input signals $\mathbf{u}_j$ $(j \neq i)$ imposed at the same moment by the other control stations, the best possible solution of the team problem can be received only after some information has been exchanged among the different control agents. Team decision theory states that this information exchange has to bring about a partially nested information structure. That is, the agents have to act in such a way, one after the other, that each agent knows completely the inputs of all agents acting before him or her as well as the resulting system behaviour. In terms of discrete-time systems, the decisions are made sequentially by means of the observation at time t, which includes the decisions made at time $t-1$. The availability of the action made at $t-1$ for decision at time t is referred to as a one-step-delay-sharing information pattern.

This result from team theory has two consequences for the design of the decentralized controller, where the decision to be made concerns the selection of the control laws. First, a nested information structure can be produced by prescribing a certain sequence in which the control stations have to be designed. Then, each decision maker acts alone on a system whose properties are fixed. The one-step-delay-sharing information pattern refers to the fact that the model $\bar{S}_i$, which describes the relevant part of the plant with input $\mathbf{u}_i$ and output $\mathbf{y}_i$, includes all control stations that have already been implemented. This model can be obtained by communicating the control laws and with them all the information about the actions of the corresponding decision maker from the preceding to the succeeding control agent or by identifying the model of the resulting plant through the ith channel $(\mathbf{u}_i, \mathbf{y}_i)$. The sequential design of decentralized controllers will be explained for the on-line tuning of PI controllers in Chapter 11. Another sequential design algorithm has already been presented in Section 6.1.

Second, the overall design problem can be decomposed into separate design problems which refer to different problems S_i and different performance indices $I(\mathbf{y}_i, \mathbf{u}_i)$, which are to be minimized with respect to S_i. The decomposition must be done in such a way that the solutions to the separate problems (I_i, S_i) comprise the solution to the global problem (I, S). This coincidence of the local with the global solution cannot be ensured by simply dividing the overall model S and the global aim I. Team decision theory has shown that it is relatively easy to find the person-by-person optimal decision rule $\mathbf{u}_i(\mathbf{y}_i)$, but that there is no simple criterion for deciding whether these rules are team optimal as well. Team problems are, in general, hard combinational problems.

To be more specific for decentralized control, a decentralization of the design is possible if the plant has a hierarchical structure and the aim is closed-loop stability. Then the control stations can be designed independently for the isolated subsystems (Section 10.1). If the overall system is symmetric, the auxiliary plant models can be set up simply by means of the subsystem models and some information about the interaction relations (Section 12.3). However, for the majority of design problems the derivation of auxiliary models S_i and aims I_i from S and I necessitates some information about the global solution. It has been shown in Section 7.3 that the global solution can be obtained by combining subsystem solutions only if the subsystem interactions are neutral. It has been shown further that the neutrality of interactions is a property which is related to the global solution and, thus, cannot be checked prior to the overall design problem. Hence, unless specific properties can be exploited, decentralized design necessitates some coordination of the separate design processes or leads to non-optimal solutions, which may be satisfactory but result in unstable closed-loop systems.

Decentralized control as a game problem

The alternative situation occurs if the subsystems are in competition with each other. The control agents have to choose the parameters of the control stations in terms of contradictory aims $I_i(\mathbf{y}_i, \mathbf{u}_i)$, which cannot be replaced by a common aim I. Game theory has shown that an 'equilibrium solution' may exist, which represents a set of control strategies $\mathbf{u}_i(\mathbf{y}_i)$ that give the best possible solution if they are used by all the decision makers.

However, for decentralized control two conceptual difficulties arise for the utilization of game-theoretic results. First, if no deterministic equilibrium exists, game theory proposes to use mixed strategies, that is different decision rules have to be used with a given probability density in order to get the best average performance index. Mixed strategies cannot be implemented as decentralized controllers since this would mean using a control law with statistically changing parameters. Second, the solution of the game problem requires knowledge of all objective functions I_i and the overall system model S. That is, the design process has a centralized rather than a decentralized information structure.

Decentralized control cannot be considered a typical game problem, but the game-theoretic approach to decentralized control shows what may happen in the 'worst case' where the control agents have no goal in common and try to counteract each other as strongly as possible.

These decision-theoretic considerations show that a complete decentralization of the design process is not possible unless the plant or the aims have specific characteristics. The design problem has to be reformulated as a problem with nested information structure or can otherwise be solved only approximately. Both ways have no completely decentralized information structure but include a certain type of coordination among the separate design problems.

Decentralization of the Design Problem

Since no complete decentralization of the design process is possible, the furthest possible decentralization is aimed at by combining the alternative ways discussed above. It leads to design procedures which are carried out at different levels of abstraction (Figure 9.1). On the subsystem level, separate design problems are solved in order to find the corresponding control stations. On the overall system level, the compatibility of the results is checked. The decentralized nature of this design scheme becomes obvious from the fact that:

- in no step a complete model of the overall system is used;
- no complete coordination of the separate design problems is made, that is no 'optimal' solution is received;
- most of the effort is made on the subsystem level using information about the corresponding subsystem.

These characteristics imply that iterative techniques, which involve communications between the control agents at every step, are avoided as far as possible. Uncertainties within the control agents' information, which are not resolved by communication, are assumed to have the worst values.

The main problem of decentralized design refers to the questions of what information about the plant has to be available for the ith control agent, and what design requirements or restrictions on the choice of the control stations have to be imposed on the design problem of the ith subsystem in order to ensure that the separately designed control stations satisfy the overall system specifications. As a general rule, the control stations obtained independently can be combined with a decentralized controller only if sufficient information about the *other* subsystems and the design aims of all the other control agents is included in the model S_i and aim A_i of the ith control agent.

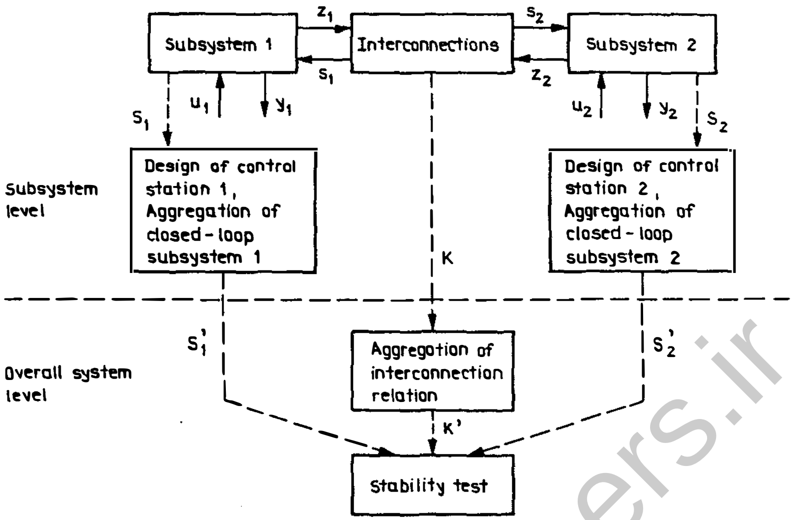


Figure 9.1 Information structure of the aggregation–decomposition method

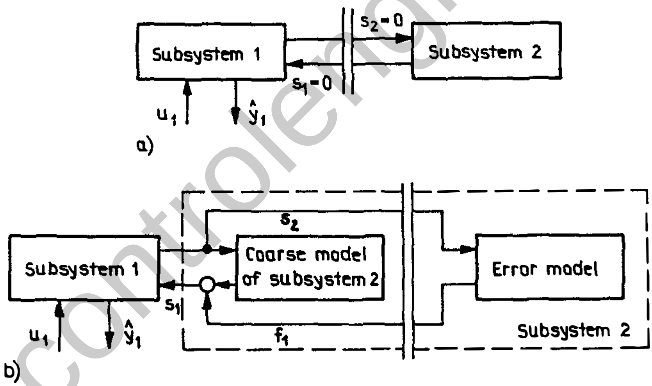


Figure 9.2 Description of the behaviour of subsystem 1: (a) weak interactions; (b) strong interactions

The principal ways of decentralizing the design tasks are depicted in Figure 9.2 for a system which consists of two subsystems.

Decentralized design for weakly coupled systems

Figure 9.2(a) shows that the task of designing control station 1 is independent of that of designing control station 2 if, during the design process, the subsystems are assumed to be decoupled ($\mathbf{s}_i = \mathbf{0}$). Then, subsystem 1 serves as the auxiliary plant of control station 1, although after the application of control station 1 to the real process, the plant of this controller includes all the subsystems as well as all the other control stations.

This rather primitive division of the overall design problem is possible whenever the overall system consists of weakly coupled subsystems. The control stations can be designed and their effect analysed in connection with the subsystem model only. That is, the design problems for all control stations are completely independent. If the interactions are sufficiently weak, the control stations will behave similarly within the interconnected system.

The main problem, which remains to be solved for this method of decentralization, is to find ways of testing the compatibility of the resulting control stations. The main part of this chapter deals with procedures in which the control stations are designed for the isolated subsystems and the stability criteria of Chapter 8 are used to analyse the closed-loop overall system. In Chapter 10, systems will be investigated whose particular kinds of coupling enable the design engineer to solve the design problems without the need for testing the compatibility of the results.

Decentralized design for strongly coupled systems

If the system cannot be decomposed into weakly coupled subsystems, the interaction of the ith subsystem with all other subsystems has to be considered while designing the ith control station. That is, the model used in the design has to include at least a coarse model of the performance of the other subsystems. As illustrated in Figure 9.2(b), the line dividing the overall model goes through subsystem 2. The coarse model simulates the 'surroundings' of subsystem 1. However, the plant description used in the design of control station 1 is by no means complete. The error model symbolizes the dependency of the signal $\mathbf{f}_1$ on $\mathbf{s}_2$, but this model is not known exactly (unless the overall system is completely known). It represents the uncertainties within subsystem 2 and the uncertain influence of control station 2 on the whole system.

The question of how to find reasonable coarse models will be answered in different ways. In Section 9.5, the overlapping decomposition is used to get weakly coupled subsystems. Here, the coarse model is obtained from the expansion and decomposition of the overall system. For PI controllers a simple modification of the subsystem model leads to a reasonable description of the subsystem under the influence of its couplings with the other subsystems (Section 11.3). For symmetric systems the coarse model is proved to be static (Section 12.1).

The Basic Algorithm of Decentralized Design

It is obvious that some kind of coordination of the independent design problems is necessary in order to ensure compatibility of the resulting control stations. Since no complete coordination in the sense of multilevel schemes (Section 1.1) is desired, this coordination should be carried out by prescribing a part of the design aims or modifying the design problems solved at the subsystem level or by testing the compatibility after the control stations have been chosen independently.

In what follows, methods for decentralized design are investigated which take into account the following practical circumstances:

- No decision maker uses a complete model of the overall system. Analytical and design tasks may exploit either precise models of a single subsystem, possibly in connection with coarse models of the adjacent subsystems, or estimations of all the subsystems and interaction relations.
- The plant is subjected to structural or parametrical uncertainties, which are caused, for example, by the disconnection of subsystems, parameter variations or changing operating conditions.

All design algorithms to be presented have the following structure.

Algorithm 9.1 (*Decentralized design of decentralized controllers*)

Given: System composed of weakly coupled subsystems; local and global design aims.

1. Design the control stations for the isolated subsystems according to local design specifications independently of one another.
2. Determine the characteristic properties of the closed-loop isolated subsystems.
3. Check whether the overall closed-loop system satisfies the global design aims.

Result: Decentralized controller.

Algorithms of this type can be developed by combining methods for designing centralized controllers (step 1) with methods for analysing interconnected systems (step 3). In the next two sections, a pole placement procedure and the **LQ** design will be used in combination with the scalar and the vector Lyapunov function approaches to stability analysis. It will be shown in what ways the stability, the stability degree and and suboptimality of the overall closed-loop system can be assessed in step 3, what local aims have to be followed in step 1 to meet these global aims, and how the design method can be extended to the design of robust decentralized controllers.

In the following the model (3.1.16) is used, but all results may be reformulated for the interconnection-oriented model (3.1.4), (3.1.6) and (3.1.12) simply by writing $\mathbf{E}_i\mathbf{L}_{ij}\mathbf{C}_{zj}$ instead of $\mathbf{A}_{ij}$.

9.2 THE AGGREGATION–DECOMPOSITION METHOD

The first method is based on the input-decentralized form (3.3.3) of the plant model

$$\begin{aligned}\dot{\mathbf{x}}_i &= \mathbf{A}_{ii}\mathbf{x}_i + \sum_{\substack{j=1\\ j\neq 1}}^{N} \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{b}_i u_i\\ \mathbf{y}_i &= \mathbf{x}_i\end{aligned} \tag{9.2.1}$$

where the single-input isolated subsystems are completely controllable. The subsystem states $\mathbf{x}_i$ are assumed to be locally measurable. A multilevel controller

$$u_i = u_i^l + u_i^g \tag{9.2.2}$$

is used, which consists of decentralized controllers

$$u_i^l = -\mathbf{k}_i'\mathbf{x}_i \tag{9.2.3}$$

and global feedback links

$$u_i^g = \sum_{\substack{j=1\\ j\neq 1}}^{N} -\mathbf{k}_{ij}'\mathbf{x}_j. \tag{9.2.4}$$

Although the controller (9.2.2)–(9.2.4) is, in principle, a centralized controller, it will be referred to as a *multilevel controller*, because its local and global feedback links are clearly distinguished and will be determined in different design steps (Figure 9.3). The following investigations hold for decentralized controllers if $\mathbf{k}_{ij}' = \mathbf{0}$ for all $i \neq j$ is used.

The closed-loop system (9.2.1)–(9.2.4) has the model

$$\dot{\mathbf{x}}_i = (\mathbf{A}_{ii} - \mathbf{b}_i\mathbf{k}_i')\mathbf{x}_i + \sum_{\substack{j=1\\ j\neq 1}}^{N} (\mathbf{A}_{ij} - \mathbf{b}_i\mathbf{k}_{ij}')\mathbf{x}_j. \tag{9.2.5}$$

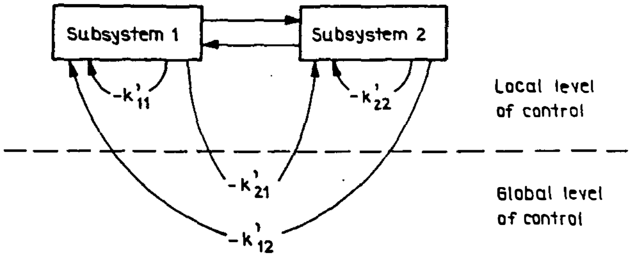


Figure 9.3 Multilevel controller

The stability of this system should be investigated by means of Algorithm 8.3. The test matrix $\mathbf{S}$ depends on the constants c_{ij}, which have to be determined by means of Lyapunov functions of the isolated subsystems (eqn (9.2.5) with $\mathbf{A}_{ij} = \mathbf{0}$). These Lyapunov functions in turn are dependent upon the controller parameters. In order to get more insight into these dependencies the subsystem state vectors are transformed according to $\tilde{\mathbf{x}}_i = \mathbf{T}_i^{-1}\mathbf{x}_i$ where $\mathbf{T}_i$ is the modal matrix of $(\mathbf{A}_{ii} - \mathbf{b}_i\mathbf{k}_i')$. Equation (9.2.5) yields

$$\dot{\tilde{\mathbf{x}}}_i = \mathbf{T}_i^{-1}(\mathbf{A}_{ii} - \mathbf{b}_i\mathbf{k}_i')\mathbf{T}_i\tilde{\mathbf{x}}_i + \sum_{\substack{j=1\\ j\neq 1}}^{N} \mathbf{T}_i^{-1}(\mathbf{A}_{ij} - \mathbf{b}_i\mathbf{k}_{ij}')\mathbf{T}_j\tilde{\mathbf{x}}_j$$

$$= \tilde{\mathbf{A}}_{ii}\tilde{\mathbf{x}}_i + \sum_{\substack{j=1\\ j\neq 1}}^{N} (\tilde{\mathbf{A}}_{ij} - \tilde{\mathbf{b}}_i\tilde{\mathbf{k}}_{ij}')\tilde{\mathbf{x}}_j \tag{9.2.6}$$

where

$$\tilde{\mathbf{A}}_{ij} = \mathbf{T}_i^{-1}\mathbf{A}_{ij}\mathbf{T}_j \qquad \tilde{\mathbf{b}}_i = \mathbf{T}_i^{-1}\mathbf{b}_i \qquad \tilde{\mathbf{k}}_{ij} = \mathbf{k}_{ij}\mathbf{T}_j \tag{9.2.7}$$

hold. $\tilde{\mathbf{A}}_{ii}$ is a diagonal matrix with the eigenvalues $-\lambda_{ij}$ $(j = 1, 2, \ldots, n_i)$ of the closed-loop subsystem on its main diagonal

$$\tilde{\mathbf{A}}_{ii} = \operatorname{diag} \; -\lambda_{ij}.$$

Since the isolated subsystems are assumed to be completely controllable, the parameter vector $\mathbf{k}_i$ of the ith control station can be chosen so that all closed-loop eigenvalues $-\lambda_{ij}$ are real and distinct.

In order to apply the stability criterion derived in Section 8.3, the Lyapunov function (8.3.3) is reformulated for $\tilde{\mathbf{x}}_i$ instead of $\mathbf{x}_i$

$$v_i(\tilde{\mathbf{x}}_i) = \sqrt{\tilde{\mathbf{x}}_i'\tilde{\mathbf{P}}_i\tilde{\mathbf{x}}_i}$$

and used with $\tilde{\mathbf{P}}_i = \mathbf{I}$, that is

$$v_i(\tilde{\mathbf{x}}_i) = \|\tilde{\mathbf{x}}_i\|. \tag{9.2.8}$$

Since the corresponding Lyapunov equation reads as

$$\tilde{\mathbf{A}}'_{ii} + \tilde{\mathbf{A}}_{ii} = 2 \text{ diag } \lambda_{ij} = -\tilde{\mathbf{Q}}_i$$

the constants c_{ij} in eqn (8.3.8) are given by

$$c_{i1} = c_{i2} = 1 \qquad c_{i3} = \min_j \lambda_{ij} \qquad c_{i4} = 1. \tag{9.2.9}$$

Now, the matrix $\mathbf{S} = (s_{ij})$ from eqn (8.3.14) can be set up:

$$\begin{aligned} s_{ii} &= \min_j \lambda_{ij} \\ s_{ij} &= -\| \tilde{\mathbf{A}}_{ij} - \tilde{\mathbf{b}}_{ij}\tilde{\mathbf{k}}'_{ij} \| \qquad (i \neq j) \end{aligned} \tag{9.2.10}$$

where instead of $b_{i1}l_{ij}b_{j2}$ the norm of the interaction matrix has been used (cf. remark after Algorithm 8.3). Theorem 8.2 and Corollary 8.1 lead to the following stability criterion.

Theorem 9.1

Assume that the local control stations (9.2.3) have been designed such that the closed-loop isolated subsystems are stable. Then the overall closed-loop system (9.2.1)–(9.2.4) is connectively stable if the matrix $\mathbf{S}$ from eqn (9.2.10) is an M-matrix.

Obviously, the test matrix $\mathbf{S}$ depends upon the stability degree of the isolated subsystems as well as the norm of the transformed interaction matrices. Both parts are interrelated since the transformation matrix $\mathbf{T}_i$ is dependent upon the local controller $\mathbf{k}_i$.

The stability condition does not claim a certain weakness of the interactions among the subsystems but does require this property within the closed-loop system. The matrix $\mathbf{S}$ is an M-matrix if the modulus of the non-diagonal elements s_{ij} is sufficiently small. Therefore, it is reasonable to choose the global feedback links so as to make $\| \tilde{\mathbf{A}}_{ij} - \tilde{\mathbf{b}}_i\tilde{\mathbf{k}}'_{ij} \|$ as small as possible

$$\min_{\tilde{\mathbf{k}}_{ij}} \| \tilde{\mathbf{A}}_{ij} - \tilde{\mathbf{b}}_{si}\tilde{\mathbf{k}}'_{ij} \|. \tag{9.2.11}$$

The problem (9.2.11) has the solution

$$\tilde{\mathbf{k}}'_{ij} = (\tilde{\mathbf{b}}'_i\tilde{\mathbf{b}}_i)^{-1}\tilde{\mathbf{b}}'_i\tilde{\mathbf{A}}_{ij} = \tilde{\mathbf{b}}_i^+\tilde{\mathbf{A}}_{ij} \tag{9.2.12}$$

where $(.)^+$ denotes the pseudoinverse (cf. Appendix 1).

If, on the other hand, the subsystems have sufficient autonomy within the plant, the global feedback links (9.2.4) are not necessary, and thus a completely decentralized controller can be used ($\tilde{\mathbf{k}}_{ij} = \mathbf{0}$).

The design steps are summarized in the following algorithms.

Algorithm 9.2 (*Aggregation–decomposition method*)

Given: Model of the plant in input-decentralized form (9.2.1); sets of eigenvalues $\{-\lambda_{ij};\ j=1,\ldots,n_i\}$ $(i=1,\ldots,N)$ of the closed-loop subsystems.

1. Design decentralized control stations (9.2.3) such that the closed-loop isolated subsystems have the prescribed set of eigenvalues.
2. Determine $\tilde{\mathbf{A}}_{ij}$ and $\tilde{\mathbf{b}}_i$ from eqn (9.2.7).
3. Determine the global feedback links according to eqn (9.2.12).
4. Check the stability of the overall closed-loop system by means of Theorem 9.1.

Result: Multilevel controller (9.2.2)–(9.2.4).

If in step 4 the stability test fails, the algorithm can be restarted with other eigenvalue prescriptions. The stability test shows that the dominant eigenvalues, which are nearest to the imaginary axis in the complex plane, have to be shifted to the left if the test is to be satisfied. Although this advice often leads to stable overall systems there is no guarantee that the overall system can be stabilized by means of this algorithm. This problem can be overcome at least for the specific class of systems described below.

Application of the Aggregation–Decomposition Method to a Specific Class of Interconnected Systems

The results presented in Theorem 9.1 will be specified for the class of systems (9.2.1) with restricted subsystem interactions. The restrictions become clear if the subsystem models are transformed into Luenberger form, that is if

$$\mathbf{A}_{ii} = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ -c_{in_i} & -c_{i,n_i-1} & \ldots & -c_{i1} \end{pmatrix} \qquad \mathbf{b}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \tag{9.2.13}$$

hold. The entries of the matrices $\mathbf{A}_{ij}$ are denoted by a_{pq}^{ij}, where i and j are fixed and $p=1,\ldots,n_i$ and $q=1,\ldots,n_j$ hold. They have to satisfy the condition

$$a_{pq}^{ij} = 0 \quad \text{for } p<q. \tag{9.2.14}$$

That is, the structure of the matrices $\mathbf{A}_{ij}$ is described by

$$[\mathbf{A}_{ij}] = \begin{pmatrix} * & 0 & \dots & 0 \\ * & * & \dots & 0 \\ \vdots & \vdots & & \vdots \\ * & * & \dots & * \\ \vdots & \vdots & & \vdots \\ * & * & \dots & * \end{pmatrix} \quad \text{if } n_i > n_j$$

or

$$[\mathbf{A}_{ij}] = \begin{pmatrix} * & 0 & \dots & 0 & \dots & 0 \\ * & * & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \\ * & * & \dots & * & \dots & 0 \end{pmatrix} \quad \text{if } n_i < n_j$$

(cf. Section 2.5 for the definition of the structure matrices). That is, couplings are forbidden between the entries x_{jk} and $x_{i,k-l}$ $(l > 0)$ of the subsystem states

$$\mathbf{x}_i = (x_{i1} \ \dots \ x_{in_i})' \qquad \mathbf{x}_j = (x_{j1} \ \dots \ x_{jn_j})'.$$

Since with $\mathbf{A}_{ii}$ in companion form (9.2.13) the state variable x_{ik} is the derivative of the variable $x_{i,k+1}$

$$\dot{x}_{ik} = x_{i,k+1} \qquad (k = 1, \dots, n_i - 1)$$

the restrictions on $\mathbf{A}_{ij}$ mean, roughly speaking, that the kth derivative of the last state variable x_{jn_j} of subsystem j must not directly influence higher derivatives of the last state variable x_{in_i} of subsystem i.

A pure decentralized controller

$$u_i = -\mathbf{k}_i'\mathbf{x}_i \tag{9.2.15}$$

is used. It is designed so that the closed-loop isolated subsystem has a prescribed set $\{-\alpha\lambda_{ij};\ j = 1, \dots, n_i\}$ of eigenvalues. The scalar $\alpha > 0$ is used later on to prescribe the stability degree for all subsystems. Since the elements c_{ij} occurring in the matrix $\mathbf{A}_{ii}$ in eqn (9.2.13) are the coefficients of the subsystem characteristic polynomial and the characteristic polynomial of the closed-loop subsystem matrix $\mathbf{A}_{ii} - \mathbf{b}_i\mathbf{k}_i'$ has to coincide with

$$\sum_{j=1}^{n_i} (p + \alpha\lambda_{ij}) = p^{n_i} + \bar{c}_{i1}p^{n_i - 1} + \dots + \bar{c}_{in_i}$$

the controller coefficients can be easily determined from

$$\mathbf{k}_i' = (\bar{c}_{in_i} - c_{in_i} \ \dots \ \bar{c}_{i1} - c_{i1}). \tag{9.2.16}$$

With the given set of subsystem eigenvalues and $\mathbf{k}_{ij} = \mathbf{0}$ $(i \neq j)$, eqn

(9.2.10) leads to

$$s_{ii} = \alpha \min_j \lambda_{ij} \quad \text{and} \quad s_{ij} = -\|\bar{\mathbf{A}}_{ij}\| \quad (i \neq j). \tag{9.2.17}$$

It will now be shown that the stability condition stated in Theorem 9.1 can be made valid by prescribing a sufficiently large stability degree for all subsystems. According to Appendix 1, the matrix $\mathbf{S}$ is an M-matrix if and only if there are scalars $d_i > 0$ such that the inequalities

$$d_i |s_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{N} d_j |s_{ij}| \qquad (i = 1, 2, \ldots, N) \tag{9.2.18}$$

hold. The diagonal elements $s_{ii}(\alpha)$ can be made arbitrarily large by choosing some large α, whereas the non-diagonal elements $s_{ij}(\alpha)$ remain bounded. The latter will become obvious from an investigation of the matrix $\bar{\mathbf{A}}_{ij} = \mathbf{T}_i^{-1}\mathbf{A}_{ij}\mathbf{T}_j$ in dependence upon α. Since $\mathbf{A}_{ii} - \mathbf{b}_i\mathbf{k}_i'$ has companion form, the modal matrix $\mathbf{T}_i$, which is used for the transformation of eqn (9.2.15) into eqn (9.2.6), is given by $\mathbf{T}_i = \mathbf{R}_i\hat{\mathbf{T}}_i$ with

$$\mathbf{R}_i = \operatorname{diag}(1 \;\; \alpha \;\; \alpha^2 \;\; \ldots \;\; \alpha^{n_i - 1}). \tag{9.2.19}$$

$\hat{\mathbf{T}}_i$ is the Vandermonde matrix

$$\hat{\mathbf{T}}_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -\lambda_{i1} & -\lambda_{i2} & \cdots & -\lambda_{in_i} \\ \vdots & \vdots & & \vdots \\ (-\lambda_{i1})^{n_i-1} & (-\lambda_{i2})^{n_i-1} & \cdots & (-\lambda_{in_i})^{n_i-1} \end{pmatrix}.$$

Hence

$$\bar{\mathbf{A}}_{ij} = \hat{\mathbf{T}}_i^{-1}\hat{\mathbf{A}}_{ij}\hat{\mathbf{T}}_j$$

holds with

$$\hat{\mathbf{A}}_{ij} = \mathbf{R}_i^{-1}\mathbf{A}_{ij}\mathbf{R}_j.$$

The matrix $\hat{\mathbf{A}}_{ij} = (\hat{a}_{pq}^{ij})$ satisfies the same restrictions as $\mathbf{A}_{ij}$:

$$\hat{a}_{pq}^{ij} = 0 \quad \text{for } p < q.$$

For all other elements,

$$\hat{a}_{pq}^{ij} = \alpha^{p-q} a_{pq}^{ij}$$

holds. All of these elements except those with $p = q$ vanish for $\alpha \to \infty$, that is

$$\hat{a}_{pp}^{ij} \to \bar{a}_{pp}^{ij}$$
$$\hat{a}_{pq}^{ij} \to \bar{a}_{pq}^{ij} = 0 \quad p \neq q$$
$$\hat{\mathbf{A}}_{ij} \to \hat{\mathbf{T}}_i^{-1}\bar{\mathbf{A}}_{ij}\hat{\mathbf{T}}_j$$

hold with

$$\bar{\mathbf{A}}_{ij} = (\bar{a}^{ij}_{pq}).$$

Hence, the stability test (9.2.18) with s_{ij} from eqn (9.2.17) can be made valid by choosing a sufficiently large parameter α.

Corollary 9.1

A composite system (9.2.1) and (9.2.13) that satisfies the structural constraints (9.2.14) on the interconnection relation is capable of being stabilized by decentralized control (9.2.15). The control stations of a stabilizing decentralized controller can be found independently of one another by means of eqn (9.2.16) provided that the scalar α has been chosen sufficiently large.

In addition to the stabilizability conditions given in Section 4.1, this corollary describes a further class of systems that can be stabilized by decentralized controllers. The stabilization is carried out by assigning each isolated subsystem a sufficiently large stability degree. The controller parameters can be found even in a decentralized way where the prescription of the parameter α is the only step which has to be done from the viewpoint of an overall system. This choice can be made without any model of the system. If the stability test fails, simply a higher value of α has to be prescribed.

Algorithm 9.3

Given: Model of the plant in input-decentralized form (9.2.1) and (9.2.13) which satisfies the restriction (9.2.14); sets of eigenvalues $\{-\lambda_{ij};\ j = 1, \ldots, n_i\}$ $(i = 1, \ldots, N)$ of the closed-loop subsystems; initial value of the parameter α.

1. Determine the parameters of the control stations (9.2.15) from eqn (9.2.16).
2. Determine $\| \mathbf{T}_i^{-1}\ \mathbf{A}_{ij}\ \mathbf{T}_j \|$.
3. Check the stability of the overall closed-loop system by proving that the matrix $\mathbf{S} = (s_{ij})$ from eqn (9.2.17) is an M-matrix. If the stability condition fails, increase the value of α and continue with step 1; otherwise stop.

Result: Decentralized controller (9.2.15) for which the closed-loop system (9.2.1), (9.2.13) and (9.2.15) is stable.

Example 9.1

To illustrate this algorithm consider the system (9.2.1) and (9.2.13) with

$$\mathbf{A}_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & 1 \end{pmatrix} \qquad \mathbf{A}_{22} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$$

$\mathbf{b}_i$ from eqn (9.2.13) and

$$\mathbf{A}_{12} = \begin{pmatrix} 0.2 & 0 \\ 0.3 & 0.4 \\ 0.2 & 0.1 \end{pmatrix} \qquad \mathbf{A}_{21} = \begin{pmatrix} 0.4 & 0 & 0 \\ 0.5 & 0.6 & 0 \end{pmatrix}.$$

Note that subsystem 1 is unstable. The closed-loop isolated subsystems should be assigned the eigenvalues $\{-\alpha, -2\alpha, -3\alpha\}$ or $\{-\alpha, -2\alpha\}$, respectively. For $\alpha = 1$ the algorithm leads to $\mathbf{k}_1' = (4 \;\; 10 \;\; 7)$, $\mathbf{k}_2' = (-1 \;\; 1)$ and

$$\mathbf{S} = \begin{pmatrix} 1 & -1.7 \\ -1.23 & 1 \end{pmatrix}$$

which is not an M-matrix. If α is increased to two, the controller parameters are $\mathbf{k}_1' = (160 \;\; 98 \;\; 19)$, $\mathbf{k}_2' = (15 \;\; 7)$. Then the test matrix

$$\mathbf{S} = \begin{pmatrix} 2 & -2.77 \\ -1.62 & 2 \end{pmatrix}$$

turns out to be an M-matrix. Hence, a stabilizing decentralized controller has been found which even ensures connective stability of the closed-loop overall system. □

The algorithm exhibits the typical characteristics of a decentralized design procedure. Nearly all design effort has to be made on the subsystem level with the subsystem model. All these steps have only the dimensionality of the subsystem. The local design aim is to reach a prescribed set of eigenvalues. It can be freely chosen and is modified merely by the global prescription of the scalar α. This modification does not affect the freedom to give the subsystem eigenvalues a relationship to each other that seems to be reasonable for the local decision maker.

This algorithm is an example for design methods in which the compatibility of the control stations is ensured by changing the subsystem design aims thriftily.

9.3 SUBOPTIMAL DECENTRALIZED CONTROLLERS

In this section, a design procedure with the structure of Algorithm 9.1 will be developed, which uses the **LQ** design at the subsystem level and checks the compatibility of the control stations with respect to the overall system stability by means of the scalar Lyapunov functions as explained in Section 8.2. Moreover, the performance of the closed-loop system will be assessed by means of a quadratic performance index.

The plant is described by the model

$$\dot{\mathbf{x}}_i = \mathbf{A}_{ii}\mathbf{x}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{B}_i\mathbf{u}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \tag{9.3.1}$$

$(i = 1, \dots, N)$. The design problem is to find control stations

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i \qquad (i = 1, \dots, N) \tag{9.3.2}$$

that minimize a given objective function

$$I = \sum_{i=1}^{N} I_i \tag{9.3.3}$$

with

$$I_i = \frac{1}{2}\int_0^{\infty} (\mathbf{x}_i'\mathbf{Q}_i\mathbf{x}_i + \mathbf{u}_i'\mathbf{R}_i\mathbf{u}_i)\, dt. \tag{9.3.4}$$

That is, the optimization problem

$$\min_{\mathbf{K}_i} I \quad \text{subject to eqns (9.3.1) and (9.3.2)} \tag{9.3.5}$$

has to be solved.

Contrary to Chapter 7, problem (9.3.5) will not be solved for the overall system but by means of a decentralized design procedure with the structure of Algorithm 9.1. In the first step the control stations (9.3.2) are designed for the isolated subsystems

$$\dot{\mathbf{x}}_i = \mathbf{A}_{ii}\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0}. \tag{9.3.6}$$

The solution is given by

$$\mathbf{K}_i = \mathbf{K}_i^{\mathrm{o}} = \mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i^{\mathrm{o}} \tag{9.3.7}$$

where $\mathbf{P}_i^{\mathrm{o}}$ is the positive definite solution of

$$\mathbf{A}_i'\mathbf{P}_i^{\mathrm{o}} + \mathbf{P}_i^{\mathrm{o}}\mathbf{A}_i - \mathbf{P}_i^{\mathrm{o}}\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i^{\mathrm{o}} + \mathbf{Q}_i = \mathbf{0} \tag{9.3.8}$$

(the symbols used here are the same as in eqns (7.3.5)–(7.3.8)).

In the second step, it is investigated whether this decentralized controller leads to a stable overall system and which deterioration of the quality of the closed-loop system as described by the performance index (9.3.3) has to be expected. The overall system (9.3.1) and (9.3.2) is described by

$$\dot{\mathbf{x}}_i = \bar{\mathbf{A}}_{ii}\mathbf{x}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{A}_{ij}\mathbf{x}_j \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \tag{9.3.9}$$

$(i = 1, \ldots, N)$ with

$$\bar{\mathbf{A}}_{ii} = \mathbf{A}_{ii} - \mathbf{B}_i\mathbf{K}_i^{\mathrm{o}} \tag{9.3.10}$$

or

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{9.3.11}$$

with

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{A}}_{11} & \mathbf{A}_{12} & \ldots & \mathbf{A}_{1N} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \ldots & \bar{\mathbf{A}}_{NN} \end{pmatrix}. \tag{9.3.12}$$

For the stability analysis the Lyapunov functions

$$v_i = \mathbf{x}_i'\mathbf{P}_i^{\mathrm{o}}\mathbf{x}_i \tag{9.3.13}$$

of the isolated closed-loop subsystems are used. Since $\mathbf{P}_i^{\mathrm{o}}$ is the solution of eqn (9.3.8), the Lyapunov equation

$$\bar{\mathbf{A}}_{ii}'\mathbf{P}_i^{\mathrm{o}} + \mathbf{P}_i^{\mathrm{o}}\bar{\mathbf{A}}_{ii} = -\mathbf{W}_i \tag{9.3.14}$$

with

$$\mathbf{W}_i = \mathbf{Q}_i + \mathbf{P}_i^{\mathrm{o}}\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i^{\mathrm{o}} \tag{9.3.15}$$

holds. Therefore, eqns (8.2.10) and (8.2.11) yield

$$\tilde{c}_{i3} = \lambda_{\min}[\mathbf{W}_i] \qquad \tilde{c}_{i4} = 2\lambda_{\max}[\mathbf{P}_i^{\mathrm{o}}]. \tag{9.3.16}$$

The interaction relation is evaluated with

$$b_{i1}l_{ij}b_{j2} = \|\mathbf{A}_{ij}\| \tag{9.3.17}$$

(cf. remark following Theorem 8.1). Theorem 8.1 leads to the following stability criterion.

Lemma 9.1

The closed-loop system (9.3.1), (9.3.2) and (9.3.7) is stable if the matrix

$$\tilde{\mathbf{S}} = (\tilde{s}_{ij}) \quad \text{with } \begin{array}{l} \tilde{s}_{ii} = \lambda_{\min}[\mathbf{W}_i] \\ \tilde{s}_{ij} = -2\lambda_{\max}[\mathbf{P}_i^{\mathrm{o}}]\,\|\mathbf{A}_{ij}\| \end{array} \tag{9.3.18}$$

is an M-matrix.

This result will now be extended to an evaluation of the overall system by means of the performance index (9.3.3). For the overall closed-loop system (9.3.1), (9.3.2) and (9.3.7) the objective function has the value

$$I^g = \mathbf{x}_0'\mathbf{P}^g\mathbf{x}_0 \tag{9.3.19}$$

where $\mathbf{P}^g$ is the solution of the Lyapunov equation

$$\bar{\mathbf{A}}'\mathbf{P}^g + \mathbf{P}^g\bar{\mathbf{A}} = -\mathbf{K}^{o\prime}\mathbf{R}\mathbf{K}^o - \mathbf{Q} = -\mathbf{W} \tag{9.3.20}$$

with $\mathbf{K}^o = \operatorname{diag} \mathbf{K}_i^o$, $\mathbf{R} = \operatorname{diag} \mathbf{R}_i$, $\mathbf{Q} = \operatorname{diag} \mathbf{Q}_i$ and $\mathbf{W} = \operatorname{diag} \mathbf{W}_i$ (cf. eqn (7.3.12)). The determination of I^g necessitates the solution of eqn (9.3.20), which has the order of the overall system, but this would disturb the decentralized structure of the design process. Therefore, I^g should be evaluated in terms of

$$I^o = \mathbf{x}_0'\mathbf{P}^o\mathbf{x}_0 \tag{9.3.21}$$

with $\mathbf{P}^o = \operatorname{diag} \mathbf{P}_i^o$ (cf. eqn (7.3.10)). As discussed in Section 7.3, without interactions ($\mathbf{A}_{ij} = \mathbf{0}$) the decentralized controller (9.3.2) and (9.3.7) is optimal with respect to the criterion (9.3.3). In order to describe how much the interactions may deteriorate the performance, a suboptimality index μ is introduced; μ is a positive scalar for which the relation

$$I^g \leqslant \frac{1}{\mu} I^o \tag{9.3.22}$$

holds. The following theorem shows that μ can be evaluated by means of an algorithm which has a decentralized information structure.

Theorem 9.2

Assume that the local control stations (9.3.3) have been designed such that the closed-loop isolated subsystems are optimal with respect to the subsystem performance indices I_i. Determine the matrix $\bar{\mathbf{S}}(\mu) = (\bar{s}_{ij})$ with

$$\begin{aligned} \bar{s}_{ii}(\mu) &= (1-\mu)\lambda_{\min}[\mathbf{W}_i] \\ \bar{s}_{ij} &= -2\lambda_{\max}[\mathbf{P}_i^o]\ \|\mathbf{A}_{ij}\|. \end{aligned} \tag{9.3.23}$$

Then the following statements hold:

(i) The overall closed-loop system (9.3.1), (9.3.2) and (9.3.7) is connectively stable if the matrix $\bar{\mathbf{S}}(0)$ is an M-matrix.
(ii) The overall closed-loop system (9.3.1), (9.3.2) and (9.3.7) is suboptimal with index μ if the matrix $\bar{\mathbf{S}}(\mu)$ is an M-matrix.

This theorem shows how the suboptimality index can be found. Moreover, it shows that a certain suboptimality of the system claims a stronger test to be passed than stability.

Proof

Part (i) states the same as Lemma 9.1. To prove part (ii) introduce

$$\mathbf{P}^{\mathrm{d}} = \frac{1}{\mu}\mathbf{P}^{\mathrm{o}} - \mathbf{P}^{g}.$$

The relation (9.3.22) is equivalent to

$$\mathbf{x}_0'\mathbf{P}^{\mathrm{d}}\mathbf{x}_0 \geqslant 0. \tag{9.3.24}$$

From eqns (9.3.14) and (9.3.20)

$$\bar{\mathbf{A}}'\mathbf{P}^{\mathrm{d}} + \mathbf{P}^{\mathrm{d}}\bar{\mathbf{A}} = \mathbf{W} + \frac{1}{\mu}\bar{\mathbf{A}}'\mathbf{P}^{\mathrm{o}} + \frac{1}{\mu}\mathbf{P}^{\mathrm{o}}\bar{\mathbf{A}} \tag{9.3.25}$$

can be derived. Since the overall system is stable all eigenvalues of $\bar{\mathbf{A}}$ have negative parts. Therefore, eqn (9.3.25) is a Lyapunov equation which leads to a positive definite solution $\mathbf{P}^{\mathrm{d}}$ if the right-hand side represents a negative definite matrix. That is, the relation (9.3.24) can be proved by showing that the matrix $\bar{\mathbf{A}}'\mathbf{P}^{\mathrm{o}} + \mathbf{P}^{\mathrm{o}}\bar{\mathbf{A}} + \mu\mathbf{W}$ is negative definite. From eqns (9.3.7) and (9.3.14) the equality

$$\begin{aligned}\bar{\mathbf{A}}'\mathbf{P}^{\mathrm{o}} + \mathbf{P}^{\mathrm{o}}\bar{\mathbf{A}} + \mu\mathbf{W} &= \operatorname{diag}\bar{\mathbf{A}}_{ii}'\,\mathbf{P}^{\mathrm{o}} + \mathbf{P}^{\mathrm{o}}\operatorname{diag}\bar{\mathbf{A}}_{ii} + \mathbf{A}_C'\mathbf{P}^{\mathrm{o}} + \mathbf{P}^{\mathrm{o}}\mathbf{A}_C + \mathbf{W}\\ &= \mathbf{A}_C'\mathbf{P}^{\mathrm{o}} + \mathbf{P}^{\mathrm{o}}\mathbf{A}_C + (\mu - 1)\mathbf{W}\end{aligned}$$

holds, where the decomposition $\mathbf{A} = \operatorname{diag}\mathbf{A}_{ii} + \mathbf{A}_C$ as introduced in eqn (7.3.24) has been used. In order to prove the inequality

$$\mathbf{x}'[\mathbf{A}_C'\mathbf{P}^{\mathrm{o}} + \mathbf{P}^{\mathrm{o}}\mathbf{A}_C + (\mu - 1)\mathbf{W}]\mathbf{x} \leqslant 0 \tag{9.3.26}$$

the following estimation is derived, where $\mathbf{x} = (\mathbf{x}_1' \ \ldots \ \mathbf{x}_N')'$ and

$$\mathbf{x}'\mathbf{W}\mathbf{x} = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\mathbf{x}_i'\mathbf{A}_{ij}\mathbf{x}_j$$

is used:

$$\mathbf{x}'[\mathbf{A}_C'\mathbf{P}^o + \mathbf{P}^o\mathbf{A}_C + (\mu - 1)\mathbf{W}]\mathbf{x}$$

$$= \sum_{i=1}^{N} -(1-\mu)\mathbf{x}_i'\mathbf{W}_i\mathbf{x}_i + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{x}_i'(\mathbf{A}_{ji}'\mathbf{P}_j^o + \mathbf{P}_i^o\mathbf{A}_{ij})\mathbf{x}_j$$

$$\leqslant \sum_{i=1}^{N} -(1-\mu)\lambda_{\min}[\mathbf{W}_i]\,\|\mathbf{x}_i\|^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} (\lambda_{\max}[\mathbf{P}_j^o]\,\|\mathbf{A}_{ji}\|$$

$$+ \lambda_{\max}[\mathbf{P}_i^o]\,\|\mathbf{A}_{ij}\|)\,\|\mathbf{x}_i\|\,\|\mathbf{x}_j\|$$

$$= -(\|\mathbf{x}_1\| \;\ldots\; \|\mathbf{x}_N\|)[\bar{\mathbf{S}}(\mu)' + \bar{\mathbf{S}}(\mu)]\,(\|\mathbf{x}_1\| \;\ldots\; \|\mathbf{x}_N\|)'. \qquad (9.3.27)$$

With the same argument as in Section 8.2 it is clear that if $\bar{\mathbf{S}}(\mu)$ is an M-matrix then the last line of the expression above is less than zero, which proves part (ii) of the theorem. □

Since $\bar{\mathbf{S}}(\mu)$ is constructed by means of norm bounds, part (ii) of Theorem 9.2 is sufficient but not necessary for μ to be the suboptimality index. The conservatism of this test can be shown if a system with neutral interconnections is considered. According to Theorem 7.2, for such systems $\mathbf{A}_C = \mathbf{S}\,\mathrm{diag}\,\mathbf{P}_i^o$ with $\mathbf{S}$ being skew-symmetric holds. Therefore, the second term in the second line of eqn (9.3.27) is zero, which proves the validity of eqn (9.3.26) for arbitrary μ. In contrast, $\bar{\mathbf{S}}(\mu)$ from eqn (9.3.23) is an M-matrix only for some interval $\mu \in (0, \bar{\mu})$. This conservatism is the price to be paid for the decentralization of the analytical process.

The application of Theorem 9.2 mainly involves steps which can be made on the subsystem level without the complete model of the overall system. Moreover, as the proof shows, the suboptimality index remains unchanged if some interaction links fail in operation. That is, the system is *connectively suboptimal with index* μ.

Sometimes another definition of the suboptimality index is used. Instead of μ in eqn (9.3.22) ε in

$$I^g \leqslant (1+\varepsilon)I^o$$

is claimed. Then Theorem 9.2 is valid with $(1+\varepsilon)$ replacing $1/\mu$.

A constructive way of determining the suboptimality index μ can be derived from the M-matrix conditions summarized in Appendix 1. The matrix

$$\bar{\mathbf{S}}(\mu) = (1-\mu)\mathrm{diag}(\lambda_{\min}[\mathbf{W}_i]) + \bar{\mathbf{S}}_C$$

with

$$\bar{\mathbf{S}}_C = \begin{pmatrix} 0 & \bar{s}_{12} & \dots & \bar{s}_{1N} \\ \bar{s}_{21} & 0 & \dots & \bar{s}_{2N} \\ \vdots & \vdots & & \vdots \\ \bar{s}_{N1} & \bar{s}_{N2} & \dots & 0 \end{pmatrix}$$

is an M-matrix if and only if the Perron root of the matrix

$$(1-\mu)^{-1}\ \mathrm{diag}(\lambda_{\min}[\mathbf{W}_i])^{-1}\bar{\mathbf{S}}_C$$

is smaller than one. Hence, part (ii) of Theorem 9.2 is satisfied for all μ with

$$\mu < 1 - \lambda_p[\mathrm{diag}(\lambda_{\min}[\mathbf{W}_i])^{-1}\bar{\mathbf{S}}_C]. \tag{9.3.28}$$

Algorithm 9.4

Given: Interconnected plant (9.3.1) with performance indices (9.3.4).

1. Solve the subsystem Riccati equations (9.3.8) and determine the controller parameters separately for all subsystems.
2. Determine the test matrix $\bar{\mathbf{S}}(0)$ according to eqn (9.3.23). Check whether $\bar{\mathbf{S}}(0)$ is an M-matrix. If the test fails, change the weighting matrices of the performance indices and continue with step 1.
3. Determine the suboptimality index μ according to inequality (9.3.28).

Result: Decentralized controller for which the overall closed-loop system has suboptimality index μ.

9.4 ROBUST DECENTRALIZED CONTROLLERS

In this section, the decentralized design is extended to overall systems with some kind of non-linear interactions. Non-linear memoryless elements $\mathbf{f}_i(\mathbf{u}_i)$ are inserted into the input channels of the subsystems so that the overall system is described by

$$\dot{\mathbf{x}}_i = \mathbf{A}_{ii}\mathbf{x}_i + \mathbf{B}_i\mathbf{f}_i(\mathbf{u}_i) + \mathbf{f}_{ki}(\mathbf{x}) \tag{9.4.1}$$

where, as usual, $\mathbf{x} = (\mathbf{x}_1' \ \dots \ \mathbf{x}_N')'$ holds. Note that the overall system has the structure of eqn (3.1.4).

It is assumed that, as in the preceding section, the decentralized control stations

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i \tag{9.4.2}$$

have been designed as optimal controllers for the linear isolated subsystems. That is,

$$\mathbf{K}_i = \mathbf{K}_i^o = \mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i^o \tag{9.4.3}$$

holds with $\mathbf{P}_i^o$ as solution of

$$\mathbf{A}_i'\mathbf{P}_i^o + \mathbf{P}_i^o\mathbf{A}_i - \mathbf{P}_i^o\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i^o + \mathbf{Q}_i = \mathbf{0}. \tag{9.4.4}$$

In order to apply the stability condition of Theorem 8.1 to the closed-loop overall system (9.4.1) and (9.4.2) the constants $\tilde{c}_{ij}$ have to be determined from Lyapunov functions of the isolated closed-loop subsystems, which are obtained from eqns (9.4.1) and (9.4.3) for $\mathbf{f}_k(\mathbf{x}) = \mathbf{0}$

$$\dot{\mathbf{x}}_i = \mathbf{A}_{ii}\mathbf{x}_i + \mathbf{B}_i\mathbf{f}_i(\mathbf{u}_i). \tag{9.4.5}$$

For the Lyapunov function

$$v_i(\mathbf{x}_i) = \mathbf{x}_i'\mathbf{P}_i\mathbf{x}_i \tag{9.4.6}$$

the scalars

$$\tilde{c}_{i1} = \lambda_{\min}[\mathbf{P}_i] \qquad \tilde{c}_{i2} = \lambda_{\max}[\mathbf{P}_i] \qquad \tilde{c}_{i4} = 2\lambda_{\max}[\mathbf{P}_i]$$

are the same as in eqn (8.2.10). For the time derivative of $v_i(\mathbf{x}_i)$ along the trajectory of the non-linear subsystem (9.4.5) the following relation is obtained

$$\begin{aligned}
\dot{v}_i|_{SS} &= \mathbf{x}_i'(\mathbf{A}_i'\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_i)\mathbf{x}_i + 2\mathbf{x}_i'\mathbf{P}_i\mathbf{B}_i\mathbf{f}_i(\mathbf{u}_i)\\
&= \mathbf{x}_i'(-\mathbf{Q}_i + \mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i)\mathbf{x}_i + 2\mathbf{x}_i'\mathbf{P}_i\mathbf{B}_i\mathbf{f}_i(\mathbf{u}_i)\\
&= \mathbf{x}_i'[-\mathbf{Q}_i + (w_i - 1)\mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i]\mathbf{x}_i\\
&\quad + \mathbf{x}_i'(2 - w_i)\mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i\mathbf{x}_i + 2\mathbf{x}_i'\mathbf{P}_i\mathbf{B}_i\mathbf{f}_i(\mathbf{u}_i)\\
&\leqslant -\lambda_{\min}[\mathbf{Q}_i + (1 - w_i)\mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i]\,\|\mathbf{x}_i\|^2 + z_i\\
&\leqslant -\tilde{c}_{i3}\|\mathbf{x}_i\|^2
\end{aligned}$$

with

$$\tilde{c}_{i3} = \lambda_{\min}[\mathbf{Q}_i + (1 - w_i)\mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i] \tag{9.4.7}$$

provided that

$$z_i = \mathbf{x}_i'(2 - w_i)\mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i\mathbf{x}_i + 2\mathbf{x}_i'\mathbf{P}_i\mathbf{B}_i\mathbf{f}_i(\mathbf{u}_i) \leqslant 0$$

holds. By using eqn (9.4.3), the first term on the right-hand side of the last equation above can be expressed as $(2 - w_i)\mathbf{u}_i'\mathbf{R}_i\mathbf{u}_i$. Consequently,

$z_i \leqslant 0$ is equivalent to

$$\left(1 - \frac{w_i}{2}\right)\mathbf{u}_i'\mathbf{R}_i\mathbf{u}_i \leqslant \mathbf{u}_i\mathbf{R}_i\mathbf{f}_i(\mathbf{u}_i). \tag{9.4.8}$$

That is, if the non-linearity of the ith subsystem satisfies the inequality (9.4.8) for some constant w_i then the Lyapunov function (9.4.6) satisfies the inequalities (8.2.7)–(8.2.9) with the constants given above. The inequality (9.4.8) will be discussed later.

For the application of the stability criterion of Theorem 8.1 the interconnection relation has to be described in a similar way to eqn (8.2.14). That is, scalars $\tilde{l}_{ij}$ have to be found so that

$$\| \mathbf{f}_{ki}(\mathbf{x}) \| \leqslant \sum_{i=1}^{N} \tilde{l}_{ij} \| \mathbf{x}_j \| \tag{9.4.9}$$

is valid. Then the test matrix $\tilde{\mathbf{S}} = (\tilde{s}_{ij})$ can be set up as in eqn (8.2.18) with $\tilde{l}_{ij}$ replacing the term $b_{i1}l_{ij}b_{j2}$

$$\begin{aligned} \tilde{s}_{ii} &= \lambda_{\min}[\mathbf{Q}_i + (1 - w_i)\mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i] - 2\lambda_{\max}[\mathbf{P}_i]\,\tilde{l}_{ii} \\ \tilde{s}_{ij} &= -2\lambda_{\max}[\mathbf{P}_i]\,\tilde{l}_{ij}. \end{aligned} \tag{9.4.10}$$

Theorem 8.1 and Corollary 8.1 lead to the following stability condition.

Theorem 9.3

Assume that the local control stations (9.4.2) have been designed for the isolated subsystems by means of an **LQ** procedure for the quadratic performance index with weighting matrices $\mathbf{Q}_i$ and $\mathbf{R}_i$. If the non-linearities $\mathbf{f}_i(\mathbf{u}_i)$ $(i = 1, \ldots, N)$ satisfy the inequalities (9.4.8) for some constants $w_i > 0$ and the matrix $\tilde{\mathbf{S}}$ is an M-matrix, then the overall closed-loop system is connectively stable.

This result leads to the following design algorithm.

Algorithm 9.5

Given: Non-linear plant (9.4.1), for which the non-linearities $\mathbf{f}_i(\mathbf{u}_i)$ satisfy the condition (9.4.8) for some scalars $w_i > 0$; weighting matrices $\mathbf{Q}_i$ and $\mathbf{R}_i$.

1. Solve the local **LQ** problems according to eqns (9.4.3) and (9.4.4).
2. Determine $\tilde{l}_{ij}$ so that eqn (9.4.9) holds.

3. Check whether the matrix $\tilde{\mathbf{S}}$ from eqn (9.4.10) is an M-matrix, which ensures the stability of the overall closed-loop system. If the stability condition fails, change the weighting matrices $\mathbf{Q}_i$ and $\mathbf{R}_i$ and continue with step 1; otherwise stop.

Result: Decentralized controller (9.4.2) which ensures the closed-loop stability.

Theorem 9.3 describes under what conditions the solutions of the independent **LQ** problems for the isolated subsystems are compatible in the sense that the implementation of the control stations results in a stable overall closed-loop system. The rather abstract mathematical condition on $\tilde{\mathbf{S}}$ to be an M-matrix will now be interpreted in control engineering terms.

First, for linear subsystems with non-linear interactions

$$\dot{\mathbf{x}}_i = \mathbf{A}_{ii}\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i + \mathbf{f}_{ki}(\mathbf{x}) \tag{9.4.11}$$

Theorem 9.3 holds with $w_i = 0$ (cf. eqn (9.4.8)).

Corollary 9.2

Consider a non-linear composite system (9.4.11) with linear subsystems and assume that the control stations (9.4.3) have been chosen for the isolated subsystems so as to minimize the quadratic performance indices I_i $(i = 1, \ldots, N)$ from eqn (9.3.4). If the matrix $\tilde{\mathbf{S}} = (\tilde{s}_{ij})$ with

$$\begin{aligned} \tilde{s}_{ii} &= \lambda_{\min}[\mathbf{Q}_i + \mathbf{P}_i\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i'\mathbf{P}_i] - 2\lambda_{\max}[\mathbf{P}_i]\,\tilde{l}_{ii} \\ \tilde{s}_{ij} &= -2\lambda_{\max}[\mathbf{P}_i]\,\tilde{l}_{ij}. \end{aligned} \tag{9.4.12}$$

is an M-matrix, then the overall closed-loop system (9.4.2), (9.4.3) and (9.4.11) is connectively stable.

The matrices $\mathbf{Q}_i$ and $\mathbf{R}_i$ can be freely chosen, but as eqn (9.4.4) shows, $\mathbf{P}_i$ is closely related to this choice and the subsystem properties. It is, therefore, impossible to use the M-matrix condition of the test matrix $\tilde{\mathbf{S}}$ to determine explicitly such weighting matrices for which the stability of the overall closed-loop system is ensured. But the trend is known: increasing the elements of $\mathbf{Q}_i$, which corresponds to a larger penalization of $\mathbf{x}_i(t)$, leads to larger elements in $\mathbf{P}_i$ and, hence, to large diagonal elements $\tilde{s}_{ii}$. However, simultaneously the non-diagonal elements $\tilde{s}_{ij}$ become smaller.

Second, if the non-linear element $\mathbf{f}_i(\mathbf{u}_i)$ in eqn (9.4.1) is replaced by a linear element

$$\mathbf{f}_i(\mathbf{u}_i) = k_i \mathbf{u}_i \tag{9.4.13}$$

the robustness of the decentralized controller against loop gain variations can be assessed. The use of $\mathbf{f}_i$ as in eqn (9.4.13) is equivalent to the use of the control station (9.4.2) with controller matrix

$$\mathbf{K}_i = -k_i \mathbf{R}_i^{-1} \mathbf{B}_i' \mathbf{P}_i \tag{9.4.14}$$

in connection with the linear subsystem (9.4.11). Hence, k_i can be interpreted as loop gain variation. The element (9.4.13) satisfies the inequality (9.4.8) if

$$\left(1 - \frac{w_i}{2}\right) \leqslant k_i \tag{9.4.15}$$

holds. If the test matrix $\tilde{\mathbf{S}}$ is an M-matrix, the system remains stable for all such k_i. That is, all loop gains can be infinitely increased or reduced by the factor $(1 - \frac{1}{2} w_i)$ without endangering the closed-loop stability. This corresponds to an infinite gain margin and a $w_i/2 \times 100\%$ gain reduction tolerance in every decentralized control loop.

Corollary 9.3

If the matrix $\tilde{\mathbf{S}}$ in eqn (9.4.13) is an M-matrix, the closed-loop system (9.4.11) and (9.4.3) is connectively stable and has an infinite gain margin and a $w_i/2 \times 100\%$ gain reduction tolerance in every input channel.

The third remark concerns the interpretation of eqn (9.4.8) for non-linear elements $\mathbf{f}_i(\mathbf{u}_i)$, which are restricted to the form

$$\mathbf{f}_i(\mathbf{u}_i) = (\mathbf{f}_{i1}(u_{i1}) \;\; f_{i2}(u_{i2}) \;\; \ldots \;\; f_{imi}(u_{im_i}))'$$

with

$$\mathbf{u}_i = (u_{i1} \;\; u_{i2} \;\; \ldots \;\; u_{im_i})'.$$

Then, eqn (9.4.8) yields

$$\frac{f_{ij}(u_{ij})}{u_{ij}} \geqslant 1 - \frac{w_i}{2}. \tag{9.4.16}$$

That is, the non-linearity has to lie inside the sector which is drawn in Figure 9.4. The boundary of the section depends on the scalar w_i, which in turn influences the constant $\tilde{c}_{i3}$ of the corresponding subsystem. The smaller w_i is, the smaller is $\tilde{c}_{i3}$. A lower bound for w_i is given by that value for which $\tilde{c}_{i3}$ in eqn (9.4.7) becomes negative.

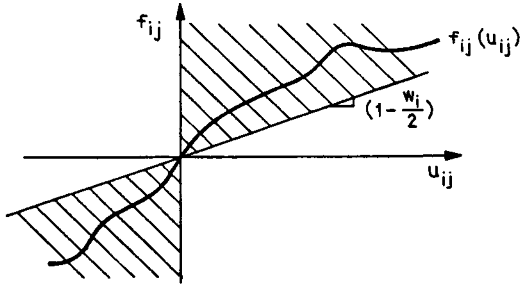


Figure 9.4 Sector describing the non-linear element

9.5 DECENTRALIZED OUTPUT FEEDBACK

The methods developed so far are applicable only if the subsystem states are available for measurement since the controllers have the form

$$\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i. \tag{9.5.1}$$

If only the subsystem outputs $\mathbf{y}_i$ can be accessed, decentralized observers have to be used for state reconstruction. Owing to the decentralized nature of the design methods, this observation problem can be solved without encountering the difficulties discussed in Section 7.4. The reason for this is given by the fact that in the decentralized design algorithm the local control stations are chosen for the isolated subsystems and so are the observers (step 1 in Algorithm 9.1).

To outline this in more detail, note that the control stations (9.5.1) are designed in all the methods presented in the Sections 9.2–9.4 for the isolated subsystems

$$\begin{aligned}\dot{\mathbf{x}}_i &= \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i\\ \mathbf{y}_i &= \mathbf{C}_i\mathbf{x}_i\end{aligned} \tag{9.5.2}$$

which are provided here with an equation for the output $\mathbf{y}_i$. The aim was to make the closed-loop system (9.5.1) and (9.5.2) stable, to move the eigenvalues to prescribed positions in the complex plane, or to minimize an objective function. If the resulting controller (9.5.1) has to be applied by means of an observer

$$\dot{\hat{\mathbf{x}}}_i = \mathbf{A}_{bi}\hat{\mathbf{x}}_i + \mathbf{B}_{bi}\mathbf{u}_i + \mathbf{E}_{bi}\mathbf{y}_i \tag{9.5.3}$$

the observer matrices $\mathbf{A}_{bi}$, $\mathbf{B}_{bi}$ and $\mathbf{E}_{bi}$ can be found for the isolated subsystem (9.5.2) as described in Section 2.2. The observation problem is a classical one. The decentralized control stations consist of the observer

(9.5.3) and the feedback

$$\mathbf{u}_i = -\mathbf{K}_i \hat{\mathbf{x}}_i. \tag{9.5.4}$$

Since the design and observation problems are centralized rather than decentralized, the design aims posed on the subsystems are at least approximately satisfied for the loop (9.5.2), (9.5.3) and (9.5.4) which includes dynamic output feedback if they are satisfied for the state feedback system (9.5.1) and (9.5.2). The reason for this is given by the separation theorem (Theorem 2.8). Accordingly, the system (9.5.2), (9.5.3) and (9.5.4) is stable if and only if the system (9.5.1) and (9.5.2) and the observer (9.5.3) are stable. The eigenvalues of (9.5.2)–(9.5.4) consist of those of (9.5.1) and (9.5.2) and the observer (9.5.3). Similar results are known from multivariable control theory for the optimality of the controllers (9.5.3) and (9.5.4).

The only modifications to the methods presented in the preceding sections concern the fact that the overall system has to be assessed with the dynamic controller (9.5.3) and (9.5.4) rather than the static feedback (9.5.1) attached. That is, the Lyapunov functions (9.2.8), (9.3.13) or (9.4.6) have to be constructed for the expanded subsystem state $(\mathbf{x}_i' \ \hat{\mathbf{x}}_i')'$. It is obvious that all further design steps proceed as above and lead to similar test matrices.

9.6 EXTENSION TO OVERLAPPING SUBSYSTEMS

It has been explained in Section 3.4 that an overlapping decomposition may lead to weakly coupled subsystems even if disjoint subsystems are strongly coupled. The overlapping part of the subsystems can be considered as an approximate description of the influence that the other subsystems impose on the given subsystem. Hence, the model describes the subsystem performance better than the model of the isolated subsystem.

The main problem when using the overlapping decomposition for design purposes is the contractability of the resulting control law. It has to be ensured that the controller can be applied to the original system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{9.6.1}$$

although it has been designed for an expansion of (9.6.1)

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u} \qquad \tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0 \\ \mathbf{y} &= \tilde{\mathbf{C}}\tilde{\mathbf{x}} \end{aligned} \tag{9.6.2}$$

(cf. Section 3.4). This possibility is described by the following property.

Definition 9.1

The control law

$$\mathbf{u} = -\tilde{\mathbf{K}}\tilde{\mathbf{x}} \tag{9.6.3}$$

for the expansion (9.6.2) is *contractable* to the control law

$$\mathbf{u} = -\mathbf{K}\mathbf{x} \tag{9.6.4}$$

for the original system (9.6.1) if $\tilde{\mathbf{x}}_0 = \mathbf{T}\mathbf{x}_0$ implies

$$\mathbf{K}\mathbf{x} = \tilde{\mathbf{K}}\tilde{\mathbf{x}} \quad \text{for all } t \geqslant 0 \tag{9.6.5}$$

where $\mathbf{T}$ is the transformation matrix used in Definition 3.2.

The contractability of the control law implies that the closed-loop system (9.6.1) and (9.6.4)

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{9.6.6}$$

is a contraction of the closed-loop system (9.6.2) and (9.6.3)

$$\dot{\tilde{\mathbf{x}}} = (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{K}})\tilde{\mathbf{x}} \qquad \tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0. \tag{9.6.7}$$

A necessary and sufficient condition for contractability can be proved if the controller matrix is represented as

$$\tilde{\mathbf{K}} = \mathbf{K}\mathbf{T}^{+} + \mathbf{L} \tag{9.6.8}$$

where $\mathbf{L}$ is a constant complementary matrix.

Theorem 9.4

The control law (9.6.3) is contractable to the control law (9.6.2) if and only if the following conditions are satisfied:

$$\mathbf{L}\mathbf{M}^{i-1}\mathbf{T} = \mathbf{0} \qquad \mathbf{L}\mathbf{M}^{i-1}\mathbf{N} = \mathbf{0} \tag{9.6.9}$$

$(i = 1, \ldots, \dim \tilde{\mathbf{x}})$.

Equation (9.6.9) has the same requirements as Theorem 3.3. The proof that eqn (9.6.5) follows from eqns (3.4.6) and (9.6.9) can be established by a comparison of the time series expansions of $\mathbf{K} \exp \mathbf{A}t$ and $\tilde{\mathbf{K}} \exp \tilde{\mathbf{A}}t$.

Theorem 3.4 together with Theorem 9.4 leads to the following corollary, which shows that stabilizing decentralized controllers of the original system (9.6.1) can be designed for the expanded system (9.6.2).

Corollary 9.4

If the system (9.6.2) is an expansion of (9.6.1) and if the control law (9.6.3) can be contracted to (9.6.4), then the closed-loop system (9.6.1) and (9.6.4) is stable if and only if the system (9.6.2) and (9.6.3) is stable.

Example 9.2

Consider the system (9.6.1) under the state partition made in Example 3.4. A decentralized controller

$$\mathbf{u}_1 = (-\tilde{\mathbf{K}}_{11} \quad -\tilde{\mathbf{K}}_{12})\tilde{\mathbf{x}}_1$$
$$\mathbf{u}_2 = (-\tilde{\mathbf{K}}_{23} \quad -\tilde{\mathbf{K}}_{24})\tilde{\mathbf{x}}_2$$

can be designed for the expanded plant (3.4.8). It represents the controller

$$\mathbf{u} = -\left(\begin{array}{cc|cc} \tilde{\mathbf{K}}_{11} & \tilde{\mathbf{K}}_{12} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{K}}_{23} & \tilde{\mathbf{K}}_{24} \end{array}\right)\begin{pmatrix} \tilde{\mathbf{x}}_1 \\ \tilde{\mathbf{x}}_2 \end{pmatrix}$$

for the expanded system (9.6.2). The controller satisfies eqn (9.6.8) with $\mathbf{T}$ used in Example 3.4 since

$$\mathbf{K} = \tilde{\mathbf{K}}\mathbf{T} = -\begin{pmatrix} \tilde{\mathbf{K}}_{11} & \tilde{\mathbf{K}}_{12} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{K}}_{23} & \tilde{\mathbf{K}}_{24} \end{pmatrix}$$

holds. Although this controller has been designed as independent control stations for the isolated subsystems of the expansion (9.6.2), it does not represent a decentralized controller for the original system. The state $\mathbf{x}_2$ has to be available for measurement for both control stations as the frequency f in the power system (Example 3.5, Figure 9.5). However, if

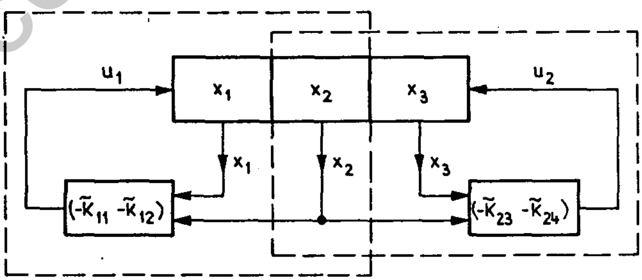


Figure 9.5 Closed-loop system of Example 9.2

an incomplete state feedback was used during the design, that is if $\tilde{\mathbf{K}}_{12}$ or $\tilde{\mathbf{K}}_{23}$ was zero, then a completely decentralized control law would result. □

BIBLIOGRAPHICAL NOTES

Several authors have tackled the problem of decentralized control in a decision-theoretic setting. Probably the best survey was given by Findeisen (1982). The notion of a non-classical information pattern has been introduced in decision theory (see, for example, Ho and Chu 1972, 1974 or Ho *et al.* 1978) and discussed in connection with decentralized design by Bailey (1978), Li and Singh (1983), Tenney and Sandell (1981a,b), Tsitsiklis and Athans (1985) or Fiorio and Villa (1986). In these references the reasons for the non-classical information pattern and ways to solve decision problems under such an information structure are described in more detail than in Section 9.1.

Special investigations of decentralized control as a dynamic team problem were carried out by Ho and Chu (1974) and Sandell and Athans (1974) who solved the **LQG** problem. Extensions were made to systems with a k-step-delay sharing information pattern, in which the control agents receive the controls made by the other agents after k time steps. All these investigations refer to systems subject to stochastic disturbances and lead to control agents, which are linked by a communication network. Although all links are allowed to have some time delay, the controller is, in principle, a centralized one.

Basic game-theoretic notions of the control of dynamic processes by more than one control agent were mentioned in the monograph by Bryson and Ho (1969). Starr and Ho (1969) and Varaiya (1970) gave solutions to decentralized control as non-zero-sum differential games with linear dynamics. Other authors considered the decentralized design as Nash or Stackelberg problems, for example Mageirou and Ho (1977). The fundamental results along these lines were reviewed by Bernussou and Titli (1982). Relevant game-theoretic results can be found in the book by Bazar and Bernhard (1989).

The idea of reducing the design complexity by designing the control stations independently of one another for the isolated subsystems is old. Bailey and Wang proposed a strategy to find weighting matrices $\mathbf{Q}_i$ and $\mathbf{R}_i$ for the isolated **LQ** problems in order to ensure a reasonable overall system performance index for the interconnected system as early as 1972. Bailey and Ramapriya (1973) derived bounds on the suboptimality of the decentralized controller in terms of the overall performance criterion. Darwish *et al.* (1979) published a method in which a reasonable prescription of the degrees of subsystem stability ensures overall system stability. Although the design steps used in these methods are the same as those explained in this chapter, a major difference is given by the fact that the analysis step necessitates a complete model of the overall closed-loop system. Therefore, the information structure inherent in these methods is not decentralized.

The main step towards decentralized design methods was made when the upper bounds of the subsystem dynamics were used in the analysis of the overall system. The aggregation–decomposition method (Section 9.2) was proposed by Šiljak (1972, 1976, 1978) and served as a paradigm, which has been used and extended by many other researchers with or without reference to Šiljak's work. A modification of the design principle to decentralized feedback of the subsystem state and the interconnection input was made by Xiao (1985). This author used a criterion based on comparison systems (Section 8.4) in order to prove the stability of the closed-loop system. Examples of applications can be found in Šiljak (1978) or Čalović *et al.* (1978). A similar technique for decentralized design has been developed by Vesely (1981) who started with the Bellman-Lyapunov equation for optimal control. The design scheme has an information structure similar to Figure 9.1. Applications to the decentralized control of multiarea power systems are reported by Vesely *et al.* (1981, 1984). Chmúrny (1989) is one of several authors who applied the aggregation–decomposition method to plants with specific structure. He considered chemical reactors for which the multilevel controller has proportional-derivative (PD) character.

The method of ensuring the compatibility of the control stations by prescribing a degree of stability for the closed-loop subsystems was investigated, for example, by Sundareshan (1977b) and Mahalanabis and Singh (1980). The results of Section 9.2 and the main idea of Example 9.1 were developed by Vukčević (1975).

The extension of the stability analysis to non-linearly coupled systems began with the work of Weissenberger (1974). Sezer and Šiljak (1981a), and Ikeda and Šiljak (1982), whose results are reviewed in Section 9.3, investigated the robustness of the decentralized controller against parametrical uncertainties or non-linearities within the subsystems. The connective stability and performance degradation under perturbations of the global information links within multilevel controllers were studied by Geromel and Bernussou (1979b) and Hassan and Singh (1979). Petkovski (1984) derived upper bounds on the perturbations $\mathrm{d}\mathbf{A}_{ij}$, $\mathrm{d}\mathbf{B}_{ij}$ of the matrices $\mathbf{A}_{ij}$ and $\mathbf{B}_i$ for which the closed-loop stability is ensured. Bahnasawi *et al.* (1990) elaborated an alternative method in which the overall system stability is tested by means of the scalar Lyapunov function approach described in Section 8.2. This stability test has been extended to non-linear interconnections and uncertainties of the subsystem model.

Whereas most of these results are based on sufficient stability conditions and, thus, give conservative results on the interaction relation, Bhattacharyya (1987) derived restrictions on structural perturbations, which have necessarily to be satisfied if the perturbed system should be stable. Lunze (1988) and Šiljak (1989) considered decentralized design problems explicitly as problems of robust control.

An analogous frequency-domain design method was derived from the Nyquist array method (Rosenbrock 1974) by Nwokah (1980). Accordingly, the decentralized control stations have to be chosen so as to ensure the diagonal dominance of the open-loop frequency response matrix. An extension of this method from scalar to multidimensional interconnection signals among the subsystems was given by Bennett and Baras (1980) and Hung and Limebeer (1984) who defined the property of block-diagonal dominance as an appropriate generalization of the dominance property mentioned above. All these results were presented in the general framework of robust decentralized control by Lunze (1988).

The idea of using coarse models of the subsystems j ($j \neq i$) in order to approximate the surrounding subsystem i (as in Figure 9.2) has been proposed by Lunze (1980c) and Litz (1983). A particular type of the coarse model is received by overlapping decomposition. The design of decentralized controllers for overlapping subsystems was investigated by Ikeda *et al.* (1981) and Ikeda and Šiljak (1984). The estimation and control of discrete-time systems subject to Gaussian noise were considered by Hodžić and Šiljak (1986).

Šiljak (1980a,b) and Ladde and Šiljak (1981) considered multi-controller configurations within the framework of overlapping decomposition, where one control station and the whole plant appear in each overlapping subsystem. They derived conditions under which the controller is reliable in the sense that some controllers may fail and can be disconnected from the plant for repair without losing the stability of the remaining system. The integrity is extended to reliability studies, where the structural changes of the control configuration are described by Markov processes.

The idea of proving closed-loop stability by means of the composite-system method and of deriving requirements on the decentralized control laws from these stability criteria can also be used for adaptive control. Results along these lines were obtained, for example, by Gavel and Šiljak (1985) who used a scalar Lyapunov function (Section 8.2) to derive an adaptation law for the decentralized controller, for which the closed-loop system with initially unknown parameters is known to be globally stable.

10

Decentralized Control of Weakly Coupled Systems

10.1 DECENTRALIZED CONTROL OF HIERARCHICALLY STRUCTURED SYSTEMS

The simplifications of the design problem which can be gained from the weakness of subsystem interactions will be investigated in this chapter. The first section is devoted to systems which are not strongly coupled in the sense of Definition 3.1 but have a hierarchical structure. It was explained in Section 3.2 that due to the absence of certain couplings the subsystems can be ordered in such a way that if there is an interaction from subsystem i to subsystem j then no interaction exists in the opposite direction.

It will now be shown that the stability analysis is simplified as a result of the hierarchical structure, but that in the controller design concerning the system dynamics such a simplification is, in general, not possible.

Algebraically, the hierarchical structure of the overall system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \operatorname{diag} \mathbf{B}_i\, \mathbf{u} \\ \mathbf{y} &= \operatorname{diag} \mathbf{C}_i\, \mathbf{x} \end{aligned} \tag{10.1.1}$$

is reflected by the property of $\mathbf{A}$ to be block triangular (cf. eqns (3.1.14) and (3.2.12))

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \dots & \mathbf{A}_{NN} \end{pmatrix}.$$

This property is preserved under decentralized control

$$\mathbf{u}_i = -\mathbf{K}_i(\mathbf{y}_i - \mathbf{v}_i) \tag{10.1.2}$$

independently of whether the control stations have dynamical parts or not. In the model of the closed-loop system (10.1.1) and (10.1.2)

$$\begin{aligned}\dot{\mathbf{x}} &= \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{v} \\ \mathbf{y} &= \bar{\mathbf{C}}\mathbf{x}\end{aligned} \tag{10.1.3}$$

the matrix $\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}(\text{diag } \mathbf{K}_i)\mathbf{C}$ has the block-triangular form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} - \mathbf{B}_1\mathbf{K}_1\mathbf{C}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} - \mathbf{B}_2\mathbf{K}_2\mathbf{C}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{N1} & \mathbf{A}_{N2} & \cdots & \mathbf{A}_{NN} - \mathbf{B}_N\mathbf{K}_N\mathbf{C}_N \end{pmatrix}$$

where the diagonal blocks represent the system matrices of the isolated subsystems

$$\begin{aligned}\dot{\mathbf{x}}_i &= (\mathbf{A}_{ii} - \mathbf{B}_i\mathbf{K}_i\mathbf{C}_i)\mathbf{x}_i + \mathbf{B}_i\mathbf{K}_i\mathbf{v}_i \\ \mathbf{y}_i &= \mathbf{C}_i\mathbf{x}_i.\end{aligned} \tag{10.1.4}$$

Therefore, Corollary 3.1 yields the following result.

Theorem 10.1

If the system has a hierarchical structure then the overall closed-loop system (10.1.3) is stable if and only if the isolated closed-loop subsystems (10.1.4) are stable.

If only the stability of the closed-loop system is involved, the control stations of a decentralized controller can be designed independently of each other for the isolated subsystems. The interactions of the subsystems must be known only to the extent that the hierarchical structure of the overall system can be proved. For the design, no information has to be available about the subsystem interconnections.

If the isolated subsystems are controllable and observable, they can be stabilized by the attached control station.

Corollary 10.1

If the overall system has a hierarchical structure then it can be stabilized by decentralized control if and only if the isolated subsystems are completely controllable and completely observable through their input $\mathbf{u}_i$ and output $\mathbf{y}_i$.

This corollary shows that systems with hierarchical structure represent a class of systems that can be stabilized by decentralized state feedback $\mathbf{u}_i = -\mathbf{K}_i\mathbf{x}_i$.

If the design aims include requirements on the dynamical I/O behaviour of the closed-loop system, simplifications of the design problem cannot be obtained so easily. Since the decentralized controller does not alter the interconnections between the subsystems, the command input $\mathbf{v}_i$ or disturbances, which affect subsystem i, do influence the output $\mathbf{y}_j$ if interconnections from subsystem i to subsystem j exists. Typically, as the design specifications refer not only to the direct couplings of $\mathbf{v}_i$ to $\mathbf{y}_i$ but also to the cross couplings from $\mathbf{v}_i$ to $\mathbf{y}_j$, specific properties of the plant or specific design requirements have to be exploited in order to make a completely decentralized design possible.

It is interesting to note that the hierarchical structure of the plant does not even ensure that the control problem can be solved by means of a sequential design procedure, which follows the direction of the interactions. In such a design procedure the ith control station is chosen before the jth controller if the interactions are directed from subsystem i to subsystem j. However, as a result of these interactions the question of whether a control station j can be found so as to satisfy requirements on the couplings from $\mathbf{v}_i$ to $\mathbf{y}_j$ depends upon the ith controller, which is already fixed. In other words, although the plant has one-directional dynamical interconnections the controller parameters are dependent in both directions. Therefore, a completely decentralized design is, in general, not possible.

Example 10.1 (*Decentralized control of a string of vehicles*)

The following example should illustrate that the design specifications concerning the dynamical I/O properties of the closed-loop system cannot be simply divided into disjoint sets for the isolated subsystems, but that due to the structure of the example system such a decomposition may be possible. Consider a string of vehicles one behind the other moving in a straight line (Figure 10.1). Each vehicle represents a subsystem of the overall plant. If the velocity of the vehicles is used as state variable x_{i1} and the distance between a vehicle and its predecessor as

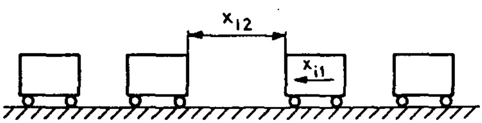


Figure 10.1 String of vehicles

state variable x_{i2} then a simple model of the form

$$\dot{x}_{11} = \frac{-1}{m_1} x_{11} + \frac{1}{m_1} u_1 \qquad x_1(0) = x_{10} \tag{10.1.5}$$

$$y_1 = z_1 = x_{11}$$

and

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \end{pmatrix} = \begin{pmatrix} -1/m_i & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix} + \begin{pmatrix} 1/m_i \\ 0 \end{pmatrix} u_i + \begin{pmatrix} 0 \\ 1 \end{pmatrix} s_i$$

$$\mathbf{y}_i = \mathbf{x}_i \qquad z_i = (1 \ \ 0)\mathbf{x}_i \qquad (i = 2, 3, \ldots, N) \tag{10.1.6}$$

with $\mathbf{x}_i = (x_{i1} \ \ x_{i2})'$ can be set up where m_i denotes the mass of the ith vehicle. The input u_i represents the accelerating force in terms of the position of the accelerator. The subsystems are serially interconnected

$$s_i = z_{i-1} \qquad (i = 2, 3, \ldots, N). \tag{10.1.7}$$

The following control aims have to be obtained:

1. The closed-loop system has to be stable.
2. Disturbances, which can be modelled as initial deviations $\mathbf{x}_i(0)$ of the vehicle velocity or the vehicle distance, have to be asympotically attenuated.
3. A distance deviation $x_{i2}(0)$ between succeeding vehicles has to be decreased monotonically (without overshoot), that is

$$x_{i2}(0)\dot{x}_{i2}(t) \leqslant 0 \tag{10.1.8}$$

should hold.

This problem of replacing the driver of a vehicle by a feedback controller is practically relevant to transportation systems with many different vehicles on a common rail network. The controller must be decentralized since no coordination between the vehicles is possible and because the vehicles are chained randomly. A static controller

$$u_i = -(k_{i1} \ \ k_{i2})\mathbf{x}_i \tag{10.1.9}$$

suffices to solve the problem since no external disturbance signals are involved. If the closed-loop system is stable, then the design requirement (point 2 above) is satisfied.

The plant is unstable, but, due to Corollary 10.1, it can be stabilized by a decentralized controller because each isolated subsystem is controllable. Moreover, the closed-loop stability is ensured if and only if all closed-loop subsystems are stable.

The design requirement (3) refers to the overall system behaviour since the distance between succeeding vehicles cannot be evaluated if the vehicles are considered separately from one another. In the model (10.1.6) this fact can be seen by the influence of s_i upon $\dot{x}_{i2}$. Therefore, the question arises of how to reformulate the design requirement (3) for the isolated subsystem. For this purpose consider the ith closed-loop subsystem for the initial state $\mathbf{x}_i(0) = (0 \;\; x_{i2}(0))'$ used in (3). Equations (10.1.6) and (10.1.9) yield

$$x_{i2}(t) = (1 \;\; 0) \exp\left[\begin{pmatrix} \frac{-1}{m_i}(1+k_{i1}) & \frac{1}{m_i}k_{i2} \\ 1 & 0 \end{pmatrix} t\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_{i2}(0).$$

Hence, inequality (10.1.8) is satisfied whenever the impulse response of the isolated closed-loop system (10.1.6) and (10.1.9) concerning the interconnection input s_i and the interconnection output z_i is non-negative

$$g_{zsi}(t) = (1 \;\; 0) \exp\left[\begin{pmatrix} \frac{-1}{m_i}(1+k_{i1}) & \frac{1}{m_i}k_{i2} \\ 1 & 0 \end{pmatrix} t\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geqslant 0. \qquad (10.1.10)$$

That is, the closed-loop overall system satisfies the design requirements (1)–(3) if and only if all isolated closed-loop subsystems are stable and have non-negative impulse responses g_{zsi}.

This shows that it is possible to divide the overall design problem into completely independent design problems. The control stations can be found as solutions of these completely independent design problems. Note that the subsystem design tasks do not refer to the original specification (3) but to quite another design requirement (10.1.10).

In the resulting decentralized design scheme, even more involved control problems can be solved with reasonable effort. If, for example, the design requirements (1)–(3) have to be satisfied despite mass variations, the decentralized design problem has simply to be extended by a robustness requirement. Instead of the model (10.1.6), the extended model

$$\begin{aligned} \begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} u_i + \begin{pmatrix} 0 \\ 1 \end{pmatrix} s_i + \begin{pmatrix} 0 \\ -1 \end{pmatrix} f_i \\ \mathbf{y}_i &= \mathbf{x}_i + \begin{pmatrix} 1 \\ 0 \end{pmatrix} f_i \qquad\qquad (10.1.11) \\ z_i &= (1 \;\; 0)\mathbf{x}_i + f_i \qquad (i = 2, 3, \ldots, N) \end{aligned}$$

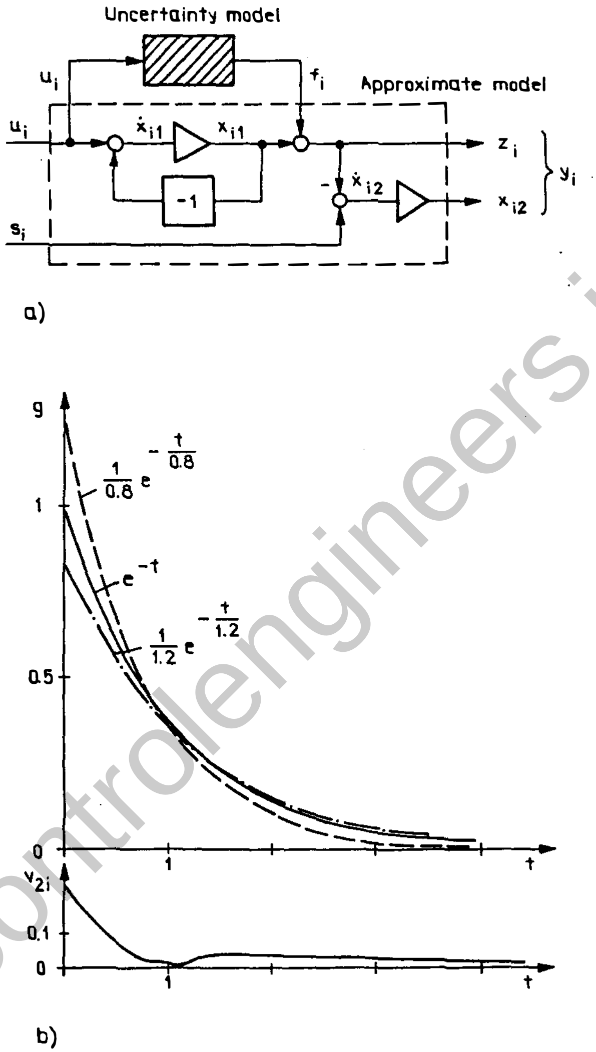


Figure 10.2 Modelling the vehicle behaviour for uncertain mass: (a) model of vehicle i; (b) determination of $V_{2i}(t)$ as the upper bound of the difference between the impulse responses of the approximate model (10.1.11) and the original system for $0.8 \leqslant m_i \leqslant 1.2$

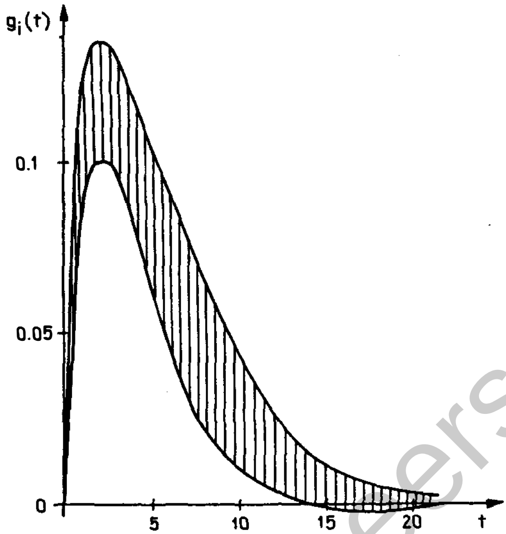


Figure 10.3 Tolerance band of the impulse response

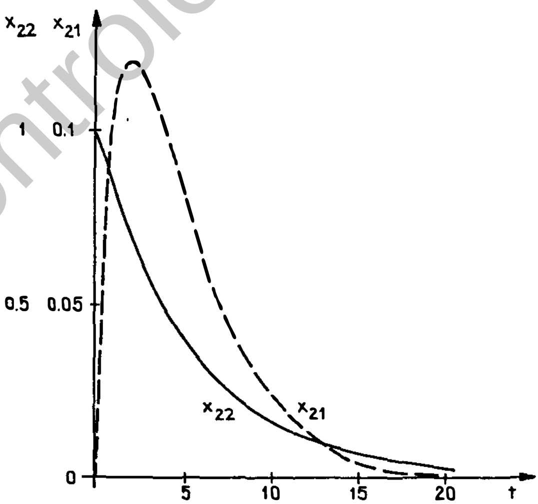


Figure 10.4 Velocity $x_{21}(t)$ and distance $x_{22}(t)$ between vehicles 1 and 2

with

$$|f_i(t)| \leqslant v_{2i}(t) * |u_i|$$

and

$$v_{2i}(t) = |1.25 \exp(-1.25t) - \exp(-t)| \tag{10.1.12}$$

(cf. Figure 10.2) has to be used. This model holds for

$$0.8 \leqslant m_i \leqslant 1.2.$$

It has the structure of Figure 2.8(b) with u_i and f_i instead of $\mathbf{d}$ and $\mathbf{f}$. The controller has to stabilize the system (10.1.11) and to ensure the non-negative definiteness of $g_{zsi}(t)$ for all model errors f_i described by eqn (10.1.12).

The solution of this extended problem can be obtained by means of methods for centralized robust control (Section 2.6). With the parameters $k_{i1} = -k_{i2} = 0.22$ the isolated closed-loop system is stable and the impulse response g_{zsi} lies within the tolerance band depicted in Figure 10.3. Hence, the requirements on the subsystems are (nearly) satisfied. Figure 10.4 shows that an initial distance deviation $x_{22}(0)$ is removed monotonically within the overall system. □

10.2 DECENTRALIZED CONTROL OF MULTI-TIMESCALE SYSTEMS

This section highlights another possibility of the decentralization of the design process where again structural properties of the plant will be exploited. The multi-time-scale system (3.5.11)

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \mathbf{A}_{00}\mathbf{x}_0 + \sum_{j=1}^{N} \mathbf{A}_{0j}\mathbf{x}_j + \sum_{j=1}^{N} \mathbf{B}_{0j}\mathbf{u}_j \\ \varepsilon_i\dot{\mathbf{x}}_i &= \mathbf{A}_{i0}\mathbf{x}_0 + \mathbf{A}_{ii}\mathbf{x}_i + \mathbf{B}_{ii}\mathbf{u}_i \qquad (i = 1, \ldots, N) \end{aligned} \tag{10.2.1}$$

consists of a slow subsystem with state $\mathbf{x}_0$ and N fast subsystems, which do not interact directly with each other but are coupled together only via the slow subsystem. A controller is to be found which minimizes the cost function

$$I = \sum_{i=0}^{N} I_i + \frac{1}{2}\int_0^\infty \mathbf{x}_0'\mathbf{Q}_0\mathbf{x}_0 \,\mathrm{d}t \tag{10.2.2}$$

with

$$I_i = \frac{1}{2}\int_0^\infty (\mathbf{x}_i'\mathbf{Q}_i\mathbf{x}_i + \mathbf{u}_i'\mathbf{R}_i\mathbf{u}_i)\,\mathrm{d}t. \tag{10.2.3}$$

Owing to the temporal separation of the dynamics of the fast and the slow subsystems it is reasonable to decompose the state and each control $\mathbf{u}_i$ into a slow and a fast part:

$$\mathbf{x}_i = \mathbf{x}_{si} + \mathbf{x}_{fi} \qquad \mathbf{u}_i = \mathbf{u}_{si} + \mathbf{u}_{fi} \qquad (i = 0, 1, \ldots, N). \tag{10.2.4}$$

First, the slow part of the system performance is considered. The assumptions $\mathbf{x}_{fi} = \mathbf{0}$, $\mathbf{u}_{fi} = \mathbf{0}$ and $\varepsilon_i = 0$ lead to eqn (3.5.12), which is written here as

$$\mathbf{x}_{si} = \mathbf{C}_{i0}\mathbf{x}_{s0} + \mathbf{D}_{i0}\mathbf{u}_{si} \qquad (i = 1, \ldots, N) \tag{10.2.5}$$

with

$$\mathbf{C}_{i0} = -\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0} \qquad \mathbf{D}_{i0} = -\mathbf{A}_{ii}^{-1}\mathbf{B}_{ii} \tag{10.2.6}$$

and to

$$\dot{\mathbf{x}}_{s0} = \mathbf{A}_s\mathbf{x}_{s0} + \mathbf{B}_s\mathbf{u}_s \tag{10.2.7}$$

with

$$\begin{aligned} \mathbf{A}_s &= \mathbf{A}_{00} - \sum_{i=1}^{N} \mathbf{A}_{0i}\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0} \\ \mathbf{B}_s &= (\mathbf{B}_{s1} \;\; \mathbf{B}_{s2} \;\; \ldots \;\; \mathbf{B}_{sN}) \\ \mathbf{B}_{si} &= \mathbf{B}_{0i} - \mathbf{A}_{0i}\mathbf{A}_{ii}^{-1}\mathbf{B}_{ii} \end{aligned} \tag{10.2.8}$$

and $\mathbf{u}_s = (\mathbf{u}_{s1}' \;\; \mathbf{u}_{s2}' \;\; \ldots \;\; \mathbf{u}_{sN}')'$ (cf. eqns (3.5.13) and (3.5.14)). For the performance index (10.2.2) the relations

$$I_s = \sum_{i=1}^{N} I_{si} + \frac{1}{2}\int_0^\infty (\mathbf{x}_{s0}'\mathbf{Q}_0\mathbf{x}_{s0})\, dt$$

with

$$\begin{aligned} I_{si} &= \frac{1}{2}\int_0^\infty (\tilde{\mathbf{x}}_{si}'\mathbf{Q}_i\tilde{\mathbf{x}}_{si} + \mathbf{u}_{si}'\mathbf{R}_i\mathbf{u}_{si})\, dt \\ &= \frac{1}{2}\int_0^\infty [\mathbf{x}_{s0}'\mathbf{C}_{i0}'\mathbf{Q}_i\mathbf{C}_{i0}\mathbf{x}_{s0} + 2\mathbf{u}_{si}'\mathbf{D}_{i0}'\mathbf{Q}_i\mathbf{C}_{i0}\mathbf{x}_{s0} \\ &\qquad + \mathbf{u}_{si}'(\mathbf{R}_i + \mathbf{D}_{i0}'\mathbf{Q}_i\mathbf{D}_{i0})\mathbf{u}_{si}]\, dt \\ &= \frac{1}{2}\int_0^\infty (\mathbf{x}_{s0}'\mathbf{Q}_{si}\mathbf{x}_{s0} + 2\mathbf{u}_{si}'\mathbf{S}_i\mathbf{x}_{s0} + \mathbf{u}_{si}'\mathbf{R}_{si}\mathbf{u}_{si})\, dt \end{aligned}$$

$(i = 1, 2, \ldots, N)$ and

$$\mathbf{Q}_{si} = \mathbf{C}_{i0}'\mathbf{Q}_i\mathbf{C}_{i0} \qquad \mathbf{S}_i = \mathbf{D}_{i0}'\mathbf{Q}_i\mathbf{C}_{i0} \qquad \mathbf{R}_{si} = \mathbf{R}_i + \mathbf{D}_{i0}'\mathbf{Q}_i\mathbf{D}_{i0} \tag{10.2.9}$$

and, finally,

$$I_s = \frac{1}{2} \int_0^\infty (\mathbf{x}_{s0}'\mathbf{Q}_s\mathbf{x}_{s0} + 2\mathbf{u}_s'\mathbf{S}\mathbf{x}_{s0} + \mathbf{u}_s'\mathbf{R}_s\mathbf{u}_s)\ dt$$

with

$$\mathbf{Q}_s = \mathbf{Q}_0 + \sum_{i=1}^{N} \mathbf{Q}_{si} \qquad \mathbf{R}_s = \operatorname{diag} \mathbf{R}_{si} \qquad \mathbf{S} = \begin{pmatrix} \mathbf{S}_1 \\ \vdots \\ \mathbf{S}_N \end{pmatrix} \tag{10.2.10}$$

is obtained. That is, for the slow part the optimization problem

$$\min_{\mathbf{u}_s = -\mathbf{K}_s\mathbf{x}_{s0}} \quad I_s \quad \text{subject to eqn (10.2.7)}$$

has to be solved.

According to Section 2.4, the solution is

$$\mathbf{u}_s = -\mathbf{R}_s^{-1}(\mathbf{B}_s'\bar{\mathbf{P}} + \mathbf{S}')\mathbf{x}_{s0} \tag{10.2.11}$$

with $\bar{\mathbf{P}}$ being the solution of the Riccati equation

$$(\mathbf{A}_s - \mathbf{B}_s\mathbf{R}_s^{-1}\mathbf{S})'\bar{\mathbf{P}} + \bar{\mathbf{P}}(\mathbf{A}_s - \mathbf{B}_s\mathbf{R}_s^{-1}\mathbf{S}) - \bar{\mathbf{P}}\mathbf{B}_s\mathbf{R}_s^{-1}\mathbf{B}_s\bar{\mathbf{P}} + \mathbf{Q}_s - \mathbf{S}'\mathbf{R}_s^{-1}\mathbf{S} = \mathbf{0}. \tag{10.2.12}$$

Owing to the structure of the matrices involved, the controller (10.2.11) can be decomposed into N state feedbacks

$$\mathbf{u}_{si} = -\mathbf{K}_{si}\mathbf{x}_{s0} \tag{10.2.13}$$

with

$$\mathbf{K}_{si} = \mathbf{R}_{si}^{-1}(\mathbf{B}_{si}'\bar{\mathbf{P}} + \mathbf{S}_i). \tag{10.2.14}$$

Second, the fast part of the overall system is investigated by means of the model (3.5.16)

$$\dot{\mathbf{x}}_{fi} = \mathbf{A}_{ii}\mathbf{x}_{fi} + \mathbf{B}_{ii}\mathbf{u}_{fi}. \tag{10.2.15}$$

For $\mathbf{u}_{si} = \mathbf{0}$, $\mathbf{x}_i = \mathbf{x}_{fi}$ $(i = 1, \ldots, N)$ and $\mathbf{x}_0 = \mathbf{x}_{f0} = \mathbf{0}$ the performance index (10.2.2) is

$$I_f = \sum_{i=1}^{N} I_{fi}$$

with

$$I_{fi} = \frac{1}{2} \int_0^\infty (x_{fi}'\mathbf{Q}_i\mathbf{x}_{fi} + \mathbf{u}_{fi}'\mathbf{R}_i\mathbf{u}_{fi})\ dt.$$

The optimization problem

$$\min_{\mathbf{u}_f} I_f$$

breaks down into the N independent problems

$$\min_{\mathbf{u}_{fi} = -\mathbf{K}_{fi}\mathbf{x}_{fi}} I_{fi} \quad \text{subject to eqn (10.2.15)}$$

which can be solved as described in Section 2.4. The solutions are

$$\mathbf{u}_{fi} = -\mathbf{K}_{fi}\mathbf{x}_{fi} \tag{10.2.16}$$

$$\mathbf{K}_{fi} = \mathbf{R}_i^{-1}\mathbf{B}_{ii}'\mathbf{P}_{ii} \tag{10.2.17}$$

($i = 1, 2, \ldots, N$) where all $\mathbf{P}_{ii}$ satisfy the corresponding Riccati equation

$$\mathbf{A}_{ii}'\mathbf{P}_{ii} + \mathbf{P}_{ii}\mathbf{A}_{ii} - \mathbf{P}_{ii}\mathbf{B}_{ii}\mathbf{R}_i^{-1}\mathbf{B}_{ii}'\mathbf{P}_{ii} + \mathbf{Q}_i = \mathbf{0}. \tag{10.2.18}$$

These equations can be solved independently of each other. They merely require the subsystem models to be available.

A suboptimal solution for the overall system can be obtained by combining eqns (10.2.13), (10.2.14), (10.2.16) and (10.2.17) with $\mathbf{x}_i = \mathbf{x}_{si} + \mathbf{x}_{fi}$, $\mathbf{x}_{f0} = \mathbf{0}$ and $\mathbf{x}_{si}$ from eqn (10.2.15). The control stations are

$$\mathbf{u}_i = \mathbf{u}_{fi} + \mathbf{u}_{si} = -\mathbf{K}_{fi}\mathbf{x}_i - \mathbf{K}_{i0}\mathbf{x}_0 \tag{10.2.19}$$

with

$$\mathbf{K}_{i0} = [(\mathbf{I} + \mathbf{K}_{fi}\mathbf{A}_{ii}^{-1}\mathbf{B}_{ii})\mathbf{K}_{si} + \mathbf{K}_{fi}\mathbf{A}_{ii}^{-1}\mathbf{A}_{i0}]. \tag{10.2.20}$$

The second part constitutes a reaction concerning the performance of the slow subsystem, and the first part represents the local feedback of the fast subsystem state (Figure 10.5). The control stations are overlapping.

The solutions to the separate design problems for the approximate models (10.2.7) and (10.2.15) exist if $(\mathbf{A}_s, \mathbf{B}_s)$ or $(\mathbf{A}_{ii}, \mathbf{B}_{ii})$, respectively, are controllable.

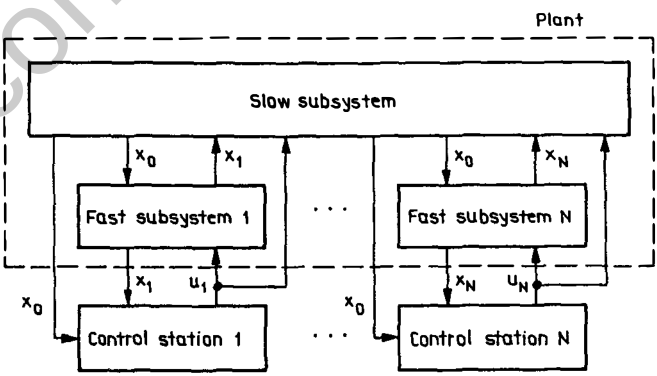


Figure 10.5 Decentralized control of a multi-timescale system

Algorithm 10.1

Given: Multi-time-scale system (10.2.1) and performance index (10.2.2).

1. Determine the matrices $\mathbf{A}_s, \mathbf{B}_s, \mathbf{Q}_s, \mathbf{S}, \mathbf{R}_s$ of the design problem for the slow performance by means of eqns (10.2.8)–(10.2.10).
2. Solve eqn (10.2.12) and determine the controller matrices $\mathbf{K}_{si}$ from eqn (10.2.14).
3. For each subsystem ($i = 1, 2, \ldots, N$) solve the corresponding eqn (10.2.18) separately and determine the controller matrix $\mathbf{K}_{fi}$ from eqn (10.2.17).
4. Determine the decentralized controller (10.2.19).

Result: Suboptimal decentralized controller (10.2.19).

Steps 1 and 2 concern the overall system under the assumption that the fast subsystems have reached their steady state, but do not involve manipulations with the complete model. Eqn (10.2.12) has the order of the slow subsystem. The result of these steps represents the overlapping part of the controller. Steps 3 and 4 have a completely decentralized information structure. All manipulations concern the isolated fast subsystems.

This method of solution leads to a suboptimal controller, but it is much simpler than the direct solution of the overall optimization problem (10.2.1) and (10.2.2), which has been explained in Chapter 7.

BIBLIOGRAPHICAL NOTES

The simplifications that can be gained from the hierarchical structure of large-scale systems were recognised early on. Özgüner and Perkins (1978) decomposed the optimal control problem on this basis and developed a sequential design procedure, where in the last step the complete overall model has to be used. The complete decentralization of the design process, which has been explained in Example 10.1, was proposed by Bakule and Lunze (1985, 1986). They used the same vehicle control example as Levine and Athans (1966) for centralized control and as the aforementioned authors for decentralized control. The practical importance of the design requirement (3) used in Example 10.1 was emphasized for example, by the detailed study of dynamical phenomena within vehicle strings made by Ullmann (1974).

If the string of vehicles is closed to become a circle, the plant loses its hierarchical structure and becomes strongly coupled. The design problems which occur in this case were described by Yoshikawa *et al.* (1983). A reasonable way towards the decentralization of the design process is the use of overlapping subsystems as described by Ikeda and Šiljak (1984). There, the ith subsystem together with its predecessor is used as a model for the design of the ith control station.

Decentralized multicontroller configurations for serially interconnected systems were considered by Bakule and Lunze (1988).

Multi-timescale systems under decentralized control were investigated by Khalil and Kokotović (1979), Özgüner (1979), whose results are based on the work of Chow and Kokotović (1976) concerning centralized control and have been reviewed in Section 10.2, Kokotović (1981) and Ladde and Šiljak (1983). Since then, timescale decomposition of large-scale systems has developed into a powerful method which also covers non-linear systems. It is a field of its own with several review papers that may be consulted for a thorough introduction, for example in the book edited by Kokotović *et al.* (1986). Those methods which are relevant to the decentralized control of linear and non-linear systems have also been reviewed by Mahmoud and Singh (1981b), Bernussou and Titli (1982) and Jamshidi (1983).

Although most of the results concern the existence of some bounds for the parameter ε such that a given property is preserved, there are also methods for estimating such bounds. For example, Grujić (1979) used the vector Lyapunov function approach to stability analysis in order to determine quantitative bounds on the fast subsystem behaviour in terms of constants, which are similarly introduced as c_{ij} and b_{ij} in Section 8.3.

11

Decentralized PI Controllers

A considerable reduction of the design problem can also be gained from specific structures of the controller to be designed. Whereas in Chapters 10 and 12 the structural properties of the plant are exploited, it will be shown in the following that simplifications of the design problem can be derived if the control law is restricted to be of PI character. This holds true even if the subsystems are strongly coupled.

The application of a decentralized PI controller is a direct consequence of the requirement of asymptotic regulation for stepwise disturbance and command signals. According to the internal model principle (Chapter 5), each control station has to have m_i integrators. The control law is given by

$$\dot{\mathbf{x}}_{ri} = \mathbf{y}_i - \mathbf{v}_i$$
$$\mathbf{u}_i = -\mathbf{K}_{Pi}(\mathbf{y}_i - \mathbf{v}_i) - \mathbf{K}_{Ii}\mathbf{x}_{ri}.$$

The problem of designing decentralized PI controllers will be considered for the additional requirements that the controller should ensure closed-loop stability even if some control stations are disconnected from the plant (Section 11.1), that the control laws should be obtained without setting up a complete model of the plant by sequential on-line tuning (Section 11.2), or that the design of the control stations should be carried out completely independently (Section 11.3).

11.1 EXISTENCE OF ROBUST DECENTRALIZED PI CONTROLLERS

For a stable linear plant

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned} \tag{11.1.1}$$

a decentralized PI controller should be designed in such a way that the closed-loop stability is maintained if some control stations are disconnected from the plant. This task concerns the stability in case of sensor or actuator failures. It also refers to the assurance of closed-loop stability during the implementation of the control stations, which under practical circumstances can never be applied strictly simultaneously. The same type of operational conditions occur if control stations are deliberately shut off. In all cases, the structural perturbations of the closed-loop system can be described by scalars e_i in the control law

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{y}_i - \mathbf{v}_i \\ \mathbf{u}_i &= -e_i\mathbf{K}_{Pi}(\mathbf{y}_i - \mathbf{v}_i) - e_i\mathbf{K}_{Ii}\mathbf{x}_{ri}. \end{aligned} \tag{11.1.2}$$

e_i indicates whether the control station i is ($e_i = 1$) or is not ($e_i = 0$) in operation. All the different operational modes of the closed-loop system (11.1.1) and (11.1.2) can be described by the vector $\mathbf{e}$

$$\mathbf{e} \in \mathscr{E} = \{(e_1 \;\; e_2 \;\; \ldots \;\; e_N)' : \; e_i \in \{0, 1\}\}. \tag{11.1.3}$$

A further extension of the design requirements can be made for controller matrices of the type

$$\mathbf{K}_{Ii} = a\hat{\mathbf{K}}_{Ii} \qquad \mathbf{K}_{Pi} = b\hat{\mathbf{K}}_{PI}. \tag{11.1.4}$$

It is a practically desirable property that the closed-loop system (11.1.1) and (11.1.2) remains stable for all $\mathbf{e} \in \mathscr{E}$ even if the loop gains are reduced, that is for all scalars a and b of given intervals $0 < a < \bar{a}$ and $0 \leqslant b < \bar{b}$. Hence, the control law has to be robust enough so as to tolerate the uncertainties described by the set $\mathscr{E}$ and the intervals $(0, \bar{a})$ and $[0, \bar{b})$.

Problem 11.1

For the stable plant (11.1.1) and given structural constraints on the control law find controller matrices $\hat{\mathbf{K}}_{Ii}$ and $\hat{\mathbf{K}}_{Pi}$ and bounds $\bar{a}$ and $\bar{b}$ such that the closed-loop system (11.1.1), (11.1.2) and (11.1.4) is stable for all $\mathbf{e} \in \mathscr{E}$ and all $0 < a < \bar{a}$, $0 \leqslant b < \bar{b}$.

In this section, the existence of such decentralized controllers is investigated. A decentralized design algorithm will be developed in Section 11.2. In both sections it is assumed that $\dim \mathbf{u}_i = \dim \mathbf{y}_i = m_i$ holds but all results can be extended to systems with $\dim \mathbf{u}_i \geqslant \dim \mathbf{y}_i$.

In the following investigations concerning the existence of a solution to Problem 11.1, it can be assumed without loss of generality that the controller is a pure I controller ($\mathbf{K}_{Pi} = \mathbf{0}$)

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{y}_i - \mathbf{v}_i \\ \mathbf{u}_i &= -\mathbf{K}_{Ii}\mathbf{x}_{ri} \end{aligned} \tag{11.1.5}$$

If an I controller (11.1.5) exists then a PI controller (11.1.2) with sufficiently small parameter b also yields a stable loop. The existence of such a controller is described by the decentralized integral controllability.

Definition 11.1

A linear system (11.1.1) is called *decentralized integral controllable* with respect to a given structural restriction on the control law if there exist a set of controller matrices $\hat{\mathbf{K}}_{Ii}$ $(i = 1, \ldots, N)$ and a bound $\bar{a}$ such that the closed-loop system (11.1.1), (11.1.4) and (11.1.5) is stable for all $\mathbf{e} \in \mathscr{E}$ and all $0 < a < \bar{a}$.

Criteria for decentralized integral controllability can be derived by means of the assertions concerning centralized PI control stated in Theorem 2.14. Equations (2.4.18)

$$\det(\mathbf{K}_s \mathbf{K}_I) > 0 \tag{11.1.6}$$

and (2.4.20)

$$\mathrm{Re}\,[\lambda_i[\mathbf{K}_s \hat{\mathbf{K}}_I]] > 0 \qquad (i = 1, 2, \ldots, n) \tag{11.1.7}$$

where $\mathbf{K}_s = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is the static transmission matrix of the plant (11.1.1), have to be used for I controllers that are subjected to the structural constraints

$$\mathbf{K}_I = -\operatorname{diag} e_i \mathbf{K}_{Ii}.$$

A compatible partitioning of the static transmission matrix $\mathbf{K}_s$ yields

$$\mathbf{K}_s = \begin{pmatrix} \mathbf{K}_{s11} & \mathbf{K}_{s12} & \ldots & \mathbf{K}_{s1N} \\ \vdots & \vdots & & \vdots \\ \mathbf{K}_{sN1} & \mathbf{K}_{sN2} & \ldots & \mathbf{K}_{sNN} \end{pmatrix}. \tag{11.1.8}$$

A necessary condition for decentralized integral controllability

The condition (11.1.6) has to be satisfied with respect to the inputs $\mathbf{u}_i$ and outputs $\mathbf{y}_i$ of those subsystems whose control stations are in operation ($e_i = 1$). If only the ith control station is attached to the plant (11.1.1), that is if $e_i = 1$ and $e_j = 0$ for $j = 1, \ldots, N$, $j \neq i$ hold, eqn (11.1.6) becomes

$$\det(\mathbf{K}_{sii} \mathbf{K}_{Iii}) > 0. \tag{11.1.9}$$

Equation (11.1.9) implies $\det(\mathbf{K}_{sii}) \neq 0$. Without loss of generality it can be assumed that

$$\det(\mathbf{K}_{sii}) > 0 \qquad (i = 1, \ldots, N) \tag{11.1.10}$$

hold. This assumption can be satisfied by appropriately defining the signs of the input and output signals. Eqns (11.1.9) and (11.1.10) yield

$$\det(\mathbf{K}_{Ii}) > 0 \qquad (i = 1, \ldots, N). \tag{11.1.11}$$

If more than one control station is in operation, the static model of the plant and the controller matrix has to be extended correspondingly. For example, if $e_1 = 0$ and $e_i = 1$ $(i = 2, \ldots, N)$ hold, eqn (11.1.6) has to be applied to the reduced plant (11.1.1) with input vector $(\mathbf{u}_2' \; \ldots \; \mathbf{u}_N')'$ and output vector $(\mathbf{y}_2' \; \ldots \; \mathbf{y}_N')'$. The static behaviour of this system is described by

$$\begin{pmatrix} \mathbf{y}_2 \\ \mathbf{y}_3 \\ \vdots \\ \mathbf{y}_N \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{s22} & \mathbf{K}_{s23} & \cdots & \mathbf{K}_{s2N} \\ \mathbf{K}_{s32} & \mathbf{K}_{s33} & \cdots & \mathbf{K}_{s3N} \\ \vdots & \vdots & & \vdots \\ \mathbf{K}_{sN2} & \mathbf{K}_{sN3} & \cdots & \mathbf{K}_{sNN} \end{pmatrix} \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{u}_3 \\ \vdots \\ \mathbf{u}_N \end{pmatrix}. \tag{11.1.12}$$

The static transmission matrix is obtained from $\mathbf{K}_s$ by deleting the first hyper row and column. Then the necessary condition (11.1.6) is

$$\det\left[\begin{pmatrix} \mathbf{K}_{s22} & \cdots & \mathbf{K}_{s2N} \\ \vdots & & \vdots \\ \mathbf{K}_{sN2} & \cdots & \mathbf{K}_{sNN} \end{pmatrix} \begin{pmatrix} \mathbf{K}_{I2} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{K}_{IN} \end{pmatrix}\right] > 0$$

which together with eqn (11.1.11) is equivalent to

$$\det \begin{pmatrix} \mathbf{K}_{s22} & \cdots & \mathbf{K}_{s2N} \\ \vdots & & \vdots \\ \mathbf{K}_{sN2} & \cdots & \mathbf{K}_{sNN} \end{pmatrix} > 0. \tag{11.1.13}$$

This inequality is identical to

$$\det[\hat{\mathbf{K}}_s(\mathbf{e})] > 0 \tag{11.1.14}$$

for $\mathbf{e} = (0 \;\; 1 \;\; 1 \;\; \ldots \;\; 1)'$ with

$$\hat{\mathbf{K}}_s(\mathbf{e}) = \begin{pmatrix} \mathbf{K}_{s11} & e_1\mathbf{K}_{s12} & \cdots & e_1\mathbf{K}_{s1N} \\ e_2\mathbf{K}_{s21} & \mathbf{K}_{s22} & \cdots & e_2\mathbf{K}_{s2N} \\ \vdots & \vdots & & \vdots \\ e_N\mathbf{K}_{sN1} & e_N\mathbf{K}_{sN2} & \cdots & \mathbf{K}_{sNN} \end{pmatrix}. \tag{11.1.15}$$

By considering other operational conditions $\mathbf{e} \in \mathscr{E}$ the inequality (11.1.14) can be proved to be necessary for decentralized integral controllability.

Theorem 11.1

Consider a stable plant (11.1.1) with static transmission matrix $\mathbf{K}_s$, whose partitioning (11.1.16) corresponds to that of the controller (11.1.2). Assume that condition (11.1.10) holds. A necessary condition for the decentralized integral controllability is given by (11.1.14) which has to be satisfied for all $\mathbf{e} \in \mathscr{E}$.

Condition (11.1.14) coincides with the requirement that the determinants of $\mathbf{K}_s$ and of all matrices which result from $\mathbf{K}_s$ by deleting corresponding hyper rows and hyper columns have to be positive. For systems with $\dim \mathbf{u}_i = \dim \mathbf{y}_i = 1$ condition (11.1.14) claims that all principal minors of $\mathbf{K}_s$ have to be positive.

It is of interest to note that the decentralized integral controllability depends only on the static transmission matrix of the plant. Therefore, only a static model of the overall system is necessary if this property is to be checked.

A sufficient condition for decentralized integral controllability

If these investigations are repeated with (11.1.7) instead of (11.1.6), condition (11.1.7) has to be replaced by the claim

$$\mathrm{Re}[\lambda_i[\mathbf{K}_{sii}\hat{\mathbf{K}}_{Ii}]] > 0 \qquad (i = 1, \ldots, m_i) \tag{11.1.16}$$

which is obviously satisfied for

$$\hat{\mathbf{K}}_{Ii} = \mathbf{K}_{sii}^{-1}. \tag{11.1.17}$$

If such controller matrices are used, condition (11.1.7) for decentralized control with $\mathbf{e} = (0\ 1\ \ldots\ 1)'$ yields

$$\mathrm{Re}\left[\lambda_i\left[\begin{pmatrix} \mathbf{K}_{s22} & \cdots & \mathbf{K}_{s2N} \\ \vdots & & \vdots \\ \mathbf{K}_{sN2} & \cdots & \mathbf{K}_{sNN} \end{pmatrix}\begin{pmatrix} \mathbf{K}_{s22}^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{K}_{sNN}^{-1} \end{pmatrix}\right]\right] > 0 \tag{11.1.18}$$

$(i = 1, \ldots, m_2 + m_3 + \cdots + m_N)$. Equations (11.1.16)–(11.1.18) together are equivalent to

$$\mathrm{Re}[\lambda_i[\hat{\mathbf{K}}_s(\mathbf{e})\operatorname{diag} \mathbf{K}_{sii}^{-1}]] > 0 \tag{11.1.19}$$

for $\mathbf{e} = (0\ 0\ \ldots\ 0)'$ or $\mathbf{e} = (0\ 1\ 1\ \ldots\ 1)'$, respectively. All other operational modes lead to the same requirement (11.1.19) for the respective vectors $\mathbf{e} \in \mathscr{E}$.

Theorem 11.2

Consider the stable plant (11.1.1) with static transmission matrix $\mathbf{K}_s$, whose partitioning (11.1.11) corresponds to that of the controller (11.1.2). A sufficient condition for decentralized integral controllability of the plant (11.1.1) is given by the requirement (11.1.19) which has to be satisfied for all $\mathbf{e} \in \mathscr{E}$.

If condition (11.1.19) holds, a solution to Problem 11.1 is given by eqns (11.1.2), (11.1.4) and (11.1.17) for sufficiently small scalars a and b.

For systems with $\dim \mathbf{u}_i = \dim \mathbf{y}_i = 1$ whose static transmission matrix satisfies the assumption (11.1.10), the conditions (11.1.14) and (11.1.19) are equivalent and identical to the requirement that all principal minors of the matrix $\mathbf{K}_s$ have to be positive.

Corollary 11.1

Consider a stable linear system (11.1.1) with $\dim \mathbf{u}_i = \dim \mathbf{y}_i = 1$. Assume that $k_{sii} > 0$ $(i = 1, \ldots, N)$ hold. This plant is decentralized integral controllable if and only if all principal minors of $\mathbf{K}_s$ are positive. A solution to Problem 11.1 is given by eqns (11.1.2) and (11.1.4) with positive $\hat{k}_{Ii}$ and sufficiently small parameters a and b.

Example 11.1 (*Decentralized voltage control of a multiarea power system*)

From the viewpoint of controlling the voltages of the feeding nodes, a multiarea power system consists of several synchronous machines feeding the load through transformers, and a distribution net (Figure 11.1(a)). The machines, including their generator voltage controllers, are considered as subsystems, the generator voltages V_{gi} and the node voltages V_{ei} being the interconnection outputs and inputs, respectively. The subsystem inputs are the command inputs $u_i = V_{ci}$ of the generator voltage controllers. The outputs of the measuring devices for the node voltages act as subsystem outputs $y_i = \tilde{V}_{ei}$ (Figure 11.1(b)).

The node voltages are to be maintained at prescribed values. This control task should be carried out by a decentralized PI controller (11.1.2). Since power plants have to be shut off during normal operation of the system because of changing power demands during the day, the voltage controllers have to be chosen so as to ensure closed-loop stability for all operation modes $\mathbf{e} \in \mathscr{E}$.

It is empirically known that the voltage/reactive power behaviour of a plant can be considered independently of the frequency/real power behaviour. Therefore, the static model of the plant, which is necessary to check the decentralized integral controllability, involves the voltages and the 'reactive part' of the currents and admittances. The mesh theorem yields

$$V_{gi} = I_i X_i + V_{ei}$$

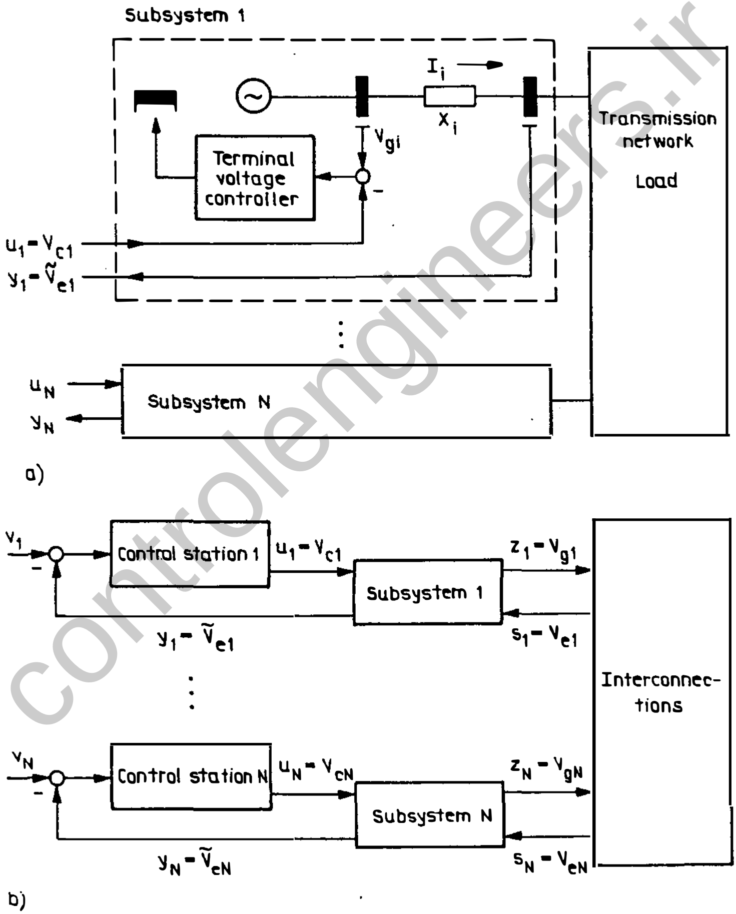


Figure 11.1 Decentralized voltage control of an electric power system: (a) electric power system as the plant of the decentralized voltage control problem; (b) structure of the decentralized control system

where I_i is the reactive current through the transformer, which has the admittance X_i (Figure 11.1(a)). In control engineering terms this equation can be expressed as

$$u_i = I_i X_i + y_i. \tag{11.1.20}$$

The transmission network including the load is described by

$$\begin{pmatrix} I_1 \\ \vdots \\ I_N \end{pmatrix} = \mathbf{Y} \begin{pmatrix} V_{e1} \\ \vdots \\ V_{eN} \end{pmatrix} \tag{11.1.21}$$

with $\mathbf{Y}$ denoting the net admittance matrix. Eqns (11.1.19) and (11.1.21) yield

$$\mathbf{u} = (\mathbf{I} + \text{diag } X_i \ \mathbf{Y})\mathbf{y}.$$

Hence, the static transmission matrix satisfies the equation

$$\mathbf{K}_s^{-1} = \mathbf{I} + \text{diag } X_i \ \mathbf{Y}. \tag{11.1.22}$$

It is well known that an (N, N) admittance matrix $\mathbf{Y} = (y_{ij})$ has the properties

$$y_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{N} |y_{ij}| \geq 0 \quad \text{and} \quad y_{ij} \leq 0 \quad \text{for} \quad i \neq j. \tag{11.1.23}$$

From the properties of M-matrices summarized in Appendix 1, the matrix $\mathbf{Y}$ is an M-matrix and so is the matrix $\mathbf{K}_s^{-1}$. Due to Theorem A1.6, $\mathbf{K}_s$ satisfies the requirements stated in Corollary 11.1. Hence, the power plant is decentralized integral controllable.

Note that this result does not depend upon the number N of subsystems and the parameters of the transformers or the transmission net. It holds true because of the structural properties of the plant, which are represented by (11.1.23). Therefore, the decentralized integral controllability is not affected by switching transmission lines on or off or by changing generator parameters. □

Example 11.2

This simple example should show that the conditions stated in Theorems 11.1 and 11.2 can only be satisfied if the subsystems have a certain autonomy, that is if the subsystem interactions are restricted. This restriction, which is described implicitly by the requirement that all principal minors of $\mathbf{K}_s$ have to be positive, can be made explicit for systems that consist of two subsystems. Consider a system with

$$\dim \mathbf{u}_i = \dim \mathbf{y}_i = \dim \mathbf{s}_i = \dim \mathbf{z}_i = 1$$

whose subsystems (3.1.4) have the static model

$$\begin{aligned} y_i &= k_{yui}u_i + k_{ysi}s_i \\ z_i &= k_{zui}u_i + k_{zsi}s_i \end{aligned}$$

($i = 1, 2$). In order to simplify the investigations, the input signals are scaled according to

$$\tilde{s}_i = k_{zsi}s_i \qquad \tilde{u}_i = k_{yui}u_i.$$

Then, the static model is

$$\begin{aligned} y_i &= \tilde{u}_i + \tilde{k}_{ysi}\tilde{s}_i \\ z_i &= \tilde{k}_{zui}u_i + \tilde{s}_i \end{aligned}$$

($i = 1, 2$). The interconnection relation (3.1.6)

$$\mathbf{s} = \mathbf{L}\mathbf{z}$$

between two subsystems with $l_{ii} = 0$ is given by $\tilde{s}_i = \tilde{l}_{ij}z_j$ with $\tilde{l}_{ij} = k_{zsi}l_{ij}$ for $i \neq j$; $i, j = 1, 2$. These equations yield the overall model

$$\mathbf{y} = \mathbf{K}_s\mathbf{u}$$

with

$$\mathbf{K}_s = \frac{1}{1 - \tilde{l}_{12}\tilde{l}_{21}} \begin{pmatrix} 1 + \tilde{l}_{12}\tilde{l}_{21}(\tilde{k}_{ys1}\tilde{k}_{zu1} - 1) & \tilde{l}_{12}\tilde{k}_{ys1}\tilde{l}_{zu2} \\ \tilde{l}_{21}\tilde{k}_{ys2}k_{zu1} & 1 + \tilde{l}_{12}\tilde{l}_{21}(\tilde{k}_{ys2}\tilde{k}_{zu2} - 1) \end{pmatrix}$$

From the condition stated in Corollary 11.1 the inequalities

$$\begin{aligned} &[1 + \tilde{l}(\tilde{k}_1 - 1)]/(1 - \tilde{l}) > 0 \\ &[1 + \tilde{l}(\tilde{k}_2 - 1)]/(1 - \tilde{l}) > 0 \\ &[1 + \tilde{l}(\tilde{k}_1 - 1)]\,[1 + \tilde{l}(\tilde{k}_2 - 1)] - \tilde{l}\tilde{k}_1\tilde{k}_2 > 0 \end{aligned}$$

are obtained with the coupling parameter $\tilde{l} = \tilde{l}_{12}\tilde{l}_{21}$ and the subsystem parameters $\tilde{k}_1 = \tilde{k}_{ys1}\tilde{k}_{zu1}$ and $\tilde{k}_2 = \tilde{k}_{ys2}\tilde{k}_{zu2}$.

These inequalities should be discussed here for $\tilde{k}_i < 1$ and $\tilde{l} < 1$. They yield three equivalent inequalities

$$\begin{aligned} &\tilde{l} < 1/(1 - \tilde{k}_1) \\ &\tilde{l} < 1/(1 - \tilde{k}_2) \\ &\tilde{l} < 1/(1 - \tilde{k}_1)(1 - \tilde{k}_2) \end{aligned}$$

which show that the condition of Corollary 11.1 gives an upper bound on $\tilde{l}$. If $\tilde{l}$ is negative, it may be arbitrarily small. Both conditions are necessary and sufficient for the two subsystems to have the autonomy which is necessary for decentralized integral controllability. □

11.2 SEQUENTIAL TUNING OF DECENTRALIZED PI CONTROLLERS

In this section a solution is given to Problem 11.1. An algorithm will be developed which does not suppose the availability of a plant model (11.1.1) but can be used by tuning one regulator at a time on the basis of simple experiments with the plant. The basis for this is the assertion that the static reinforcement of the plant or the plant with some control stations attached provides sufficient information for the determination of the controller parameters.

Tuning Rule for the kth Control Station

A basic step in this tuning algorithm concerns the following problem. Consider the plant (11.1.1), which for notational convenience only, has no direct throughput ($\mathbf{D} = \mathbf{0}$). Decompose the model into the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \sum_{i=1}^{N} \mathbf{B}_{si}\mathbf{u}_i \\ \mathbf{y}_i &= \mathbf{C}_{si}\mathbf{x} \qquad (i = 1, \ldots, N). \end{aligned} \tag{11.2.1}$$

Assume that the control stations $1, 2, \ldots, k-1$ are already implemented so that the resulting closed-loop system is stable. For $\mathbf{v}_i = \mathbf{0}$ $(i = 1, \ldots, k-1)$ this partially controlled system is described by

$$\begin{aligned} \dot{\bar{\mathbf{x}}}_k &= \bar{\mathbf{A}}\bar{\mathbf{x}}_k + \sum_{i=k}^{N} \bar{\mathbf{B}}_{si}\mathbf{u}_i \\ \mathbf{y}_i &= \bar{\mathbf{C}}_{si}\bar{\mathbf{x}}_k \qquad (i = k, \ldots, N) \end{aligned} \tag{11.2.2}$$

with $\bar{\mathbf{x}}_k = (\mathbf{x}' \;\; \mathbf{x}'_{\mathrm{r}1} \;\; \ldots \;\; \mathbf{x}'_{\mathrm{r},k-1})'$,

$$\bar{\mathbf{A}}_k = \begin{pmatrix} \mathbf{A} - \Sigma_{i=1}^{k-1} \mathbf{B}_{si}\mathbf{K}_{\mathrm{P}i}\mathbf{C}_{si} & -\mathbf{B}_{s1}\mathbf{K}_{\mathrm{I}1} & -\mathbf{B}_{s2}\mathbf{K}_{\mathrm{I}2} & \ldots & -\mathbf{B}_{s,k-1}\mathbf{K}_{\mathrm{I},k-1} \\ \mathbf{C}_{s1} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \mathbf{C}_{s,k-1} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{pmatrix} \tag{11.2.3}$$

$$\bar{\mathbf{B}}_{si} = \begin{pmatrix} \mathbf{B}_{si} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \qquad \bar{\mathbf{C}}_{si} = (\mathbf{C}_{si} \;\; \mathbf{0} \;\; \ldots \;\; \mathbf{0}).$$

The problem is to determine the kth control station

$$\begin{aligned} \dot{\mathbf{x}}_{rk} &= \mathbf{y}_k - \mathbf{v}_k \\ \mathbf{u}_k &= -\mathbf{K}_{Pk}(\mathbf{y}_k - \mathbf{v}_k) - \mathbf{K}_{Ik}\mathbf{x}_{rk} \end{aligned} \tag{11.2.4}$$

so that the system (11.2.3) and (11.2.4) is stable.

As shown in Figure 11.2, the system (11.2.3) forms the resulting plant for the kth control station. Hence, the problem of choosing the parameters of the control station (11.2.4) is a problem of centralized control. It will now be solved by means of Theorem 2.14. Accordingly, the static transmission matrix $\bar{\mathbf{K}}_{skk}$ between $\mathbf{u}_k$ and $\mathbf{y}_k$ of the system (11.2.3) has to be known. This matrix is also called the *steady-state tracking gain matrix*. If the model (11.2.3) were available it could be determined analytically according to

$$\bar{\mathbf{K}}_{skk} = -\bar{\mathbf{C}}_{sk}\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}_{sk}. \tag{11.2.5}$$

An alternative way is to make experiments with the resulting plant (11.2.4) in order to measure the entries of $\bar{\mathbf{K}}_{skk}$. This experiment can be made by the authority of the kth subsystem alone. If the system (11.2.4) is subjected to m_k different step signals $\mathbf{u}_k = \tilde{\mathbf{u}}_i\sigma(t)$ with linearly independent vectors $\tilde{\mathbf{u}}_i$ ($i = 1, \dots, m_k$), the steady-state values $\tilde{\mathbf{y}}_i$ of the output can be measured. Then, $\bar{\mathbf{K}}_{skk}$ follows from

$$\bar{\mathbf{K}}_{skk} = (\tilde{\mathbf{y}}_1 \ \dots \ \tilde{\mathbf{y}}_{m_k})(\tilde{\mathbf{u}}_1 \ \dots \ \tilde{\mathbf{u}}_{k-1})^{-1}. \tag{11.2.6}$$

According to Theorem 2.14 the controller (11.2.4) with

$$\mathbf{K}_{Ik} = a_k\bar{\mathbf{K}}_{skk}^{-1} \qquad \mathbf{K}_{Pk} = b_k\hat{\mathbf{K}}_{Pk} \tag{11.2.7}$$

ensures the stability of the system (11.2.3) and (11.2.4) for sufficiently small values of a_k and b_k. Equation (11.2.7) provides a description of

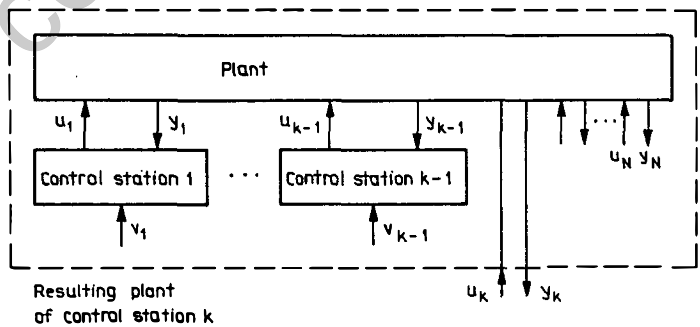


Figure 11.2 Plant with $k-1$ control stations

$\mathbf{K}_{Ik}$ in terms of $\bar{\mathbf{K}}_{skk}$ while $\hat{\mathbf{K}}_{Pk}$ can be chosen arbitrarily. Then, the controller matrices are fixed up to the two unknown scalars a_k and b_k. What 'sufficiently small' means for the choice of the tuning parameters has to be decided depending upon the main time constants and static reinforcement of the system (11.2.3). Hence, the design problem can be solved only if the static model of the plant (11.2.3) and a rough estimate of the dynamical behaviour are known.

Since it is known that closed-loop stability is assured not only for specific values but also for intervals, it is desirable to find values of a_k and b_k for which the I/O performance of the system (11.2.3) and (11.2.4) is reasonable. To this end, the intervals $(0, \bar{a}_k)$ and $[0, \bar{b}_k)$ can be estimated by means of experiments. Beginning with small values the tuning parameters are increased until the system output indicates that the system (11.2.3) and (11.2.4) approaches the stability border. Then parameter values can be selected for which the system has a reasonable I/O performance.

The Tuning Algorithm

Equation (11.2.7) explains how to choose the parameters of the kth control station provided that the already-existing control stations produce a stable closed-loop system (11.2.4). It remains to investigate under what conditions these tuning rules can be used sequentially. It has to be shown that the tuning of the kth controller leads to a closed-loop system for which the $(k+1)$th control station can be designed in the same manner.

The basis for this is provided by eqn (2.4.20) which, in the context of the sequential design, has the form

$$\mathrm{Re}\,[\lambda_i\,[\bar{\mathbf{K}}_{skk}\mathbf{K}_{Ik}]\,] > 0.$$

Since $\mathbf{K}_{Ik}$ is chosen as prescribed by eqn (11.2.7), this condition is satisfied if and only if the matrix $\bar{\mathbf{K}}_{skk}$ is non-singular. Hence, it has to be investigated under what conditions $\bar{\mathbf{K}}_{sii}$ $(i = 1, \ldots, N)$ is non-singular. To this end, it will be shown that the static model

$$\mathbf{y}_k = \bar{\bar{\mathbf{K}}}_{skk}\mathbf{u}_k \tag{11.2.8}$$

of the system (11.2.4) does not depend on the parameters of the control stations that have already been attached to the plant.

Equation (11.2.8) describes the plant in connection with the first $(k-1)$ control stations for $\mathbf{v}_i = \mathbf{0}$ $(i = 1, \ldots, k-1)$ and $\mathbf{u}_i = \mathbf{0}$ $(i = k-1, \ldots, N)$. For this set of external signals, the existing control stations ensure that $\mathbf{y}_i(t) \rightarrow \mathbf{0}$ holds for $t \rightarrow \infty$ $(i = 1, \ldots, k-1)$. The static plant model

$$\begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_k \\ \vdots \\ \mathbf{y}_N \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{s11} & \cdots & \mathbf{K}_{s1N} \\ \vdots & & \vdots \\ \mathbf{K}_{sk1} & \cdots & \mathbf{K}_{skN} \\ \vdots & & \vdots \\ \mathbf{K}_{sN1} & \cdots & \mathbf{K}_{sNN} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \\ \vdots \\ \mathbf{u}_N \end{pmatrix}$$

yields

$$\mathbf{0} = \begin{pmatrix} \mathbf{K}_{s11} & \cdots & \mathbf{K}_{s,k-1} \\ \vdots & & \vdots \\ \mathbf{K}_{sk-1,1} & \cdots & \mathbf{K}_{sk-1,k-1} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{k-1} \end{pmatrix} + \begin{pmatrix} \mathbf{K}_{s1k} \\ \vdots \\ \mathbf{K}_{sk-1,k} \end{pmatrix} \mathbf{u}_k$$

$$\begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{s11} & \cdots & \mathbf{K}_{s,k-1} \\ \vdots & & \vdots \\ \mathbf{K}_{sk-1,1} & \cdots & \mathbf{K}_{sk-1,k-1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{K}_{s1k} \\ \vdots \\ \mathbf{K}_{sk-1,k} \end{pmatrix} \mathbf{u}_k \tag{11.2.9}$$

and

$$\mathbf{y}_k = \mathbf{K}_{skk}\mathbf{u}_k + (\mathbf{K}_{sk1} \ \ldots \ \mathbf{K}_{sk,k-1}) \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{k-1} \end{pmatrix}. \tag{11.2.10}$$

From eqns (11.2.9) and (11.2.10) the relation

$$\bar{\mathbf{K}}_{skk} = \mathbf{K}_{skk} + (\mathbf{K}_{sk1} \ \ldots \ \mathbf{K}_{sk,k-1}) \times \begin{pmatrix} \mathbf{K}_{s11} & \cdots & \mathbf{K}_{s,k-1} \\ \vdots & & \vdots \\ \mathbf{K}_{sk-1,1} & \cdots & \mathbf{K}_{sk-1,k-1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{K}_{s1k} \\ \vdots \\ \mathbf{K}_{sk-1,k} \end{pmatrix} \tag{11.2.11}$$

is obtained.

Lemma 11.1

The static transmission matrix $\bar{\mathbf{K}}_{skk}$ of the system (11.2.4), which consists of the plant (11.2.1) and $k-1$ decentralized control stations, is given by eqn (11.2.11) provided that the system (11.2.4) is stable.

It is of interest to note that the matrix $\bar{\mathbf{K}}_{skk}$ does not depend on the controller parameters. This independence results from the fact that I controllers ensure asymptotic regulation independently of the controller parameters provided that the closed-loop system is stable. It can be used here to show that for decentralized integral controllable plants it is sufficient to make the closed-loop system stable in the $(k-1)$th tuning step in order to ensure the non-singularity of $\bar{\mathbf{K}}_{skk}$ and, thus, the applicability of the next tuning step. According to Theorem 11.1, decentralized integral controllability of the plant (11.2.1) implies

$$\det(\mathbf{K}_{s11}) > 0$$

and

$$\det\left(\begin{array}{c|ccc} \mathbf{K}_{s11} & \mathbf{K}_{s12} & \cdots & \mathbf{K}_{s1N} \\ \hline \mathbf{K}_{s21} & \mathbf{K}_{s22} & \cdots & \mathbf{K}_{s2N} \\ \vdots & \vdots & & \vdots \\ \mathbf{K}_{sk-1,1} & \mathbf{K}_{sk-1,2} & \cdots & \mathbf{K}_{sk-1,k-1} \end{array}\right) > 0.$$

Using the determinant relation

$$\det\begin{pmatrix} \mathbf{R} & \mathbf{S} \\ \mathbf{T} & \mathbf{U} \end{pmatrix} = \det \mathbf{R} \det(\mathbf{U} - \mathbf{T}\mathbf{R}^{-1}\mathbf{S}) \quad \text{for } \det \mathbf{R} \neq 0$$

with respect to the indicated decomposition of $\mathbf{K}_s$, these inequalities are seen to be equivalent to

$$\det(\bar{\mathbf{K}}_{skk}) > 0.$$

This proves the following theorem.

Theorem 11.3

If the plant (11.2.1) is stable and satisfies the necessary condition for decentralized integral controllability stated in Theorem 11.1, a decentralized controller, which ensures the closed-loop stability in every design step, can be found sequentially by using the controller matrices as described in eqn (11.2.7) with sufficiently smaller tuning parameters a_k and b_k.

This result leads to the following tuning algorithm.

Algorithm 11.1 (*Decentralized tuning*)

Given: Stable plant that is decentralized integral controllable; $k = 1$.

1. Make experiments with the plant including the control stations $1 \ldots k-1$ in order to determine the static transition matrix $\bar{\mathbf{K}}_{skk}$ (cf. eqn (11.2.6)).
2. Determine the controller matrices according to eqn (11.2.7) with arbitrary $\hat{\mathbf{K}}_{Pk}$ and sufficiently small tuning parameters a_k and b_k.

3. Implement the kth control station and make experiments with the closed-loop system with different tuning parameters until a reasonable I/O behaviour of the system is reached.
4. If $k = N$ holds, stop; otherwise, let $k = k + 1$ and continue with step 1.

Result: Decentralized PI controller which ensures the sequential stability of the closed-loop system.

Although the controller is sequentially stable, it is not sure that the closed-loop system remains stable if arbitrary control stations are disconnected from the plant. This stronger design specification, which is included in Problem 11.1, can be satisfied if, in addition to the requirement of Theorem 11.3, the decentralized integral controllability of the plant (11.2.1) is ensured due to the sufficient condition stated in Theorem 11.2. Then, a solution to Problem 11.1 is received as a result of Algorithm 11.1 if all the tuning parameters are chosen sufficiently small.

Example 11.3 (*Decentralized control of a glass furnace*)

Consider an electrically heated glass furnace with the structure of Figure 11.3. A mixture of sand, limestone and soda is brought into the furnace. In the left-hand part, layers of heated mixture and glass develop, whereas after passing the partition wall a homogeneous molten glass mass is produced. The chemical and physical processes which accompany the melting are complex. It is, therefore, impossible to set up a precise model of the plant. The decentralized controllers should be tuned sequentially.

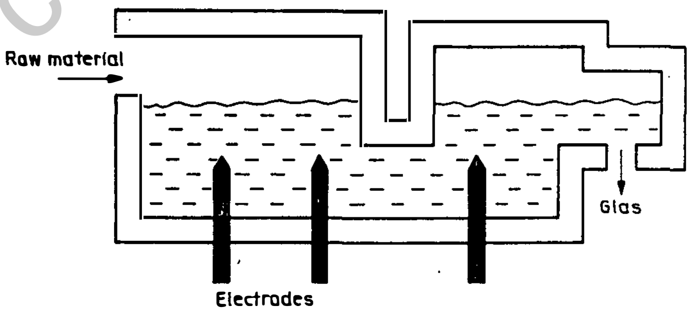


Figure 11.3 Electrically heated glass furnace

In order to check the decentralized integral controllability of the plant, the static transition matrix

$$\mathbf{K}_s = \begin{pmatrix} 2.12 & 4.18 & 1.67 \\ 1.82 & 4.54 & 1.82 \\ 1.67 & 4.18 & 2.12 \end{pmatrix}$$

is measured. All minors of this matrix are positive. The control stations

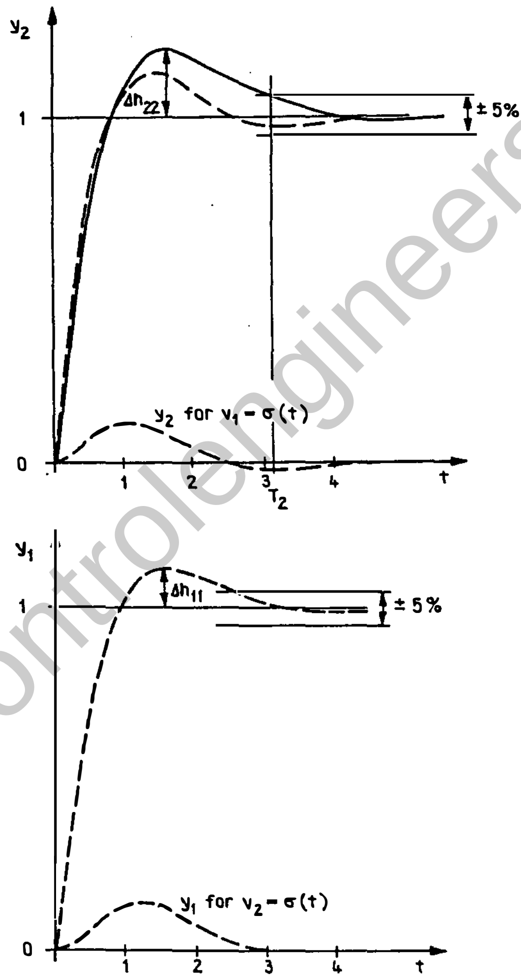


Figure 11.4 Command step responses of the glass furnace under decentralized control: ———, after implementation of control station 2 only; – – –, complete decentralized closed-loop system

(11.2.4) with $\dim \mathbf{y}_i = \dim \mathbf{u}_i = 1$ and, hence, scalar parameters k_{Pk} and k_{Ik} are tuned in the sequence 2–1–3. With $k_{P2} = 2.67$ and $k_{I2} = 6.79$ the command step response of the second subsystem has an overshoot of 19% and a settling time of 3.1 hours (Figure 11.4). Then control station 1 is applied with $k_{P1} = 8.14$ and $k_{I1} = 10.37$ leading to a 15% overshoot in this loop. The third controller is connected to the plant with $k_{P3} = 8.14$ and $k_{I3} = 10.37$. Simultaneously, the overshoot of the other control loops reduces (Figure 11.4). In the closed-loop system the cross couplings are small as shown by $y_2(t)$ in the case of the command step in the first loop. □

11.3 DECENTRALIZED DESIGN OF DECENTRALIZED PI CONTROLLERS

In this section, a design method will be presented which is particularly suitable for decentralized PI control of plants with $\dim \mathbf{y}_i = \dim \mathbf{u}_i = 1$ and which proceeds in a completely decentralized manner. It is motivated by the results presented in Theorem 11.1 and Lemma 11.1. Accordingly, the decentralized controller (11.1.2) can be chosen so as to ensure closed-loop stability for all the operation conditions $\mathbf{e} \in \mathscr{E}$ only if the subsystems possess a certain autonomy. This suggests that the ith control station should be designed by means of a model which describes the ith subsystem with a reasonable accuracy, and that the influences of the other control stations should be considered as model uncertainties. If all control stations are robust enough to tolerate the uncertainties of the models for which they are designed, then the closed-loop system will be stable for all $\mathbf{e} \in \mathscr{E}$.

The question of how to evaluate the influence of the control stations with index i $(i \neq k)$ on the resulting plant of the kth regulator (Figure 11.2) can be answered in the following way. The plant (11.2.1) can be described from the point of view of the ith subsystem as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}_{si}u_i \\ y_i &= \mathbf{c}_{si}'\mathbf{x}. \end{aligned} \tag{11.3.1}$$

If control stations are attached to the plant, this model is no longer true. The deviation between the model (11.3.1) and the behaviour of the plant under the influence of the other regulators depends on the controller parameters and cannot be determined in advance. However, as pointed out by Lemma 11.1, the static deviation is independent of the controller parameters and merely dependent on which control stations are in operation. The static reinforcement $\bar{k}_{sii}$ can be represented in terms of the matrix $\mathbf{K}_s$ and the vector $\mathbf{e}$ in a form which is a slight generalization of eqn (11.2.11). For $\bar{k}_{s11}$ this relation is

$$\bar{k}_{s11}(\mathbf{e}) = k_{s11} + \delta k_{s11}(\mathbf{e}) \tag{11.3.2}$$

with

$$\delta k_{s11}(\mathbf{e}) = (k_{s12} \ \dots \ k_{s1N}) \times \begin{pmatrix} k_{s22} & e_2 k_{s23} & \dots & e_2 k_{s2N} \\ \vdots & \vdots & & \vdots \\ e_N k_{sN2} & e_N k_{sN3} & \dots & k_{sNN} \end{pmatrix}^{-1} \begin{pmatrix} e_2 k_{s21} \\ \vdots \\ e_N k_{sN1} \end{pmatrix}. \tag{11.3.3}$$

That is, the static reinforcement between u_i and y_i is known in advance for all $\mathbf{e} \in \mathscr{E}$, no matter what controller parameters will be chosen by the authorities of the other subsystems. k_{sii} can be evaluated by

$$|\delta k_{sii}(\mathbf{e})| \leqslant \bar{k}_i \tag{11.3.4}$$

with some reasonably chosen scalar $\bar{k}_i$. This suggests the use of the model

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}_{si}(1 + k_i)u_i \\ y_i &= \mathbf{c}'_{si}\mathbf{x} \end{aligned} \tag{11.3.5}$$

for the resulting plant of control station i where k_i represents the model uncertainties that are brought about by the connection or disconnection of the other control stations to the plant. Since the static reinforcement of the model (11.3.2)

$$(1 + k_i)\mathbf{c}'_{si}\mathbf{A}^{-1}\mathbf{b}_{si} = (1 + k_i)k_{sii}$$

has to be equal to $\bar{k}_{sii}$, $k_i = \delta k_{sii}(\mathbf{e})$ and, thus,

$$|k_i| \leqslant \bar{k}_i \tag{11.3.6}$$

hold. This gives an answer to the question of how to model the ith subsystem under the influence of all other subsystems and all other control stations (cf. Section 9.1).

With the model (11.3.5) and (11.3.6) the problem of designing the ith control station

$$\begin{aligned} \dot{x}_{ri} &= y_i - v_i \\ u_i &= -k_{Pi}(y_i - v_i) - k_{Ii}x_{ri} \end{aligned} \tag{11.3.7}$$

is a problem of robust centralized control. The model (11.3.5) and (11.3.6), whose structure is depicted in Figure 11.5, describes the set of all linear systems with the multiplicative uncertainty k_i. The control station has to be chosen so that the closed-loop system (11.3.5) and (11.3.7) satisfies the design specifications for all k_i from eqn (11.3.6). A method of solving this problem was described in Section 2.6.

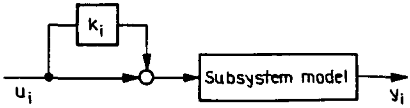


Figure 11.5 Structure of the model (11.3.5) and (11.3.6)

Algorithm 11.2 (*Decentralized design of PI controllers*)

Given: Stable, decentralized integral controllable plant (11.1.1) with dim $\mathbf{u}_i = \dim \mathbf{y}_i = 1$; specifications on the dynamical I/O behaviour of the closed-loop system.

1. Derive the model (11.3.5) and (11.3.6) for all subsystems from the model (11.1.1).
2. For $i = 1, 2, \ldots, N$ design the ith control station (11.3.7) as the robust controller for the plant (11.3.5) and (11.3.6).

Result: Decentralized PI controller.

Note that in step 2 the control stations are designed completely independently of one another with different models of the overall plant. Owing to the robustness of all these control stations for all operation modes $\mathbf{e} \in \mathscr{E}$, not only the closed-loop stability but also the satisfaction of the dynamical requirements is ensured.

The weak point of this algorithm is the assumption that the uncertainties of the model (11.3.5) can be completely described by a static error model whose transmission factor k_i is bounded by eqn (11.3.6) (Figure 11.5). This assumption implies that the variety of the dynamical properties, which is observed via the ith channel (u_i, y_i) for different $\mathbf{e} \in \mathscr{E}$, can be described by a single unknown factor. Decentralized integral controllable plants tend to satisfy this assumption because of the autonomy of its subsystems, which is claimed in Corollary 11.1.

Example 11.1 (cont.)

The design algorithm can be demonstrated by considering a voltage control problem for a multiarea power system with twenty feeding nodes. The matrix $\mathbf{K}_s$ of the system has been determined by load flow calculations. For subsystem 1, eqns (11.3.2) and (11.3.3) yield

$$0.338 \leqslant \bar{k}_{s11}(\mathbf{e}) \leqslant 0.375$$

and

$$\bar{k}_{s11}(\mathbf{e}) = 0.356(1 + k_1) \qquad |k_1| \leqslant 0.06.$$

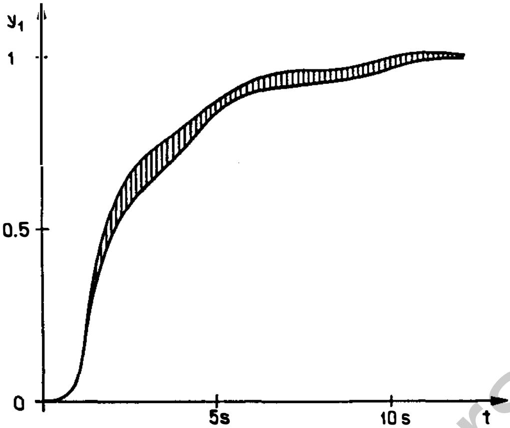


Figure 11.6 Tolerance band of the command step response of the closed-loop system

A twenty-fifth-order model and this uncertainty bound has been used to design control station 1. With the parameters $k_{I1} = k_{P1} = 0.834$ the band shown in Figure 11.6 covers all command step responses of the system (11.3.5)–(11.3.7) (for $i = 1$). That is, the step response of the ith subsystem within the overall closed-loop system can be expected to remain in this band for all operational modes $\mathbf{e} \in \mathcal{E}$.

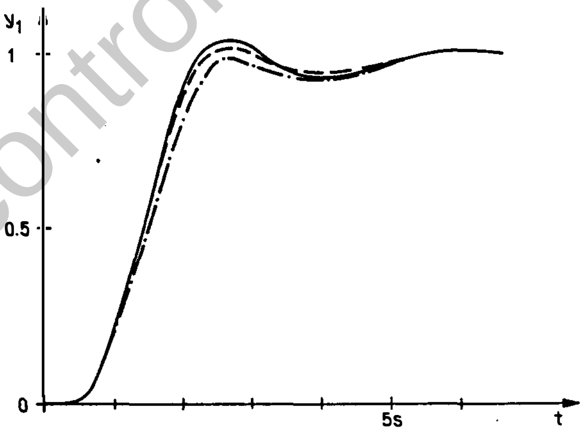


Figure 11.7 Step response of the resulting plant of control station 1 for different operational conditions (normed for final value equal to one): ———, $e_i = 0$ ($i = 1, \dots, N$); – – –, $e_2 = e_4 = e_5 = 1$; – · – · –, $e_2 = e_5 = 1$

The last assertion is only true as long as the model (11.3.5) provides a reasonable approximation of the ith subsystem under the influence of the other subsystems and control stations within the overall system. To show that this approximation is acceptable, Figure 11.7 shows the response y_1 to step input u_1 of the resulting plant of control station 1 for a selected set of operating conditions $\mathbf{e} \in \mathscr{E}$. The step responses have been multiplied by $1/\bar{k}_{s11}(\mathbf{e})$ for the purpose of comparison. The small differences between the curves show clearly that the dynamics of the closed-loop system can be really described by a model (11.3.5) with multiplicative uncertainty k_1. □

BIBLIOGRAPHICAL NOTES

Problem 11.1 was investigated independently by Lunze (1983a,b), Locatelli *et al.* (1983, 1986) and Grosdidier *et al.* (1984), who introduced the notion of decentralized integral controllability. Locatelli *et al.* (1983, 1986) also investigated an alternative design strategy in which the closed-loop stability is ensured by PI controllers with sufficiently high gains rather than sufficiently low gains as explained in Section 11.1. Grosdidier and Morari (1986, 1987) investigated the integral controllability in terms of interaction measures of the plant and presented a methodology for computer-aided controller structure selection and sequential tuning. The voltage control problem was investigated intensively by Gamaleja *et al.* (1984). A thorough explanation of Example 11.1 and its extension to a real power system with twenty feeding nodes can also be found in the monograph by Lunze (1988).

Specific centralized design methods for decentralized PI controllers were proposed, for example, by Locatelli *et al.* (1986) and Yanchevsky (1987), who used the optimal control principle explained in Chapter 7. A solution of the initialization problem, which takes advantage of the PI structure of the controller and thus, is more efficient than Algorithm 7.2, was proposed by Guardabassi *et al.* (1982). Petkovski (1981) applied Košut's method (1970) for replacing a centralized controller by a decentralized one to PI control and demonstrated it by studying the control of a plate-type absorption column.

Decentralized tuning regulators were investigated by Davison (1978) who considered the larger class of polynomial disturbance and command signals. This generalization yields similar results where the steady-state tracking matrices represent the transfer function matrix for polynomial inputs under consideration. If the controllers have to ensure only the sequential stability of the closed-loop system, the requirement given in

Theorem 11.1 can be relaxed by the claim that only a certain sequence of principal minors of $\mathbf{K}_s$ have to be positive. An extensive study of applications was described by Davison and Tripathi (1980). Example 11.3 is an extract of the work by Hunger (1989) and Hunger and Jumar (1989).

Algorithm 11.3 was elaborated by Lunze (1985). Its application to a large power system is described in detail by Lauckner and Lunze (1984). Figure 11.6 was obtained by means of a method for determining the smallest possible tolerance bands proposed by Lunze and Zscheile (1985).

12

Strongly Coupled Symmetric Composite Systems

This chapter is devoted to symmetric composite systems, whose structural properties are the identity of the subsystem dynamics and the symmetry of the interconnections. The motivation for studying such systems is threefold. First, it will be shown that symmetry within the whole system gives rise to substantial simplifications of the modelling, analytical and design problems. Since these results are based on the structural properties of the overall system, they hold for arbitrary albeit identical dynamical properties of the subsystems, arbitrarily strong interactions and an unrestrained number of subsystems.

Second, symmetric composite systems make it possible to reformulate the problem of decentralized control of the overall system as a problem of robust centralized control of an auxiliary plant. In this way, an explicit relation between the complexity of the overall system and the *necessary* robustness that has to be ensured by the subsystem authorities can be developed.

Third, as shown in Section 12.4 for the stability analysis, the results on symmetric systems can be extended to systems that consist of similar rather than identical subsystems.

The investigations of symmetric composite systems are relevant to technological processes whose subsystems behave similarly as, from a technological point of view, they participate in doing the same task. Multiarea power systems will be considered here for illustration.

12.1 MODELS OF SYMMETRIC COMPOSITE SYSTEMS

The plant consists of N subsystems each of which is described by the state space model (3.1.4)

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i + \mathbf{E}\mathbf{s}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i &= \mathbf{C}\mathbf{x}_i \\ \mathbf{z}_i &= \mathbf{C}_z\mathbf{x}_i \qquad (i = 1, 2, \ldots, N) \end{aligned} \tag{12.1.1}$$

with identical matrices for all subsystems. The matrices have no index but should not be confused with the overall system matrices in eqn (3.1.1). The interconnections are described by eqn (3.1.6)

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{12.1.2}$$

where the interconnection matrix $\mathbf{L}$ is block symmetric

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_d & \mathbf{L}_q & \dots & \mathbf{L}_q \\ \mathbf{L}_q & \mathbf{L}_d & \dots & \mathbf{L}_q \\ \vdots & \vdots & & \vdots \\ \mathbf{L}_q & \mathbf{L}_q & \dots & \mathbf{L}_d \end{pmatrix}. \tag{12.1.3}$$

Equations (12.1.1)–(12.1.3) reflect the assumptions that the subsystems are identical and coupled in a symmetric way. However, no restrictions are imposed on the dynamical properties of the subsystems and the sign and strength of the interconnections. The following investigations hold for an arbitrarily large number N of subsystems.

Definition 12.1

A system that can be represented by a model of the form (12.1.1)–(12.1.3) is called a *symmetric composite system.*

It is only for notational convenience that external disturbances are ignored and that direct throughput from $\mathbf{u}_i$ and $\mathbf{s}_i$ to $\mathbf{y}_i$ and $\mathbf{z}_i$ is not considered here. All parameters occurring in eqns (12.1.1)–(12.1.3) are assumed to be independent of the number N of subsystems although Section 12.5 presents a counterexample. Then, the interconnection matrix of a system with $N_1 < N$ subsystems can be obtained from the matrix $\mathbf{L}$ for the larger system by deleting the last $(N - N_1)$ rows and columns.

Performance of the Subsystems

Equations (12.1.1)–(12.1.3) yield the overall system description

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A} + \mathbf{E}\mathbf{L}_d\mathbf{C}_z & \mathbf{E}\mathbf{L}_q\mathbf{C}_z & \dots & \mathbf{E}\mathbf{L}_q\mathbf{C}_z \\ \mathbf{E}\mathbf{L}_q\mathbf{C}_z & \mathbf{A} + \mathbf{E}\mathbf{L}_d\mathbf{C}_z & \dots & \mathbf{E}\mathbf{L}_q\mathbf{C}_z \\ \vdots & \vdots & & \vdots \\ \mathbf{E}\mathbf{L}_q\mathbf{C}_z & \mathbf{E}\mathbf{L}_q\mathbf{C}_z & \dots & \mathbf{A} + \mathbf{E}\mathbf{L}_d\mathbf{C}_z \end{pmatrix} \mathbf{x} + \text{diag}\ \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \text{diag}\ \mathbf{C}\mathbf{x} \tag{12.1.4}$$

where $\mathbf{x} = (\mathbf{x}_1', \ldots, \mathbf{x}_N')'$ and $\dim \mathbf{x} = Nn$ hold; diag $\mathbf{C}$ is the abbreviation for a block-diagonal matrix with N times the matrix $\mathbf{C}$ on the main diagonal. To obtain more insight into this model the transformation

$$\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x} \tag{12.1.5}$$

with

$$\mathbf{T} = \frac{1}{N}\left(\begin{array}{cccc|c} (N-1)\mathbf{I} & -\mathbf{I} & \ldots & -\mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & (N-1)\mathbf{I} & \ldots & -\mathbf{I} & -\mathbf{I} \\ \vdots & \vdots & & \vdots & \vdots \\ -\mathbf{I} & -\mathbf{I} & \ldots & (N-1)\mathbf{I} & -\mathbf{I} \\ \hline \mathbf{I} & \mathbf{I} & \ldots & \mathbf{I} & \mathbf{I} \end{array}\right)$$

$$\mathbf{T}^{-1} = \left(\begin{array}{cccc|c} \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} & \ldots & \mathbf{0} & \mathbf{I} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & \mathbf{I} \\ \hline -\mathbf{I} & -\mathbf{I} & \ldots & -\mathbf{I} & \mathbf{I} \end{array}\right) \tag{12.1.6}$$

is applied where the left upper hyper block of $\mathbf{T}$ consists of $(N-1)(N-1)$ identity matrices $\mathbf{I}$ of dimension n. The model (12.1.4) can be written as

$$\dot{\tilde{\mathbf{x}}} = \left(\begin{array}{cccc|c} \mathbf{A}_s & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_s & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{A}_s & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{A}_0 \end{array}\right)\tilde{\mathbf{x}}$$

$$+\frac{1}{N}\left(\begin{array}{cccc|c} (N-1)\mathbf{B} & -\mathbf{B} & \ldots & -\mathbf{B} & -\mathbf{B} \\ -\mathbf{B} & (N-1)\mathbf{B} & \ldots & -\mathbf{B} & -\mathbf{B} \\ \vdots & \vdots & & \vdots & \vdots \\ -\mathbf{B} & -\mathbf{B} & \ldots & (N-1)\mathbf{B} & -\mathbf{B} \\ \hline \mathbf{B} & \mathbf{B} & \ldots & \mathbf{B} & \mathbf{B} \end{array}\right)\mathbf{u} \tag{12.1.7}$$

$$\tilde{\mathbf{x}}(0) = \mathbf{T}\mathbf{x}_0$$

$$\mathbf{y} = \left(\begin{array}{cccc|c} \mathbf{C} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{C} \\ \mathbf{0} & \mathbf{C} & \ldots & \mathbf{0} & \mathbf{C} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{C} & \mathbf{C} \\ \hline -\mathbf{C} & -\mathbf{C} & \ldots & -\mathbf{C} & \mathbf{C} \end{array}\right)\tilde{\mathbf{x}}$$

with

$$\begin{aligned}\mathbf{A}_s &= \mathbf{A} + \mathbf{E}(\mathbf{L}_d - \mathbf{L}_q)\mathbf{C}_z \\ \mathbf{A}_0 &= \mathbf{A} + \mathbf{E}[\mathbf{L}_d + (N-1)\mathbf{L}_q]\mathbf{C}_z.\end{aligned} \tag{12.1.8}$$

Because of the structure of eqn (12.1.7), the subsystem output $\mathbf{y}_i$ depends merely on the subvectors $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{x}}_N$ of $\tilde{\mathbf{x}}$. The free motion and the I/O behaviour of the system concerning all inputs $\mathbf{u}_j$ $(j = 1, \ldots, N)$ and some output $\mathbf{y}_i$ can be described by the model

$$\begin{aligned}\dot{\tilde{\tilde{\mathbf{x}}}}_i &= \begin{pmatrix}\mathbf{A}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_0\end{pmatrix}\tilde{\tilde{\mathbf{x}}}_i + \begin{pmatrix}\frac{N-1}{N}\mathbf{B} \\ \frac{1}{N}\mathbf{B}\end{pmatrix}\mathbf{v}_i + \begin{pmatrix}-\frac{1}{N}\mathbf{B} \\ \frac{1}{N}\mathbf{B}\end{pmatrix}\sum_{\substack{j=1 \\ j\neq i}}^{N}\mathbf{u}_j \\ \tilde{\tilde{\mathbf{x}}}_i(0) &= \begin{pmatrix}\frac{N-1}{N}\mathbf{x}_{i0} + \frac{1}{N}\Sigma_{j=1, j\neq i}^{N}\mathbf{x}_{j0} \\ \Sigma_{j=1}^{N}\mathbf{x}_{j0}\end{pmatrix} \\ \mathbf{y}_i &= (\mathbf{C}\ \ \mathbf{C})\tilde{\tilde{\mathbf{x}}}_i \qquad (i = 1, \ldots, N).\end{aligned} \tag{12.1.9}$$

Hence, in a symmetric composite system the response of a single subsystem to all initial conditions and inputs can be *exactly* described by a model of order twice the subsystem order. The dynamical order is independent of the number N of subsystems although the overall system has the dynamical order Nn.

Equation (12.1.9) indicates the way in which the couplings within the whole system influence the subsystem behaviour. The model (12.1.9) consists of two parts in parallel, each of which represents a subsystem model with feedback from the interconnection output $\mathbf{z}_i$ to the interconnection input $\mathbf{s}_i$ (Figure 12.1). That is, the interactions between the subsystems act as static feedback and have only the limited potential of a static output feedback for stabilizing or destabilizing the isolated subsystems. The question of which unstable subsystems may be stabilized or which stable subsystems may be destabilized by symmetric interconnections can be answered by using the corresponding results on centralized static output feedback.

Equation (12.1.5) yields

$$\tilde{\mathbf{x}}_N = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i. \tag{12.1.10}$$

That is, $\tilde{\mathbf{x}}_N$ represents the average of the subsystem states $\mathbf{x}_i$. In the model (12.1.9) the subsystem-specific component of $\mathbf{y}_i(t)$ (upper part in Figure 12.1) is separated from the effects of the 'average' behaviour of the overall system (lower part).

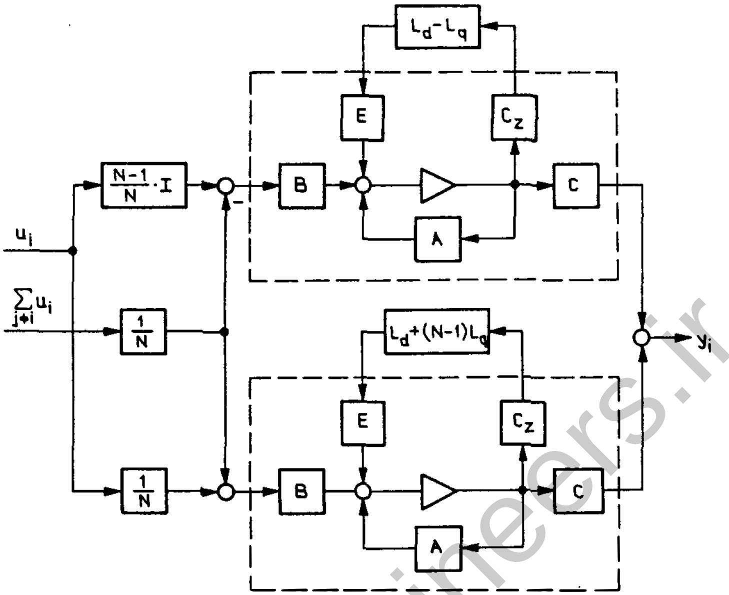


Figure 12.1 Structure of the subsystem model: the dashed lines indicate isolated subsystems

Approximate Models

If the number N of subsystems is large and the matrix $\mathbf{A}_0$ is stable then the equation

$$\begin{aligned} \dot{\hat{\mathbf{x}}}_i &= \mathbf{A}_s \hat{\mathbf{x}}_i + \mathbf{B}\mathbf{u}_i \qquad \hat{\mathbf{x}}_i(0) = \mathbf{x}_{i0} \\ \hat{\mathbf{y}}_i &= \mathbf{C}\hat{\mathbf{x}} \end{aligned} \tag{12.1.11}$$

represents a reasonable approximation of the ith subsystem under the influence of the interactions and all control inputs $\mathbf{u}_j$ ($j = 1, \ldots, N$). In this model, only the 'subsystem state' $\hat{\mathbf{x}}_i = \tilde{\mathbf{x}}_i$ and subsystem input $\mathbf{u}_i$ occurs (Figure 12.2). The feedback

$$\mathbf{s}_i = (\mathbf{L}_d - \mathbf{L}_q)\mathbf{z}_i \tag{12.1.12}$$

can be interpreted as a coarse model of the influence that the other subsystems exercise upon the ith subsystem (cf. Figure 9.2(b)). If $\mathbf{L}_d = \mathbf{L}_q$ holds, the model (12.1.11) coincides with the isolated subsystem (eqn (12.1.1) for $\mathbf{s}_i = \mathbf{0}$).

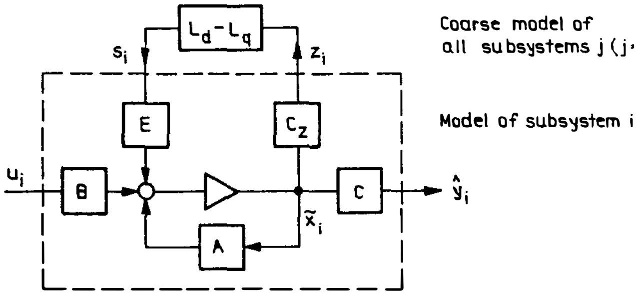


Figure 12.2 Approximate model of the behaviour of the subsystem *i* operating within a symmetric composite system

12.2 ANALYSIS OF SYMMETRIC COMPOSITE SYSTEMS

Stability Analysis

According to eqn (12.1.7), the stability of the nNth-order overall system can be tested by considering the two nth-order matrices $\mathbf{A}_s$ and $\mathbf{A}_0$ defined in eqn (12.1.8).

Theorem 12.1

The symmetric composite system (12.1.1)–(12.1.3) is stable if and only if all the eigenvalues of the matrices $\mathbf{A}_s$ and $\mathbf{A}_0$ from eqn (12.1.8) have negative real parts.

Although in general the stability of the isolated subsystems is neither necessary nor sufficient for overall system stability, for symmetric composite systems Theorem 12.1 leads to the following corollary.

Corollary 12.1

If the interconnection matrix $\mathbf{L}$ in eqn (12.1.3) has the property $\mathbf{L}_d = \mathbf{L}_q$, then the stability of the subsystem matrix $\mathbf{A}$ is a necessary condition for the stability of the symmetric composite system (12.1.1)–(12.1.3).

The simplicity of the stability analysis is based on the structure of symmetric composite systems and does not depend on some weakness of interactions. A necessary *and* sufficient condition for the connective stability can be derived as a corollary of Theorem 12.1.

Corollary 12.2

The symmetric composite system (12.1.1)–(12.1.3) is connectively stable if and only if all eigenvalues of the matrices $\mathbf{A}$ and

$$\mathbf{A}_c(i) = \mathbf{A} + \mathbf{E}\mathbf{L}_d\mathbf{C}_z + i\mathbf{E}\mathbf{L}_q\mathbf{C}_z \tag{12.2.1}$$

$(i = -1, +1, 2, \ldots, N-1)$ have negative real parts.

Because of the symmetry of the system it does not matter which subsystems are disconnected. Only N rather than 2^N modes of operation have to be investigated. Moreover, the matrices $\mathbf{A}_c(i)$ to be tested are only of nth order.

As explained in Section 8.5, a system can be characterized as cooperative (concerning its stability properties) if the interactions between the subsystems are beneficial for stability. While a general characterization of cooperative interactions has not yet been found, cooperation between the subsystems of a symmetric composite system can be thoroughly examined.

Definition 12.2

A symmetric composite system (12.1.1)–(12.1.3) is said to have an *asymptotically cooperative structure* (or *eventually cooperative structure*) if there is an integer $\tilde{N}$ such that the system is stable whenever the number N of subsystems exceeds $\tilde{N}$ $(N > \tilde{N})$.

A preliminary characterization of an asymptotically cooperative structure can be derived from Theorem 12.1.

Lemma 12.1

A symmetric composite system (12.1.1)–(12.1.3) has an asymptotically cooperative structure if and only if

(i) all eigenvalues of the matrix $\mathbf{A}_s$ in eqn (12.1.8) have negative real parts;
(ii) there exists an integer $\tilde{N}$ such that the matrix $\mathbf{A}_c(N)$ has only eigenvalues with negative real parts for all $N > \tilde{N}$.

Note that the matrix $\mathbf{A}_c(N)$ is the system matrix of the lower part in the model of Figure 12.1 for a system with $N+1$ subsystems. Part (ii) shows that cooperation can only be expected under very restrictive conditions. All eigenvalues of $\mathbf{A}_c(N)$ have to remain in the left-hand side of the complex plane if the integer N becomes arbitrarily large. For systems with $\dim \mathbf{s}_i = \dim \mathbf{z}_i = 1$ the following explicit characterization of cooperation confirms this assertion. Obviously, $\mathbf{A}_c(N)$ is the system matrix of the closed loop consisting of the plant

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} + \mathbf{e}l_d\mathbf{c}_z)\mathbf{x} + \mathbf{e}\mathbf{s} \\ \mathbf{z} &= \mathbf{c}_z\mathbf{x} \end{aligned} \tag{12.2.2}$$

and the feedback

$$\mathbf{s} = \mathbf{N}l_q\mathbf{z} \tag{12.2.3}$$

(cf. lower part of Figure 12.1). Let the poles and zeros of the system (12.2.2) be denoted by λ_i ($i = 1, \ldots, n$) or λ_{0i} ($i = 1, \ldots, n_0$), respectively. Then root locus theory leads to the following result.

Theorem 12.2

Consider a symmetric composite system (12.1.1)–(12.1.3) whose interconnection matrix $\mathbf{L}$ does not depend on N and for which the pairs $(\mathbf{A} + \mathbf{e}l_d\mathbf{c}_z, \mathbf{e})$ and $(\mathbf{A} + \mathbf{e}l_q\mathbf{c}_z, \mathbf{c}_z)$ are controllable or observable, respectively. The system has an asymptotically cooperative structure if and only if the following conditions are satisfied:

(i) All eigenvalues of the matrix $\mathbf{A}_s$ in eqn (12.1.8) have negative real parts.
(ii) All zeros λ_{0i} of the system (12.2.2) have negative real parts.
(iii) Either the condition $n - n_0 = 1$ or the relations $n - n_0 = 2$ and

$$\sum_{i=1}^{n_0} \lambda_{0i} - \sum_{i=1}^{n} \lambda_i < 0 \tag{12.2.4}$$

are satisfied.
(iv) The inequality

$$l_q\mathbf{c}_z\mathbf{e} < 0 \tag{12.2.5}$$

holds.

This theorem shows that cooperation among the subsystems in the sense of Definition 12.2 not only necessitates a certain sign of $\mathbf{l}_q$ as prescribed by eqn (12.2.5) but imposes severe restrictions on the I/O behaviour of the subsystems with regard to their interconnection signals s_i and z_i as stated in parts (ii) and (iii). However, as can be seen from the matrices

$\mathbf{A}_s$ and $\mathbf{A}_c(N)$ used in Lemma 12.1, the isolated subsystems need not be stable.

Existence of Fixed Modes

The controllability and observability of the symmetric composite system and the existence of fixed modes can be easily investigated by applying Theorem 2.1 to the overall system description (12.1.9). The existence of fixed modes can be analysed by means of the test described in Theorem 4.1. As a result of the structure of the matrices occurring in eqn (12.1.9) the following results can be obtained.

Theorem 12.3

For the symmetric composite system (12.1.1)–(12.1.3) the following statements are equivalent:

(i) The system (12.1.1)–(12.1.3) is completely controllable through $\mathbf{u} = (\mathbf{u}_1', \ldots, \mathbf{u}_N')'$ and completely observable through $\mathbf{y} = (\mathbf{y}_1', \ldots, \mathbf{y}_N')'$.

(ii) The pairs $(\mathbf{A}_s, \mathbf{B})$ and $(\mathbf{A}_0, \mathbf{B})$ are controllable and the pairs $(\mathbf{A}_s, \mathbf{C})$ and $(\mathbf{A}_0, \mathbf{C})$ are observable with $\mathbf{A}_s$ and $\mathbf{A}_0$ from eqn (12.1.8).

(iii) The system (12.1.1)–(12.1.3) has no decentralized fixed modes.

That is, for symmetric composite systems, controllability and observability of the overall system is not merely necessary but even sufficient for the absence of fixed modes under decentralized control. Moreover, these properties can be tested by considering only the low-order pairs given in part (ii) of the theorem.

Example 12.1 (*Decentralized voltage control of a multiarea power system*)

The results presented so far are relevant to systems whose subsystems behave similarly as, from a technological point of view, they participate in doing the same task. An obvious example of such systems is a multiarea power system as described in Example 11.1, which consists of several similar power generators which feed the same power distribution

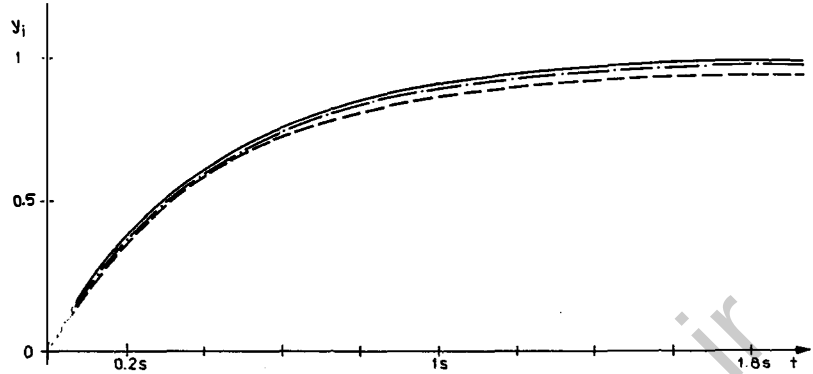


Figure 12.3 Step response of the synchronous machine i subject to set point change of u_i ($u_j = 0$ for $j \neq i$): ———, approximate model; $- \cdot - \cdot -$, $N = 2$; $- - -$, $N = 40$

net in order to satisfy the power demands. Since the power generators are of only a few different designs, power systems consist at least in part of identical subsystems or can be approximately dealt with as symmetric composite systems (for such an extension see Section 12.4).

The model used for the voltage control problem has been described in Example 11.1. It is considered here for a symmetric net and identical generators which are described by eqns (12.1.1)–(12.1.3) with $n_i = 3$, $m_i = r_i = m_{si} = r_{zi} = 1$

$$\mathbf{A} = \begin{pmatrix} -1.94 & -0.16 & 0 \\ 2.58 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0.9 \\ -1 \\ 0 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} -0.33 \\ -0.015 \\ 2 \end{pmatrix}$$

$$\mathbf{C} = (0 \;\; 0 \;\; 1) \qquad \mathbf{C}_z = (2.55 \;\; 0 \;\; 0)$$

$$\mathbf{L}_d = 0.655 \qquad \mathbf{L}_q = 0.053.$$

Hence,

$$\mathbf{A}_s = \begin{pmatrix} -2.45 & -0.16 & 0 \\ 2.56 & 0 & 0 \\ 3.07 & 0 & -2 \end{pmatrix}$$

$$\mathbf{A}_0 = \begin{pmatrix} -2.49-(N-1)0.0446 & -0.16 & 0 \\ 2.55-(N-1)0.002 & 0 & 0 \\ 3.34+(N-1)0.27 & 0 & -2 \end{pmatrix}$$

hold. The plant is stable for a large range of the subsystem number N, but it is not asymptotically cooperative, because for large subsystem numbers ($N > 1275$) it becomes unstable. This result coincides with the experience of electrical engineers who know about the difficulties of voltage control in large systems. Figure 12.3 shows the step response of the plant with $N = 5$ subsystems. Clearly, the approximation obtained by the model (12.1.11) is quite good. □

12.3 DECENTRALIZED CONTROL OF SYMMETRIC COMPOSITE SYSTEMS

The aim is to design a decentralized controller

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{F}\mathbf{x}_{ri} + \mathbf{G}\mathbf{y}_i + \mathbf{H}\mathbf{v}_i \\ \mathbf{u}_i &= -\mathbf{K}_x\mathbf{x}_{ri} - \mathbf{K}_y\mathbf{y}_i + \mathbf{K}_v\mathbf{v}_i \qquad (i = 1, \ldots, N) \end{aligned} \tag{12.3.1}$$

so that the closed-loop system (12.1.1), (12.1.2) and (12.3.1) satisfies the design specifications (1)–(3) given in Section 1.2. It is assumed that the set of admissible command signals $\mathbf{v}_i$ and the dynamical requirements (3) are the same for all subsystems because the plant is symmetric. The use of identical control stations and, thus, the preservation of the symmetry of the system after the implementation of the control stations is desirable for two reasons. First, disturbances should be rejected 'locally', that is within the subsystem they enter. Command following should be attained with the least possible excitation of the other subsystems. Second, the conformity of the subsystem performance leads to severe conceptual simplifications of the design task.

Equations (12.1.1), (12.1.2) and (12.3.1) yield

$$\begin{pmatrix} \dot{\mathbf{x}}_i \\ \dot{\mathbf{x}}_{ri} \end{pmatrix} = \begin{pmatrix} \mathbf{A} - \mathbf{B}\mathbf{K}_y\mathbf{C} & -\mathbf{B}\mathbf{K}_x \\ \mathbf{G}\mathbf{C} & \mathbf{F} \end{pmatrix} \begin{pmatrix} \mathbf{x}_i \\ \mathbf{x}_{ri} \end{pmatrix} + \begin{pmatrix} \mathbf{B}\mathbf{K}_v \\ \mathbf{H} \end{pmatrix} \mathbf{v}_i + \begin{pmatrix} \mathbf{E} \\ \mathbf{0} \end{pmatrix} \mathbf{s}_i \tag{12.3.2}$$

$$\mathbf{y}_i = (\mathbf{C} \;\; \mathbf{0}) \begin{pmatrix} \mathbf{x}_i \\ \mathbf{x}_{ri} \end{pmatrix} \tag{12.3.2}$$

$$\mathbf{z}_i = (\mathbf{C}_z \;\; \mathbf{0}) \begin{pmatrix} \mathbf{x}_i \\ \mathbf{x}_{ri} \end{pmatrix}$$

or, in short,

$$\begin{aligned} \dot{\bar{\mathbf{x}}}_i &= \bar{\mathbf{A}}\bar{\mathbf{x}}_i + \bar{\mathbf{B}}\mathbf{v}_i + \bar{\mathbf{E}}\mathbf{s}_i \\ \mathbf{y}_i &= \bar{\mathbf{C}}\bar{\mathbf{x}}_i \\ \mathbf{z}_i &= \bar{\mathbf{C}}_z\bar{\mathbf{x}}_i \end{aligned} \tag{12.3.3}$$

with $\bar{\mathbf{x}}_i = (\mathbf{x}_i', \mathbf{x}_{ri}')'$ and $\dim \bar{\mathbf{x}}_i = \bar{n} = n + n_r$. Like the plant the closed-loop system consists of identical subsystems (12.3.3) that are symmetrically interconnected via the relations (12.1.2) and (12.1.3). Therefore, models analogous to eqns (12.1.4), (12.1.8) and (12.1.9) can be found as follows. By combining eqns (12.1.2), (12.1.3) and (12.3.3) the model of the closed-loop overall system

$$\dot{\bar{\mathbf{x}}} = \begin{pmatrix} \bar{\mathbf{A}} + \bar{\mathbf{E}}\mathbf{L}_d\bar{\mathbf{C}}_z & \bar{\mathbf{E}}\mathbf{L}_q\bar{\mathbf{C}}_z & \cdots & \bar{\mathbf{E}}\mathbf{L}_q\bar{\mathbf{C}}_z \\ \bar{\mathbf{E}}\mathbf{L}_q\bar{\mathbf{C}}_z & \bar{\mathbf{A}} + \bar{\mathbf{E}}\mathbf{L}_d\bar{\mathbf{C}}_z & \cdots & \bar{\mathbf{E}}\mathbf{L}_q\bar{\mathbf{C}}_z \\ \vdots & \vdots & & \vdots \\ \bar{\mathbf{E}}\mathbf{L}_q\bar{\mathbf{C}}_z & \bar{\mathbf{E}}\mathbf{L}_q\bar{\mathbf{C}}_z & \cdots & \bar{\mathbf{A}} + \bar{\mathbf{E}}\mathbf{L}_d\bar{\mathbf{C}}_z \end{pmatrix} \bar{\mathbf{x}} + \text{diag } \bar{\bar{\mathbf{B}}}\mathbf{v}$$

$$\mathbf{y} = \text{diag } \bar{\mathbf{C}}\bar{\mathbf{x}} \tag{12.3.4}$$

is obtained where $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_1', \ldots, \bar{\mathbf{x}}_N')'$ and $\dim \bar{\mathbf{x}} = N\bar{n}$ hold. By means of the transformation

$$\bar{\bar{\mathbf{x}}} = \mathbf{T}\bar{\mathbf{x}} \tag{12.3.5}$$

with $\mathbf{T}$ from eqn (12.1.6) the analogue of eqn (12.1.7) can be obtained

$$\dot{\bar{\bar{\mathbf{x}}}} = \left(\begin{array}{ccc|c} \bar{\mathbf{A}}_s & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \cdots & \bar{\mathbf{A}}_s & \mathbf{0} \\ \hline \mathbf{0} & \cdots & \mathbf{0} & \bar{\mathbf{A}}_0 \end{array}\right) \bar{\bar{\mathbf{x}}} + \frac{1}{N} \left(\begin{array}{ccc|c} (N-1)\bar{\bar{\mathbf{B}}} & \cdots & -\bar{\bar{\mathbf{B}}} & -\bar{\bar{\mathbf{B}}} \\ \vdots & & \vdots & \vdots \\ -\bar{\bar{\mathbf{B}}} & \cdots & (N-1)\bar{\bar{\mathbf{B}}} & -\bar{\bar{\mathbf{B}}} \\ \hline \bar{\bar{\mathbf{B}}} & \cdots & \bar{\bar{\mathbf{B}}} & \bar{\bar{\mathbf{B}}} \end{array}\right) \mathbf{v}$$

$$\bar{\bar{\mathbf{x}}}(0) = \mathbf{T}\bar{\mathbf{x}}_0 \tag{12.3.6}$$

$$\mathbf{y} = \left(\begin{array}{ccc|c} \bar{\mathbf{C}} & \cdots & \mathbf{0} & \bar{\mathbf{C}} \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \cdots & \bar{\mathbf{C}} & \bar{\mathbf{C}} \\ \hline -\bar{\mathbf{C}} & \cdots & -\bar{\mathbf{C}} & \bar{\mathbf{C}} \end{array}\right) \bar{\bar{\mathbf{x}}}$$

with

$$\begin{aligned} \bar{\mathbf{A}}_s &= \bar{\mathbf{A}} + \bar{\mathbf{E}}(\mathbf{L}_d - \mathbf{L}_q)\bar{\mathbf{C}}_z \\ \bar{\mathbf{A}}_0 &= \bar{\mathbf{A}} + \bar{\mathbf{E}}[\mathbf{L}_d + (N-1)\mathbf{L}_q]\bar{\mathbf{C}}_z. \end{aligned} \tag{12.3.7}$$

Each subsystem output $\mathbf{y}_i$ of the closed-loop overall system is determined by

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A}_s & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{A}}_0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{N-1}{N}\bar{\bar{\mathbf{B}}} \\ \frac{1}{N}\bar{\bar{\mathbf{B}}} \end{pmatrix} \mathbf{v}_i + \begin{pmatrix} -\frac{1}{N}\bar{\bar{\mathbf{B}}} \\ \frac{1}{N}\bar{\bar{\mathbf{B}}} \end{pmatrix} \sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{v}_j$$

$$\mathbf{y}_i = (\bar{\mathbf{C}} \;\; \bar{\mathbf{C}})\mathbf{x}. \tag{12.3.8}$$

All phenomena encountered in the whole system can be studied by means of the model (12.3.8). Therefore, the index of $\bar{\bar{\mathbf{x}}}$ will be dropped.

Reduction of the Design Complexity

The model (12.3.8) can be thought of as the closed-loop system that consists of the plant

$$\dot{\tilde{\mathbf{x}}} = \begin{pmatrix} \mathbf{A}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_0 \end{pmatrix}\tilde{\mathbf{x}} + \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}\mathbf{u}$$

$$\tilde{\mathbf{y}} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}\tilde{\mathbf{x}} \qquad (12.3.9)$$

$$\mathbf{y}_i = (\mathbf{C} \;\; \mathbf{C})\tilde{\mathbf{x}}$$

and the decentralized controller with two control stations

$$\dot{\tilde{\mathbf{x}}}_r = \begin{pmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{pmatrix}\tilde{\mathbf{x}}_r + \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{pmatrix}\tilde{\mathbf{y}} + \begin{pmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix}\tilde{\mathbf{v}}$$

$$\tilde{\mathbf{u}} = \begin{pmatrix} -\mathbf{K}_x & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_x \end{pmatrix}\tilde{\mathbf{x}}_r + \begin{pmatrix} -\mathbf{K}_y & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_y \end{pmatrix}\tilde{\mathbf{y}} + \begin{pmatrix} \mathbf{K}_v & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_v \end{pmatrix}\tilde{\mathbf{v}} \qquad (12.3.10)$$

with $\tilde{\mathbf{x}} = (\mathbf{x}_s' \;\; \mathbf{x}_0')'$, $\tilde{\mathbf{u}} = (\mathbf{u}_s' \;\; \mathbf{u}_0')'$, $\tilde{\mathbf{y}} = (\mathbf{y}_s' \;\; \mathbf{y}_0')'$ and $\tilde{\mathbf{v}} = (\mathbf{v}_s' \;\; \mathbf{v}_0')'$

$$\mathbf{v}_s = \frac{N-1}{N}\mathbf{v}_i - \frac{1}{N}\sum_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{v}_j \qquad (12.3.11)$$

$$\mathbf{v}_0 = \frac{1}{N}\sum_{j=1}^{N} \mathbf{v}_j$$

(cf. Figure 12.4).

Lemma 12.2

The decentralized controller (12.3.1) satisfies the design specification (1)–(3) for the plant (12.1.1)–(12.1.3) if and only if the decentralized controller (12.3.10) and (12.3.11) meets these requirements in connection with the plant (12.3.9).

Hence, the symmetry of the system brings about a considerable reduction of the design complexity. Only a low-order auxiliary plant (12.3.9) and a decentralized controller with merely two control stations have to be considered. This holds true for an arbitrary number of subsystems and arbitrarily strong interactions between the subsystems. The number N merely influences the parameters of the plant (12.3.9) (cf. $\mathbf{A}_0$ in eqn (12.1.8)) and the feedforward action of the controller (cf. eqn (12.3.11)).

Stabilizability of the Plant and Stability of the Closed-loop System

Since the closed-loop system (12.3.9) and (12.3.10) consists of two independent loops, the overall closed-loop system is stable if and only if both loops are stable. As is clear from Figure 12.4 these two loops consist of different plants but have identical feedback.

Theorem 12.4

The decentralized controller (12.3.1) ensures the stability of the closed-loop system (12.3.1) and (12.1.1)–(12.1.3) if and only if the control station

$$\begin{aligned} \dot{\mathbf{x}}_r &= \mathbf{F}\mathbf{x}_r + \mathbf{G}\mathbf{y} + \mathbf{H}\mathbf{v} \\ \mathbf{u} &= \mathbf{K}_x\mathbf{x}_r - \mathbf{K}_y\mathbf{y} + \mathbf{K}_v\mathbf{v} \end{aligned} \tag{12.3.12}$$

(cf. eqn (12.3.1) with subscripts dropped) ensures closed-loop stability simultaneously for the plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_s\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{12.3.13}$$

and the plant

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_0\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}. \end{aligned} \tag{12.3.14}$$

This theorem states that in view of the stability requirement (1) the problem of designing the decentralized controller (12.3.1) for the overall system (12.1.1)–(12.1.3) can be replaced by the task of designing a centralized controller (12.3.12) that simultaneously stabilizes the two auxiliary plants (12.3.13) and (12.3.14). This is a problem of robust centralized control.

A well-known result concerning robust feedback control states that two plants can be simultaneously stabilized by a common controller if and only if a single auxiliary plant, which is derived from the two original plants, can be stabilized by a *stable* controller. This, in turn, can be checked for systems with dim $\mathbf{y}$ = dim $\mathbf{u} = 1$ by proving the 'parity interlacing property' (for details see Lunze (1988)).

Theorem 12.5

Consider a symmetric composite system (12.1.1)–(12.1.3) with dim $\mathbf{u}_i$ = dim $\mathbf{y}_i = 1$. Assume that the matrix $\mathbf{A}_s$ or $\mathbf{A}_0$ defined in eqn

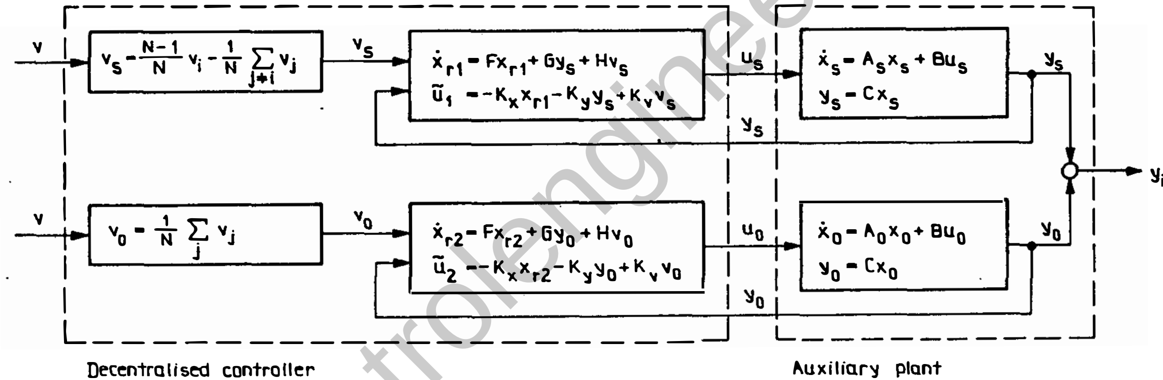


Figure 12.4 The auxiliary closed-loop system (12.3.9)–(12.3.12)

(12.1.8) is stable. Then there exists a stabilizing decentralized controller (12.3.1) if and only if the system

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \mathbf{B} \\ \mathbf{B} \end{pmatrix}\mathbf{u} \qquad (12.3.15)$$

$$\mathbf{y} = (\mathbf{C} \;\; \mathbf{C})\mathbf{x}$$

has the 'parity interlacing property', that is the total number of the real poles that lie to the right of each of the real right-half plane zeros in the complex plane are all odd or all even.

A comparison of the statements of Theorems 12.5 and 12.3 is now in order. Theorem 12.3 implies that a symmetric composite system can be stabilized by a decentralized controller if the systems (12.3.13) and (12.3.14) are completely controllable and completely observable. Theorem 12.5 says that the plant (12.3.1)–(12.3.3) can be stabilized by a decentralized controller (12.3.1) if and only if the auxiliary plants (12.3.13) and (12.3.14) can be stabilized by the same controller. Therefore, one may conjecture that the systems (12.3.13) and (12.3.14) can be simultaneously stabilized if they are controllable and observable.

This conjecture is not true. As will be shown by the counterexample below, a stabilizing decentralized controller (12.3.1) with identical control stations may not exist although the plant has no decentralized fixed modes. That is, the use of identical control stations for all subsystems represents a restriction which, however reasonable it is for symmetric composite systems, may be too restrictive to solve the stabilization problem.

Example 12.2

This example provides a symmetric system which has no decentralized fixed modes but which cannot be stabilized by a decentralized controller (12.3.1) with identical control stations. Consider the model (12.1.1)–(12.1.3) with $N = 2$,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1.25 & 0.5 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{C} = (-1.5 \;\; 1) \qquad \mathbf{C}_z = (-0.75 \;\; 2.5)$$

$$\mathbf{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The subsystem (12.1.1) and the overall system (12.1.1)–(12.1.3) are unstable. Eqn (12.1.8) yields

$$\mathbf{A}_s = \begin{pmatrix} 0 & 1 \\ -0.5 & -2 \end{pmatrix} \qquad \mathbf{A}_0 = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

As $\mathbf{A}_s$ is stable, the existence of a stabilizing decentralized controller (12.3.1) can be tested by Theorem 12.5. As the system (12.3.15) has real right-half plane poles at 1.0 and 2.0 and zeros at 0.21 and 1.35, it does not satisfy the conditions stated in Theorem 12.5. Hence, there does not exist any decentralized controller (12.3.1) with identical control stations that stabilizes the overall system.

On the other hand, the pairs $(\mathbf{A}_s, \mathbf{B})$ and $(\mathbf{A}_0, \mathbf{B})$ are controllable and the pairs $(\mathbf{A}_s, \mathbf{C})$ and $(\mathbf{A}_0, \mathbf{C})$ are observable. By Theorem 12.3, there exists a decentralized controller that stabilizes the overall system. That is, the example system can be stabilized by some decentralized controller but not by a decentralized controller with identical control stations.

Incidentally, a solution to the stabilization problem can be easily found for this example. As the example system (12.1.4) is completely controllable and observable through a single channel $(\mathbf{u}_i, \mathbf{y}_i)$ it can even be stabilized by a single control station. This controller, however, destroys the symmetry within the system and is, from a practical point of view, undesirable (cf. the discussion of the principally 'non-symmetric' stabilization method presented in Section 6.1). □

Simplification of the model (12.3.9)–(12.3.11)

Whereas the conceptual simplifications of the design problem elaborated so far are true for arbitrary symmetric composite systems, further simplifications can be made under the assumptions that the closed-loop system will certainly be stable (i.e. $\bar{\mathbf{A}}_s$ and $\bar{\mathbf{A}}_0$ are known to be stable) and that the plant consists of a large number of subsystems. Since for $N \to \infty$ eqn (12.3.9) yields $\bar{\bar{\mathbf{v}}}_s(t) \to \mathbf{v}_i(t)$ and $\bar{\bar{\mathbf{v}}}_0(t) \to 0$ the loop at the bottom of Figure 12.4 is not excited by $\mathbf{v}_j$ $(j \neq i)$ for large N. The performance of the model (12.3.9)–(12.3.11) can be described by the closed-loop system that consists of the plant (12.1.11) and the controller (12.3.12) (with $\mathbf{y} = \hat{\mathbf{y}}$, $\mathbf{u} = \mathbf{u}_i$, $\mathbf{v} = \mathbf{v}_i$). That is, for plants with a finite but large number of subsystems the model (12.1.11) and (12.3.12) or, equivalently, (12.3.12) and (12.3.13) represents an *approximate* description of the closed-loop system. The control of this plant is no longer decentralized but centralized. This observation suggests the design of the decentralized controller (12.3.1) in relation to the design requirement (3) in three design steps as follows:

Algorithm 12.1

Given: Symmetric composite system (12.1.1)–(12.1.3); design requirements (1)–(3) (Section 1.2).

1. Derive the approximate model (12.1.11).
2. Design a centralized controller (12.3.12) for the approximate model (12.1.11) to satisfy the dynamical requirements (3).
3. Check the dynamical behaviour of the closed-loop system by means of the exact model (12.3.9)–(12.3.11).

Result: Decentralized controller (12.3.1) with the same parameters as (12.3.12).

This design method is especially useful if the plant has many subsystems and the stability requirement (1) presents no real difficulty during the design process. Since the approximate model (12.1.11) is not identical to the isolated subsystem (eqn (12.1.1) with $\mathbf{s}_i = \mathbf{0}$) but includes the 'coarse model' (12.1.12) of the interactions with the other subsystems, the interactions between the subsystems are not ignored during the design as is done, for example, in the aggregation–decomposition method (Section 9.2).

Decentralized PI Controllers

The results obtained so far can be made more specific for decentralized PI control

$$\begin{aligned} \dot{\mathbf{x}}_{ri} &= \mathbf{y}_i - \mathbf{v}_i \\ \mathbf{u}_i &= -\mathbf{K}_P(\mathbf{y}_i - \mathbf{v}_i) - \mathbf{K}_I\mathbf{x}_{ri} \qquad (i = 1, 2, \ldots, N) \end{aligned} \tag{12.3.16}$$

of a stable symmetric composite system (12.1.1)–(12.1.3) subject to step commands. According to Theorem 12.4, every control station (12.3.16) has simultaneously to stabilize the plants (12.3.13) and (12.3.14). Theorem 2.14 leads to the following result.

Theorem 12.6

Consider a stable symmetric composite system (12.1.1)–(12.1.3) with $\dim \mathbf{y}_i = \dim \mathbf{u}_i = m$.

(i) A necessary condition for the stability of the closed-loop system (12.1.1)–(12.1.3) and (12.3.16) is given by

$$\det(\mathbf{K}_{ss}\mathbf{K}_{s0}) > 0 \tag{12.3.17}$$

with

$$\mathbf{K}_{ss} = -\mathbf{C}\mathbf{A}_s^{-1}\mathbf{B} \qquad \mathbf{K}_{s0} = -\mathbf{C}\mathbf{A}_0^{-1}\mathbf{B}. \tag{12.3.18}$$

(ii) The closed-loop system (12.1.1)–(12.1.3) and (12.3.16) is stable for controller matrices

$$\mathbf{K}_I = a\hat{\mathbf{K}}_I \qquad \mathbf{K}_P = b\hat{\mathbf{K}}_P \tag{12.3.19}$$

with $0 < a \leqslant \bar{a}$ and $0 \leqslant b \leqslant \bar{b}$ for sufficiently small $\bar{a}$ and $\bar{b}$ if and only if

$$\mathrm{Re}[\lambda_i[\mathbf{K}_{s0}\hat{\mathbf{K}}_I]] > 0 \qquad \mathrm{Re}[\lambda_i[\mathbf{K}_{ss}\hat{\mathbf{K}}_I]] > 0 \tag{12.3.20}$$

$(i = 1, 2, \ldots, m)$ hold.

Relation (12.3.17) states a condition that must be necessarily satisfied for a decentralized PI controller to exist. The second part of the theorem gives a constructive method of ensuring closed-loop stability. If $\dim \mathbf{y}_i = \dim \mathbf{u}_i = 1$ holds, condition (12.3.17) reads as

$$k_{ss}k_{s0} > 0. \tag{12.3.21}$$

It is necessary and sufficient for the stabilizability. A reasonable choice of $\hat{k}_I$ and $\hat{k}_P$ is given by

$$\hat{k}_I = \hat{k}_P = k_{ss}^{-1}. \tag{12.3.22}$$

Example 12.1 (cont.)

As the set points for the node voltages can be assumed to change stepwise, decentralized PI controllers (12.3.16) are used. There exist such controllers which ensure closed-loop stability because $k_{s0}k_{ss} = 0.45 > 0$ holds (cf. (12.3.17) and (12.3.18)).

Algorithm 12.1 is used to find suitable parameters for the decentralized voltage controllers:

Step 1

The model (12.1.11) is found by using eqn (12.1.8).

Step 2

The parameters of a centralized PI controller (eqn (12.3.16) without index i and for a single-input single-output system) are determined as optimal output feedback $\mathbf{u} = -k_I\mathbf{x}_r - k_P\mathbf{y}$ of the model (12.1.11) that has been expanded by the dynamical part of the controller (12.3.16) with $\mathbf{v} = \mathbf{0}$ and $\mathbf{x}_r(0) = \mathbf{x}_{r0}$. For the performance index

$$\int_0^\infty (q_{11}x_r^2 + q_{22}x^2 + u^2)\, dt \to \min_{k_I, k_P}$$

different controllers can be obtained depending upon the choice of $\mathbf{Q} = \text{diag } q_{ii}$. The solution for $\mathbf{Q} = 10\mathbf{I}$ ($k_I = 3.16$, $k_P = 2.17$) is used because it leads to a command step response of the closed-loop approximate model (12.1.11) and (12.3.16) with reasonable overshoot and settling time (Figure 12.5).

Step 3

With these controller parameters the decentralized controller (12.3.16) ensures closed-loop stability (Theorem 12.4). To investigate the command response exactly, the model (12.3.9)–(12.3.11) is used. As shown in Figure 12.6(a), for a wide range of the subsystem number N the command step response of the overall system is very close to the approximation obtained in step 2. The cross-couplings (y_1 for step input at v_2) are small and decrease with increasing number of subsystems (Figure 12.6(b)). One of these curves has been drawn in Figure 12.6(a) to illustrate the magnitude of the cross couplings in relation to the influence of v_1 on y_1. □

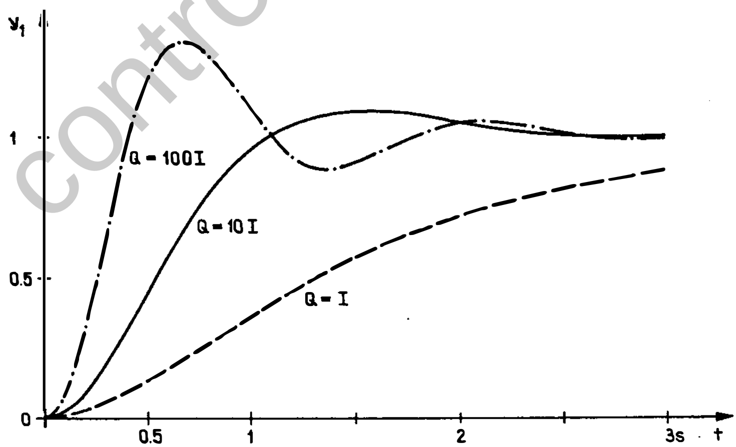


Figure 12.5 Command step response of the closed-loop approximate model

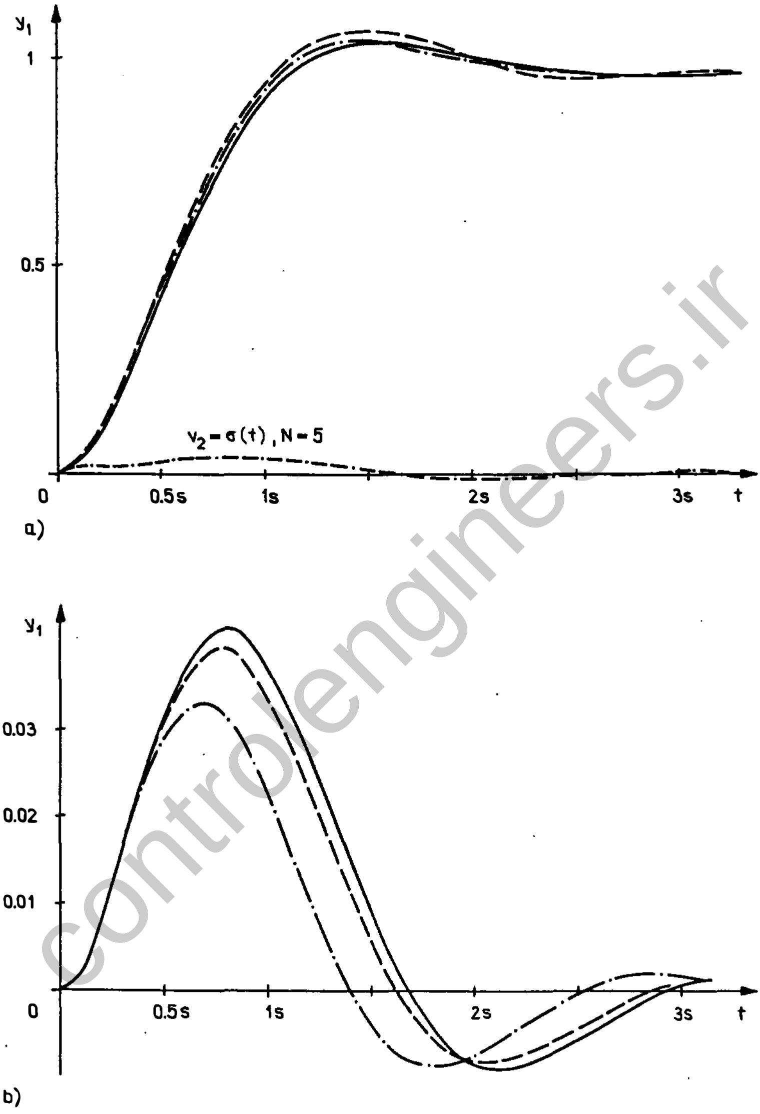


Figure 12.6 Step response $y_1(t)$ of the closed-loop overall system: (a) for $v_1(t) = \sigma(t)$: ———, approximate model; – – –, $N = 2$; – · – · –, $N = 11$; (b) for $v_2(t) = \sigma(t)$: ———, $N = 2$; – – –, $N = 5$; – · – · –, $N = 11$

Summary and Discussion of the Design Method

The results of this section clearly show that symmetry within a complex system provides the basis for major conceptual simplifications of the design of decentralized controllers:

- The problem of designing a decentralized controller (12.3.1) with N control stations that stabilizes the overall system is equivalent to the problem of designing a robust centralized controller (12.3.12) that simultaneously stabilizes the auxiliary plants (12.3.12) and (12.3.14) (Theorem 12.4).
- The I/O behaviour of the closed-loop overall system (12.1.1)–(12.1.3) and (12.3.1) is *exactly* described by the model (12.3.9)–(12.3.11), which includes just two auxiliary subsystems and two control stations.
- The I/O behaviour of the closed-loop overall system (12.1.1)–(12.1.3) and (12.3.1) is *approximately* described by the centralized closed-loop system (12.1.11) and (12.3.12) with $\hat{\mathbf{y}} = \mathbf{y}$.

These results should be compared with the methods described in Chapter 9, which likewise aim at replacing the overall design task by problems that can be solved at the subsystem level. Since in the latter methods the subsystem interactions are *ignored*, the control stations obtained independently will control the overall plant only if the subsystems are 'weakly' coupled. In contrast to this, the results derived in this chapter suggest an alternative way of reducing the design complexity. They show that the overall design problem can be reduced by *exploiting* the structural properties of the interactions rather than ignoring them. Therefore, the reductions are possible for arbitrarily strongly coupled subsystems and are particularly useful for plants with many subsystems.

12.4 STABILITY ANALYSIS OF SYSTEMS THAT ARE COMPOSED OF SIMILAR SUBSYSTEMS

In this section, an extension of the investigations of symmetric composite systems to systems is given, which consist of similar rather than identical subsystems. It will be shown that the structural property of the overall system to be nearly symmetric can be used to develop a method for analysing the stability of composite systems that is quite different from that described in Chapter 8. The main idea is outlined in

Figure 12.7. Whereas the composite-system method is elaborated to solve Problem 8.1, the following problem will be considered here.

Problem 12.1

Suppose that the subsystems can be described by identical approximate models and individual upper bounds of the modelling errors. Check the stability of the symmetric core $\hat{S}$ which is composed of the approximate models. Under what conditions on the model error bounds $\bar{S}$ does the stability of $\hat{S}$ guarantee the stability of the overall system?

Systems that are Symmetrically Composed of Similar Subsystems

The assumptions outlined in Problem 12.1 are reflected by the subsystem model that consists of the two parts

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i + \mathbf{E}\mathbf{s}_i + \mathbf{G}\mathbf{f}_i \qquad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i &= \mathbf{C}\mathbf{x}_i + \mathbf{H}\mathbf{f}_i \\ \mathbf{z}_i &= \mathbf{C}_z\mathbf{x}_i + \mathbf{H}_z\mathbf{f}_i \\ \mathbf{d}_i &= \mathbf{C}_d\mathbf{x}_i + \mathbf{D}_d\mathbf{u}_i + \mathbf{F}_d\mathbf{s}_i \end{aligned} \tag{12.4.1}$$

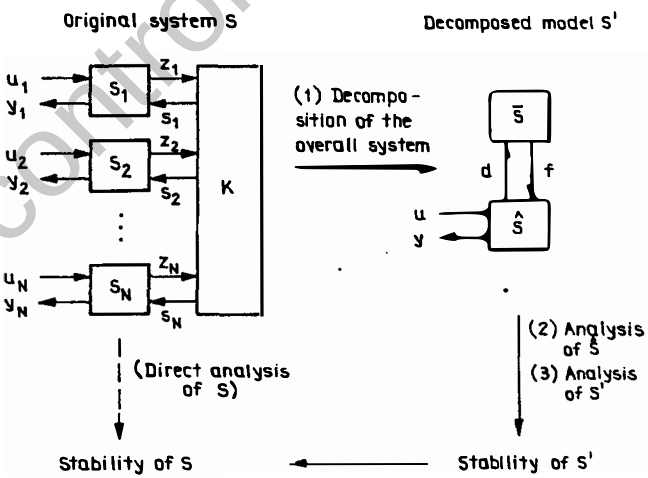


Figure 12.7 Stability analysis of systems that are composed of similar subsystems

and

$$\mathbf{f}_i = \mathbf{G}_{2i} * \mathbf{d}_i \qquad (i = 1, 2, \ldots, N) \tag{12.4.2}$$

where $\mathbf{f}_i$ and $\mathbf{d}_i$ are m_f- and r_d-dimensional signals between part 1 and part 2 of the model (Figure 12.8). Eqn (12.4.1) is referred to as the approximate model of the subsystems. Although it has identical properties for all subscripts i, the subsystems are not assumed to behave exactly in the same way as in Sections 12.1–12.3. Eqn (12.4.2) represents an individual error model, which describes the deviation of the subsystem behaviour from that of the approximate model (12.4.1). The error is described by some upper bound $\mathbf{V}_{2i}(t)$

$$|\mathbf{G}_{2i}(t)| \leqslant \mathbf{V}_{2i}(t) \quad \text{for all } t. \tag{12.4.3}$$

Hence,

$$|\mathbf{f}_i(t)| \leqslant \mathbf{V}_{2i} * |\mathbf{d}_i|. \tag{12.4.4}$$

holds (cf. Section 2.6).

Equations (12.4.1) and (12.4.4) reflect the assumption of similarity of the subsystem behaviour and make the consideration of incompletely known large-scale systems possible. The error bound (12.4.4) may include both uncertainties of the subsystem dynamics and deviations of the I/O behaviour of the individual subsystem from that of the approximate model (12.4.1).

As in Section 12.1 the interconnections are assumed to be symmetric and described by eqns (12.1.2) and (12.1.3)

$$\mathbf{s} = \mathbf{L}\mathbf{z} \tag{12.4.5}$$

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_d & \mathbf{L}_q & \cdots & \mathbf{L}_q \\ \mathbf{L}_q & \mathbf{L}_d & \cdots & \mathbf{L}_q \\ \vdots & \vdots & & \vdots \\ \mathbf{L}_q & \mathbf{L}_q & \cdots & \mathbf{L}_d \end{pmatrix}. \tag{12.4.6}$$

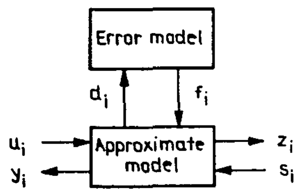


Figure 12.8 Subsystem model (12.4.1)–(12.4.3)

Definition 12.3

A system that can be described by some model (12.4.1), (12.4.4), (12.4.5) and (12.4.6) is said to be *symmetrically composed of similar subsystems.*

The Symmetric Core of the Overall System

If the approximate models (12.4.1) are combined via the symmetric interactions (12.4.5) and (12.4.6)

$$\begin{aligned}\dot{\mathbf{x}} &= [\operatorname{diag}\mathbf{A} + (\operatorname{diag}\mathbf{E})\mathbf{L}(\operatorname{diag}\mathbf{C}_z)]\,\mathbf{x} + (\operatorname{diag}\mathbf{B})\mathbf{u}\\ &\quad + [\operatorname{diag}\mathbf{G} + (\operatorname{diag}\mathbf{E})\mathbf{L}(\operatorname{diag}\mathbf{H}_z)]\,\mathbf{f}\\ \mathbf{y} &= \operatorname{diag}\mathbf{C}\,\mathbf{x} + \operatorname{diag}\mathbf{H}\,\mathbf{f}\\ \mathbf{z} &= [\operatorname{diag}\mathbf{C}_d + (\operatorname{diag}\mathbf{F}_d)\mathbf{L}(\operatorname{diag}\mathbf{C}_z)]\,\mathbf{x} + \operatorname{diag}\mathbf{D}_d\,\mathbf{u}\\ &\quad + (\operatorname{diag}\mathbf{F}_d)\mathbf{L}(\operatorname{diag}\mathbf{H}_z)\mathbf{f}\end{aligned} \tag{12.4.7}$$

are obtained.

Definition 12.4

The system described by eqns (12.4.1), (12.4.5) and (12.4.6) or, equivalently, eqn (12.4.7) is called the *symmetric core* of the system (12.4.1), (12.4.4), (12.4.5) and (12.4.6).

The symmetric core can be treated in the same way as the system (12.1.4). The transformation $\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x}$ with $\mathbf{T}$ from eqn (12.1.6) yields the model $\hat{S}$:

$$\dot{\tilde{\mathbf{x}}} = \left(\begin{array}{ccc|c} \mathbf{A}_s & \dots & \mathbf{0} & \mathbf{0}\\ \vdots & & \vdots & \vdots\\ \mathbf{0} & \dots & \mathbf{A}_s & \mathbf{0}\\ \hline \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_0 \end{array}\right)\tilde{\mathbf{x}} + \frac{1}{N}\left(\begin{array}{ccc|c} (N-1)\mathbf{B} & \dots & -\mathbf{B} & -\mathbf{B}\\ \vdots & & \vdots & \vdots\\ -\mathbf{B} & \dots & (N-1)\mathbf{B} & -\mathbf{B}\\ \hline \mathbf{B} & \dots & \mathbf{B} & \mathbf{B} \end{array}\right)\mathbf{u}$$

$$+\frac{1}{N}\left(\begin{array}{ccc|c} (N-1)\mathbf{G}_s & \dots & -\mathbf{G}_s & -\mathbf{G}_s\\ \vdots & & \vdots & \vdots\\ -\mathbf{G}_s & \dots & (N-1)\mathbf{G}_s & -\mathbf{G}_s\\ \hline \mathbf{G}_0 & \dots & \mathbf{G}_0 & \mathbf{G}_0 \end{array}\right)\mathbf{f}$$

$$\mathbf{y} = \left(\begin{array}{ccc|c} \mathbf{C} & \dots & \mathbf{0} & \mathbf{C}\\ \vdots & & \vdots & \vdots\\ \mathbf{0} & \dots & \mathbf{C} & \mathbf{C}\\ \hline -\mathbf{C} & \dots & -\mathbf{C} & \mathbf{C} \end{array}\right)\tilde{\mathbf{x}} + \operatorname{diag}\mathbf{H}\mathbf{f} \tag{12.4.8}$$

$$\mathbf{d} = \left(\begin{array}{ccc|c} \mathbf{C}_s & \cdots & \mathbf{0} & \mathbf{C}_0 \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{C}_s & \mathbf{C}_0 \\ \hline -\mathbf{C}_s & \cdots & -\mathbf{C}_s & \mathbf{C}_0 \end{array}\right)\tilde{\mathbf{x}} + \operatorname{diag} \mathbf{D}_d \mathbf{u}$$

$$+ \begin{pmatrix} \mathbf{F}_d\mathbf{L}_d\mathbf{H}_z & \cdots & \mathbf{F}_d\mathbf{L}_q\mathbf{H}_z \\ \mathbf{F}_d\mathbf{L}_q\mathbf{H}_z & \cdots & \mathbf{F}_d\mathbf{L}_q\mathbf{H}_z \\ \vdots & & \vdots \\ \mathbf{F}_d\mathbf{L}_q\mathbf{H}_z & \cdots & \mathbf{F}_d\mathbf{L}_d\mathbf{H}_z \end{pmatrix}\mathbf{f}$$

where

$$\begin{aligned} \mathbf{A}_s &= \mathbf{A} + \mathbf{E}(\mathbf{L}_d - \mathbf{L}_q)\mathbf{C}_z \\ \mathbf{A}_0 &= \mathbf{A} + \mathbf{E}\mathbf{L}_d\mathbf{C}_z + (N-1)\mathbf{E}\mathbf{L}_q\mathbf{C}_z \\ \mathbf{G}_s &= \mathbf{G} + \mathbf{E}(\mathbf{L}_d - \mathbf{L}_q)\mathbf{H}_z \\ \mathbf{G}_0 &= \mathbf{G} + \mathbf{E}\mathbf{L}_d\mathbf{H}_z + (N-1)\mathbf{E}\mathbf{L}_q\mathbf{H}_z \\ \mathbf{C}_s &= \mathbf{C}_d + \mathbf{F}_d(\mathbf{L}_d - \mathbf{L}_q)\mathbf{C}_z \\ \mathbf{C}_0 &= \mathbf{C}_d + \mathbf{F}_d\mathbf{L}_d\mathbf{C}_z + (N-1)\mathbf{F}_d\mathbf{L}_q\mathbf{C}_z. \end{aligned} \tag{12.4.9}$$

For the symmetric core all simplifications of the stability analysis hold which have been derived for symmetric composite systems in the sections above.

Lemma 12.3

Assume that the symmetric core (12.4.1), (12.4.5) and (12.4.6) or, equivalently, (12.4.8) is completely controllable through $\mathbf{u}$ and completely observable through $\mathbf{y}$. Then the symmetric core is I/O-stable if and only if the matrices $\mathbf{A}_0$ and $\mathbf{A}_s$ in eqn (12.4.9) are stable.

After the transformation above, the overall system (12.4.1), (12.4.2), (12.4.5) and (12.4.6) has the structure depicted in Figure 12.9 and is represented by eqn (12.4.8) and

$$|\mathbf{f}| \leqslant \operatorname{diag} \mathbf{V}_{2i} * |\mathbf{d}|. \tag{12.4.10}$$

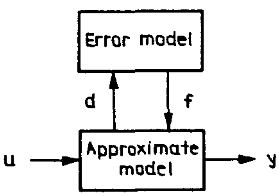


Figure 12.9 Structure of the model (12.4.12) and (12.4.13)

The Stability Criterion

In the following, the matrix $\mathbf{M}$ denotes $\int_0^\infty |\mathbf{G}(t)|\, dt$ where $\mathbf{M}$ and $\mathbf{G}$ may have identical indices. As a preliminary result the following lemma can be derived from Theorem 2.17.

Lemma 12.4

Consider the system composed of

$$\begin{aligned} \mathbf{y} &= \mathbf{G}_{yu} * \mathbf{u} + \mathbf{G}_{yf} * \mathbf{f} \\ \mathbf{d} &= \mathbf{G}_{du} * \mathbf{u} + \mathbf{G}_{df} * \mathbf{f} \end{aligned} \tag{12.4.11}$$

and

$$\mathbf{f} = \mathbf{G}_2 * \mathbf{d} \tag{12.4.12}$$

where the matrix $\mathbf{G}_2(t)$ is known to satisfy the inequality

$$|\mathbf{G}_2(t)| \leqslant \mathbf{V}_2(t) \quad \text{for all } t \tag{12.4.13}$$

for a given matrix $\mathbf{V}_2(t)$. Suppose that the systems (12.4.11) and (12.4.12) for $\mathbf{G}_2 = \mathbf{V}_2$ are I/O-stable, that is the matrices $\mathbf{M}_{yu}$, $\mathbf{M}_{yf}$, $\mathbf{M}_{du}$, $\mathbf{M}_{df}$ and

$$\mathbf{M}_2 = \int_0^\infty \mathbf{V}_2(t)\, dt$$

have finite elements. Then the overall system (12.4.11)–(12.4.13) is I/O-stable if

$$\lambda_\text{p}[\mathbf{M}_2\ \mathbf{M}_{df}] < 1. \tag{12.4.14}$$

As the model (12.4.8) and (12.4.10) of the overall system has the same structure as the system (12.4.11) and (12.4.12), the I/O stability of the system (12.4.1), (12.4.4), (12.4.5) and (12.4.6) can be investigated as follows.

Algorithm 12.2 (*Stability analysis of systems that are composed of similar subsystems*)

Given: System (12.4.1), (12.4.4), (12.4.5) and (12.4.6).

1. Determine the symmetric core (12.4.8).
2. Check the I/O stability of the symmetric core (12.4.8) as described in Lemma 12.3.

3. Check the I/O stability of (12.4.2) by proving that

$$\mathbf{M}_{2i} = \int_0^\infty \mathbf{V}_{2i}(t)\ \mathrm{d}t \qquad (i = 1, 2, \ldots, N) \tag{12.4.15}$$

is finite.

4. Check the stability condition (12.4.14). The matrix $\mathbf{G}_{df} = (\mathbf{G}_{dfij})$ represents the impulse responses of the symmetric core (12.4.8) between $\mathbf{f}_j$ and $\mathbf{d}_i$ or, equivalently, of the system

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{A}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} \dfrac{N-1}{N}\mathbf{G}_s \\ \dfrac{1}{N}\mathbf{G}_0 \end{pmatrix}\mathbf{f}_i + \begin{pmatrix} -\dfrac{1}{N}\mathbf{G}_s \\ \dfrac{1}{N}\mathbf{G}_0 \end{pmatrix}\sum_{\substack{j=1 \\ j\neq i}}^{N}\mathbf{f}_j \tag{12.4.16}$$

$$\mathbf{d}_i = (\mathbf{C}_s\ \ \mathbf{C}_0)\mathbf{x} + \mathbf{F}_d\mathbf{L}_d\mathbf{H}_z\mathbf{f}_i + \mathbf{F}_d\mathbf{L}_q\mathbf{H}_z\sum_{\substack{j=1 \\ j\neq i}}^{N}\mathbf{f}_j$$

(cf. eqn (12.4.8) for $\mathbf{u} = \mathbf{0}$). Hence,

$$\mathbf{d}_i = \sum_{j=1}^{N}\mathbf{G}_{dfij} * \mathbf{f}_j \tag{12.4.17}$$

with

$$\mathbf{G}_{dfii}(t) = \mathbf{F}_d\mathbf{L}_d\mathbf{H}_z\delta(t) + \frac{N-1}{N}\mathbf{C}_s\exp(\mathbf{A}_s t)\mathbf{G}_s + \frac{1}{N}\mathbf{C}_0\exp(\mathbf{A}_0 t)\mathbf{G}_0 \tag{12.4.18}$$

$$\mathbf{G}_{dfij}(t) = \mathbf{F}_d\mathbf{L}_q\mathbf{H}_z\delta(t) - \frac{1}{N}\mathbf{C}_s\exp(\mathbf{A}_s t)\mathbf{G}_s + \frac{1}{N}\mathbf{C}_0\exp(\mathbf{A}_0 t)\mathbf{G}_0$$

for all $i \neq j$.

Result: If the conditions tested in steps 2–4 are satisfied, the system (12.4.1), (12.4.4), (12.4.5) and (12.4.6) is stable.

The main idea of this stability test is summarized in the following theorem.

Theorem 12.7

The system (12.4.1), (12.4.4), (12.4.5) and (12.4.6) which is symmetrically composed of similar subsystems is stable if the following conditions are satisfied:

(i) The matrices $\mathbf{A}_0$ and $\mathbf{A}_s$ of eqn (12.4.9) are stable.
(ii) $\mathbf{M}_{2i}$ ($i = 1, \ldots, N$) from eqn (12.4.15) are finite.
(iii) The condition

$$\lambda_p[(\text{diag } \mathbf{M}_{2i})\mathbf{M}_{df}] < 1 \tag{12.4.19}$$

is satisfied where $\mathbf{M}_{df} = (\mathbf{M}_{dfij})$ consists of the matrices

$$\mathbf{M}_{dfij} = \int_0^\infty |\mathbf{G}_{dfij}(t)| \, \mathrm{d}t \quad \text{with } \mathbf{G}_{dfij} \text{ from eqn (12.4.18).}$$

In this stability criterion, conceptual and numerical simplifications of the stability analysis of the composite system (12.4.1), (12.4.4), (12.4.5) and (12.4.6) are not gained from breaking down the subsystem interconnections as in the composite-system method. Instead, the simplifications result from the extraction of an approximate model (12.4.8), which preserves the structural properties of the overall system and whose stability analysis can be reduced to the analysis of the nth order matrices $\mathbf{A}_s$ and $\mathbf{A}_0$. Note that the subsystems remain coupled throughout the analysis. As a result of this alternative basis, the stability conditions of Theorem 12.7 have important *characteristics* as follows:

- Although the stability condition (12.4.19) has been derived from the small-gain-type stability criterion given in Lemma 12.4, Theorem 12.7 by no means represents a small-gain theorem for the overall system (12.4.1), (12.4.4), (12.4.5) and (12.4.6). Strong interactions remain effective within the symmetric core (12.4.8) of the system. The requirements of a 'weak' coupling concerns the connection between the symmetric core (12.4.8) and the error model (12.4.2). Theorem 12.7 may be used successfully for strongly coupled systems and systems with a very large number of subsystems (for examples see below).
- The stability analysis takes into account the cooperative effects of the interactions. Therefore, no assumptions concerning the stability of the isolated subsystems have to be made.
- The simplicity of the stability test is comparable with that of composite-system method tests. Besides the stability test for the nth order matrices $\mathbf{A}_s$ and $\mathbf{A}_0$, only the Perron root of the test matrix $(\text{diag } \mathbf{M}_{2i})\mathbf{M}_{df}$ of order Nm_f has to be determined.

Several extensions of Theorem 12.5 are straightforward and will be briefly outlined here. At first, the model (12.4.1)–(12.4.6) can be formulated with direct throughput between $\mathbf{u}_i$ and $\mathbf{y}_i$ etc. Then, in principle, the same results will be obtained. Second, an extension to non-linear systems is possible if eqn (12.4.2) is replaced by a non-linear error model (cf. Section 8.5). The extension to non-symmetric couplings can be developed if the interconnection relation is decomposed into

$$\mathbf{s} = \mathbf{L}\mathbf{z} + \mathbf{w} \tag{12.4.20}$$

with $\mathbf{L}$ from eqn (12.4.6) and

$$|\mathbf{w}| \leqslant \bar{\mathbf{L}}\,|\mathbf{z}|. \tag{12.4.21}$$

Then the overall system (12.4.1), (12.4.4), (12.4.6), (12.4.20) and (12.4.21) can be brought into the form (12.4.11)–(12.4.13) and an extended stability criterion elaborated.

Example 12.3 (*Load–frequency behaviour of a multiarea power system*)

The stability criterion is used to prove the stability of a multiarea power system (Example 3.5). The symmetry assumption (12.4.6) concerning the interconnection relation is satisfied as a result of the structure of the system. In order to show this, the interconnection output z_i of the ith area is defined by

$$z_i = \int_0^t p_{\mathrm{b}i}(\tau)\,\mathrm{d}\tau.$$

This makes some modifications of the model (3.4.9)–(3.4.13) necessary. The subsystems models (3.4.12) have to be expanded by the integrator in order to determine the newly defined output z_i. Subsystem 1 as described by eqns (3.4.9) and (3.4.10) reduces to the static relation

$$s_i = \frac{1}{T_0}\sum_{j=2}^{N} z_j \qquad (i = 2, \ldots, N)$$

where eqn (3.4.13) has been used. Hence, the subsystem (3.4.9) and (3.4.10) is no longer used and the other $N-1$ subsystems are renumbered. The new model is given by eqn (3.4.12) for $i = 1, 2, \ldots, N$ and the interaction relation

$$\mathbf{s} = \mathbf{L}\mathbf{z}$$

$$\mathbf{L} = \frac{1}{NT}\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

which obviously satisfies eqn (12.1.3).

For the simplicity of demonstration, the areas are assumed to include the same amount of rotating masses, that is $T_i = T$, $T_0 = NT$. Then

$$\mathbf{A}_s = \mathbf{A} \qquad \mathbf{A}_0 = \mathbf{A} + \frac{1}{T}\mathbf{E}\mathbf{C}_z$$

result, and the stability of the symmetric part (12.4.8) is independent of the number of subsystems.

The model of a 200 MW block shown in Figure 12.10 is used as the approximate model (12.4.1) of the power generators. For the parameters given in the figure and $k_{si} = -15$, $k_{li} = -2.5$, eqn (12.4.1) holds with $n = 5$, $m = 1$, $m_s = 1$, $r = 1$, $r_z = 1$ and

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0.83 & -3.33 & 0 & 0 & -3.33 & 0 \\ 0 & 5 & -5 & 0 & 0 & 0 \\ 0 & 2.5 & -2.12 & -0.12 & 0 & 0 \\ 0 & 0 & 0 & 3.33 & -3.33 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ 3.33 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} -15 \\ -50 \\ 0 \\ 0 \\ 0 \\ -2.5 \end{pmatrix} \tag{12.4.22}$$

$$\mathbf{C} = (0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)$$
$$\mathbf{C}_z = (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1).$$

Equation (12.4.9) with $T = 16.25$ yields the stable matrix

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -0.98 \\ 0.83 & -3.33 & 0 & 0 & -3.33 & -3.2 \\ 0 & 5 & -5 & 0 & 0 & 0 \\ 0 & 2.5 & -2.12 & -0.12 & 0 & 0 \\ 0 & 0 & 0 & 3.33 & -3.33 & 0 \\ 0 & 0 & 0 & 1 & 0 & -0.16 \end{pmatrix}.$$

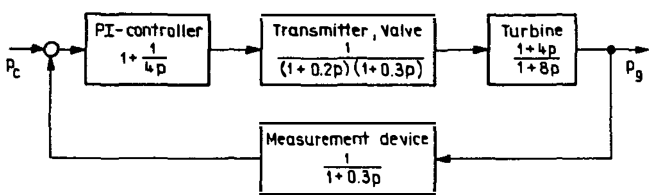


Figure 12.10 Model of the power station

Power stations with different nominal power

First, consider different nominal values of the generated power as the reason for deviations between the approximate model (12.4.1) and the actual model of the areas. As the primary control factor k_{si} is adjusted to the maximum area power, different values of the nominal power are reflected by different parameters k_{si} in the model. Hence, the model (12.4.1), (12.4.4), (12.4.5) and (12.4.6) is used with the structure of Figure 12.11, where $\bar{k}_{2i}$ is the difference between the primary control factor and the value -15 used in the approximate model. For this structure $\mathbf{H} = 0$, $\mathbf{H}_z = 0$, $\mathbf{C}_d = 0$, $\mathbf{D}_d = 0$, $\mathbf{F}_d = 1$ and $\mathbf{G} = \mathbf{E}$ hold. Equation (12.4.10) yields $\mathbf{G}_0 = \mathbf{G}_s = \mathbf{G}$, $\mathbf{C}_s = 0$, $\mathbf{C}_d = (0\ 0\ 0\ 0\ 0\ 0.065)$. As the state corresponding to the last row of $\mathbf{A}_s$ is observable through neither $\mathbf{y}_i$ nor d_i it may be deleted for I/O considerations and

$$\mathbf{A}_s = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0.83 & -3.33 & 0 & 0 & -3.33 \\ 0 & 5 & -5 & 0 & 0 \\ 0 & 2.5 & -2.12 & -0.12 & 0 \\ 0 & 0 & 0 & 3.33 & -3.33 \end{pmatrix}$$

is stable. Hence, the symmetric core is stable for an arbitrary number N of areas. Eqn (12.4.18) yields $\mathbf{M}_{dfij} = 1.35$ for all i, j. The overall system is stable if eqn (12.4.19) holds:

$$\frac{1.35}{N} \lambda_P \left[\operatorname{diag} \bar{k}_{2i} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \right] < 1.$$

By using the inclusion theorem (eqn (A1.10)) for the Perron root,

$$\frac{1}{N} \sum_{i=1}^{N} \bar{k}_{2i} < 0.735 \tag{12.4.23}$$

is obtained as a sufficient condition for the stability of the overall system.

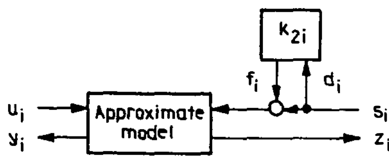


Figure 12.11 Subsystem model for areas with different nominal values of the generated power

Equation (12.4.23) can be considered as a quantitative expression of the assumption that the subsystems are similar. It demands that the *average* model uncertainty bound is less than 0.735. *Some* approximation errors $\bar{k}_{2i}$ may be very large, particularly if the whole system consists of many subsystems. For example, a system with $N = 11$ areas, where only six behave like the approximate model (200 MW blocks, $\bar{k}_{2i} = 0$) but four have 400 MW blocks (k_{si} doubled, thus $\bar{k}_{2i} = 1$) and one has a 1000 MW block ($\bar{k}_{2i} = 4$), is proved to be stable by Theorem 12.7 since $8/11 < 0.735$.

Power stations with different dynamics

Second, consider power generators with different dynamics. In this case the error model $\mathbf{v}_{2i}(t)$ describes the difference between the impulse response of the real generator and that of the model of Figure 12.10 (Figure 12.12). The approximate model (12.4.1) has the parameters of eqn (12.4.22) and $\mathbf{C}_d = 0$, $\mathbf{D}_d = 1$, $\mathbf{H}_z = 0$, $\mathbf{G} = (0\ 0\ 0\ 0\ 0\ 1)'$, $\mathbf{H} = 1$, $\mathbf{F}_d = -15$. Equation (12.4.9) yields $\mathbf{C}_s = 0$, $\mathbf{C}_0 = (0\ 0\ 0\ 0\ 0\ 0.975)$, $\mathbf{G}_s = \mathbf{G}_0 = \mathbf{G}$. Again, the last row and column of $\mathbf{A}_s$ can be deleted. The stability condition of Theorem 12.7 leads to

$$\frac{1}{N}\sum_{i=1}^{N}\int v_{2i}(t)\ dt < 0.6. \tag{12.4.24}$$

That is, the stability can be proved even if the overall system has a large variety of subsystems. The deviation of the power generator dynamics from the approximate model has only to satisfy the global restriction given in condition (12.4.24). □

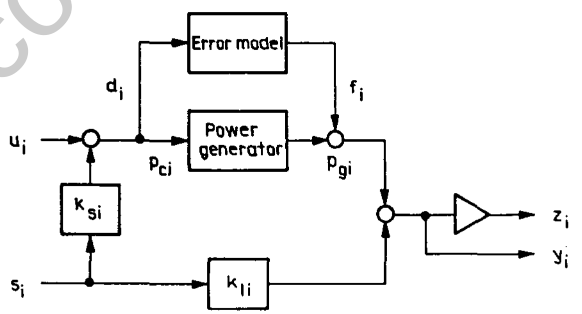


Figure 12.12 Subsystem model for areas with different power generated dynamics

This example shows that the stability can be proved for an arbitrary number N of subsystems provided that the subsystems are sufficiently similar, where 'sufficiently similar' is described by conditions (12.4.23) and (12.4.24).

BIBLIOGRAPHICAL NOTES

Symmetric relations between the components of natural phenomena or technical artifacts have long attracted the interest of engineers and artists (Wille 1988). Engineers have used the simplifications which are brought about by unification of the elements empirically. In particular, to the author's knowledge there are no systems-theoretic investigations of symmetric dynamical systems. Papers on symmetrically interconnected systems are rare even in the field of modelling. Aoki (1979), Baliga and Rao (1980) and Bergen (1979) considered systems composed of first-order subsystems and demonstrated that symmetry in the interconnection relations may lead to considerable simplifications of the stability analysis. The more general class of symmetric composite systems as considered here were investigated by Lunze (1986). The extension to systems with similar subsystems and non-symmetric interactions was described by Lunze (1989b).

The decentralized control of symmetric composite systems was investigated by Lunze (1989a). The results on the simultaneous stabilization of two plants by the same controller, which have been used in Section 12.3, were due to Vidyasagar and Viswanadham (1982). Under the circumstances given in Section 12.3 their criterion can be applied by means of the parity interlacing property investigated by Youla *et al.* (1974).

The exploitation of symmetry properties in model aggregation was considered by Lunze (1989c).

A further class of systems that consist of identical subsystems was investigated by Abraham and Lunze (1991). With the application to a multizone crystal growth furnace they developed a method for designing the control stations of a decentralized controller as robust controllers of an auxiliary system which includes three subsystems.

13

A Survey of the Results and Open Problems in Feedback Control of Large-scale Systems

Decentralized information processing has the following substantial conceptual advantages over centralized processing:

- The computations are much easier because less information is needed and manipulated.
- Parallel computation is possible without the need for complete synchronization.
- The computation is reliable as far as failures in the computing elements are concerned.

Hence, decentralized schemes can be applied by means of low-cost computing facilities.

However, distributed information processing leads to the fundamental problem of decomposing the overall analytical and design problems adequately. The resulting subproblems have to be relatively autonomous if strong coordination is to be avoided. The advance made in large-scale systems theory has to be assessed in regard to its contribution to the solution of this fundamental problem. The following critical evaluation of what has been achieved and what remains to be done refers to the basic ideas, which have been described in detail in this book, and takes into account the recent results mentioned in the bibliographical notes.

Modelling and Analysis of Interconnected Systems

The outstanding characteristic of large-scale systems is the fact that such systems consist of different interacting subsystems. This is reflected by the models explained in Chapter 3. The first question stated in Section 1.2 asks in which way the overall system performance depends on the properties of the subsystems and the interaction relations. In particular, it has to be investigated whether the subsystems are weakly or strongly coupled and whether the couplings turn out to be cooperative or competitive in nature. Theory has shown that interconnected systems can be classified into three main groups as follows.

Systems that are not strongly coupled in the sense of Definition 3.1

Owing to the absence of coupling links, the subsystems can be grouped so that they interact only in one direction. Analytical and design problems can be simplified considerably by splitting them up into separate problems which concern only the subsystems.

Systems with weakly coupled subsystems

Although the subsystems interact completely and are thus called 'strongly coupled' in the sense of Definition 3.1, the interactions influence the subsystem behaviour weakly. Interactions are weak if the links have low reinforcements or if the timescales of the subsystems are quite different. Analytical and design problems can be decomposed according to the subsystem structure of the overall system, but the resulting subproblems are still dependent upon each other. However, the subsystem interactions are weak enough to be ignored during the solution of the resulting subproblems.

Systems with strongly coupled subsystems

Analytical and design problems cannot be divided along the boundaries of the subsystems. Each subsystem has to be considered within its own environment or some approximation of it. Overlapping decomposition (Section 3.4) or specific methods, which utilize the symmetric structure of the system (Chapter 12) or the static non-interaction of PI controllers (Chapter 11), can be used to derive subproblems with weak interdependence.

Three aspects must be mentioned for the assessment of existing decomposition methods:

1. The methods used for decomposing the overall system utilize structural properties. This is particularly obvious for the graph-theoretic methods described in Section 3.2. However, the results

of Chapters 11 and 12 also utilize particular phenomena which are encountered in the class of systems under consideration and appear quite independently of the parameter values of the system. Hence, the results can be easily applied to systems with high dimension and uncertainties.

2. Whereas the methods for splitting up a given system at given boundaries are known, the question of where to place the boundaries of the subsystems in a practical example has not been satisfactorily answered. It is still a matter of engineering intuition to select those parts of the overall system that represent subsystems with weak interactions, that belong to the slow or fast part of the system, or that have to be used as the overlapping parts. The ultimate success of a given decomposition cannot be seen until the control stations have been designed and it is clear whether the couplings between the subsystems turn out to be weak under decentralized feedback. Therefore, future investigations of reasonable decompositions have to start from the closed-loop system, as has been done in Chapter 11 for PI control systems.

3. It is an open question of how to set up models of the form described in Chapter 3. First, implicit models of the form

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

with singular matrix $\mathbf{E}$ are typical intermediate results of theoretical process analysis. They have to be transformed into the model (3.1.1) only because the analytical and design methods start from this explicit representation. It has to be investigated how this transformation can be carried out while utilizing the sparsity of the matrices occurring in the models or how to start the analysis directly with the implicit model above. Second, almost nothing has been done to establish modelling and identification strategies which avoid setting up a model of the overall system but end up directly with separate models of the subsystems including their specific surroundings. Instead of deriving the models of the subsystems from the overall system description, such new methods should select that part or those properties of the overall system that have to be referred to in the analysis of a certain subsystem or the design of a certain control station. Models that describe these properties can be set up at the subsystem level by the corresponding control authorities. As an example, the model (12.1.11) of the subsystem behaviour of a symmetric composite system under the influence of all subsystems can be set up without prior identification of the overall system (12.1.4). In order to be certain that the model (12.1.11) really does describe the subsystem i under the influence of all other subsystems it has to be proved that the overall system is symmetric. This test can be done without knowing the overall system description (12.1.4).

Centralized Design Methods

Analytical and design methods, whose main ideas are known from multivariable theory, have been satisfactorily extended to decentralized control:

- The effects of the on-line information structure constraints of decentralized control became clear with the existence of decentralized fixed modes. Structural investigations showed why more fixed modes occur for decentralized control than for centralized control (Section 4.2). No separation theorem holds for decentralized observers. The signalling phenomenon must be avoided by explicitly prescribing the linearity of the control laws.
- The design principles of pole assignment and optimal control have been extended to decentralized controllers (Chapters 6 and 7).

These results give satisfactory answers to the second question posed in Section 1.2 on how the overall system behaves under the control of several independent control agents. However, the applicability of the centralized design principles to large systems poses major difficulties because a complete model of the overall system has to be known, most of the manipulations have to be carried out with this high-dimensional model, and the design follows control aims which have to be formulated for the overall system. These design methods are, therefore, applicable only for 'small' systems, where decentralized controllers are to be used and where the dimensionality and uncertainty of the plant does not pose serious difficulties in the analytical and design problems.

Decentralized Design

New problems occur if the information structure used in the off-line procedures of analysis and design is restricted. Several independent control agents are involved although the subproblems, which result from a decomposition of the overall problem, are interdependent. The basic question asks how the conflicts between the solutions of the subproblems can be resolved.

Depending on the aims of the subsystem and the kind of subsystem interactions the subproblems can be competitive or cooperative. In the former case, major conflicts occur and pose the main difficulties for the solution. In the latter case, conflict resolution represents a minor problem, so that no or only a simple coordination is necessary.

Conflict resolution within a decentralized information structure has to be made possible by appropriately organizing the analytical or design process. There are three main ways of constructing analytical and design schemes with decentralized information structure as follows.

Decentralization by direct organizational measures

If the control agents agree to design and implement the control stations one after the other, each design step has a centralized information structure. Since only one control station is considered at a time, the subsystem authority can design the subsystem controller as a centralized feedback. The methods explained in Sections 10.1 and 11.2 illustrate this fact. However, these methods use the hierarchical structure of the plant or consider only the stability requirement. For general problems and structural requirements on the I/O performance of the overall closed-loop system it is still uncertain in the kth design step whether the sequentially designed decentralized controller will eventually satisfy the design specifications stated for the overall system. It is an open question of how to derive design specifications for the kth design step from the given overall system specifications.

Decentralization by ignoring weak interactions

After the plant has been split into weakly coupled subsystems, the design problem is decomposed accordingly and solved while ignoring the subsystem interactions. The control stations obtained from independent design problems are used together as a decentralized controller, and it has to be analysed how the uncertainties that occur due to the neglect of the 'weak' couplings can affect the overall system behaviour. As a basis for this, new criteria for the stability of interconnected systems (Chapter 8), the evaluation of the I/O performance in terms of quadratic performance indices (Section 9.3) or methods for dealing with multi-time-scale systems have been elaborated (Section 10.2). The criteria yield, explicitly or implicitly, quantitative upper bounds on the strength of the couplings for which interactions can be considered weak.

Decentralization by exploitation of structural properties

These methods are based on the structural properties of the system rather than the decomposition into weakly coupled subsystems. 'Structural' properties are considered here in the broadest sense where they refer to all phenomena of the system that do not strongly depend on the system parameters. Results along this line were presented in Chapters 11 and 12, where the effects of integral control or the symmetry of the plant were exploited. The methods are applicable to strongly coupled systems. This approach to decentralized design seems to have great potential, which has not been fully exploited.

These methods have the following characteristics:

- The complexity of the overall analytical and design problems is reduced, where the resulting subproblems are often problems that can be solved by multivariable feedback theory. It is the way of decomposing and dividing the overall problem into these subproblems which is new.
- Although the investigations start from the overall system model, most of the results can be applied without using this complete model but by exploiting structural properties of the whole system, the existence of which can be checked or empirically assumed to be satisfied.

Application Aspects

High dimensionality, large model uncertainties, and information structure constraints concerning *a priori* and measurement data characterize the complexity of large-scale systems, where in dependence upon the practical application at hand one aspect or another dominates. All the methods described in this book take account of these aspects to a different extent and thus have their specific fields of application.

The survey of the large body of analytical and design methods shows that the practically important question of which method should be applied to the practical problem at hand is quite open. Engineering intuition is needed in order to assess modelling and measurement information concerning its importance for the solution of a given problem and to structure the plant and the analytical and design processes accordingly.

In summary, the theory of feedback control of large-scale systems contributes to the application of two distinct but related areas: the control of large, physically distributed dynamical processes, and distributed computing for implementing control algorithms. It provides the control engineer with the methodological background and the analytical and design algorithms that facilitate a rapid solution of complex control problems by using modern computing facilities.

Appendices

APPENDIX 1: MATRICES

This appendix contains the properties of matrices, non-negative matrices, and M-matrices which are used in this book. For detailed information see Gantmacher (1958), Zurmühl (1964), Ortega and Rheinboldt (1970) and Berman and Plemmons (1973).

Eigenvalues

The eigenvalues $\lambda_i[\mathbf{A}]$ $(i = 1, \ldots, n)$ of an (n, n) matrix $\mathbf{A}$ are defined to be real or complex numbers for which vectors $\mathbf{u}_i$ exist such that

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i \qquad (i = 1, \ldots, n) \tag{A1.1}$$

holds; $\mathbf{u}_i$ are also called (right) eigenvectors. There exist other vectors, called left eigenvectors (or reciprocal eigenvectors), $\mathbf{v}_i$ such that

$$\mathbf{v}_i'\mathbf{A} = \lambda_i \mathbf{v}_i' \qquad (i = 1, \ldots, n) \tag{A1.2}$$

holds. The eigenvalues are the solution of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{A1.3}$$

The eigenvectors can be scaled so that the modal matrices

$$\mathbf{U} = (\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \ldots \ \ \mathbf{u}_n) \quad \text{and} \quad \mathbf{V} = (\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \ldots \ \ \mathbf{v}_n) \tag{A1.4}$$

satisfy the relation

$$\mathbf{V}'\mathbf{U} = \mathbf{I}. \tag{A1.5}$$

Minimum and maximum eigenvalues are abbreviated as

$$\lambda_{\min}[\mathbf{A}] = \min_i \ \lambda_i[\mathbf{A}] \qquad \lambda_{\max}[\mathbf{A}] = \max_i \ \lambda_i[\mathbf{A}]$$

respectively, where the eigenvalues are assumed to be real. The sensitivity of the eigenvalue $\lambda_i[\mathbf{A}(p)]$ with respect to some parameter p is described by

$$\left.\frac{d\lambda_i}{dp}\right|_{p=\hat{p}} = \frac{\mathbf{v}_i'(d\mathbf{A}/dp)\mathbf{u}_i}{\mathbf{v}_i'\mathbf{u}_i} \tag{A1.6}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i'$ satisfy eqns (A1.1) and (A1.2) for $\mathbf{A} = \mathbf{A}(\hat{p})$.

The similarity transformation with $\mathbf{U}, \mathbf{V}$ from eqn (A1.4)

$$\mathbf{V}'\mathbf{A}\mathbf{U} = \text{diag } \lambda_i[\mathbf{A}] \tag{A1.7}$$

yields a diagonal matrix if all eigenvalues are distinct. If $\mathbf{A}$ has complex eigenvalues, the matrices $\mathbf{U}$, $\mathbf{V}$ and diag λ_i have complex entries. Then the transformation

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \tilde{\mathbf{A}} \tag{A1.8}$$

is used where for distinct real eigenvalues λ_i the ith column of $\mathbf{T}$ is $\mathbf{u}_i$ and $\tilde{a}_{ii} = \lambda_i$ holds, whereas for a distinct complex conjugate pair $\lambda_{i/i+1} = -\delta_i \pm i\omega_i$ the matrix $\mathbf{T}$ has the columns Re $[\mathbf{u}_i]$ and Im $[\mathbf{u}_i]$ and $\tilde{\mathbf{A}}$ the main diagonal block

$$\begin{pmatrix} -\delta_i & \omega_i \\ -\omega_i & -\delta_i \end{pmatrix}.$$

Then both $\mathbf{T}$ and $\tilde{\mathbf{A}}$ are real-valued matrices.

Non-negative Matrices

A matrix $\mathbf{A} = (a_{ij})$ is called non-negative ($\mathbf{A} \geqslant \mathbf{0}$) or positive ($\mathbf{A} > \mathbf{0}$) if all elements of $\mathbf{A}$ are real and non-negative ($a_{ij} \geqslant 0$) or positive ($a_{ij} > 0$), respectively. An (n, n) matrix $\mathbf{A}$ is called reducible if there exists a permutation matrix $\mathbf{P}$ that transforms $\mathbf{A}$ into the form

$$\mathbf{P}\mathbf{A}\mathbf{P}' = \begin{pmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} \end{pmatrix}$$

where $\bar{\mathbf{A}}_{11}$ and $\bar{\mathbf{A}}_{22}$ are square. Otherwise, $\mathbf{A}$ is called irreducible.

Theorem A1.1 (*Frobenius–Perron theorem*)

Every irreducible non-negative (n, n) matrix $\mathbf{A}$ has a positive eigenvalue $\lambda_p[\mathbf{A}]$ that is a simple root of the characteristic equation (A1.1) and is not exceeded by the moduli of all the other eigenvalues of $\mathbf{A}$

$$\lambda_p[\mathbf{A}] \geqslant |\lambda_i[\mathbf{A}]|. \tag{A1.9}$$

The eigenvector corresponding to $\lambda_p[\mathbf{A}]$ is strictly positive and unique up to a multiplicative constant.

$\lambda_p[\mathbf{A}]$ is called the *Perron root* of $\mathbf{A}$. Theorem A1.1 can be extended to reducible matrices. Then there exist eigenvalues $\lambda_i[\mathbf{A}]$ with $|\lambda_i[\mathbf{A}]| = \lambda_p[\mathbf{A}]$. All these eigenvalues are simple roots of the characteristic equation.

If $\mathbf{A}$ is irreducible then $\lambda_p[\mathbf{A}]$ can be determined by means of the following inclusion. For $\mathbf{x} = (x_1 \ \dots \ x_n)' > 0$ and $\mathbf{y} = (y_1 \ \dots \ y_n)' = \mathbf{Ax}$

$$\min_i \frac{y_i}{x_i} \leqslant \lambda_p[\mathbf{A}] \leqslant \max_i \frac{y_i}{x_i} \tag{A1.10}$$

holds. Used in an iterative way eqn (A1.10) represents the basis of a simple algorithm for determining $\lambda_p[\mathbf{A}]$ as follows.

Algorithm A1.1

Given: Irreducible non-negative matrix $\mathbf{A}$.

1. Let $\mathbf{x} = (1 \ \dots \ 1)'$.
2. Determine $\mathbf{y} = \mathbf{Ax}$ and $l_l = \min_i y_i/x_i$ and $l_u = \max_i y_i/x_i$.
3. If $l_u - l_l < \varepsilon$ for given threshold ε, stop; otherwise, let $\mathbf{x} = \mathbf{y}$ and continue with step 2.

Result: Perron root $\lambda_p[\mathbf{A}] = l_l$.

Theorem A1.2

For two (n, n) matrices $\mathbf{A}$ and $\mathbf{B}$ with $\mathbf{A} \geqslant |\mathbf{B}| \geqslant \mathbf{0}$ the relation

$$\max_i |\lambda_i[\mathbf{B}]| \leqslant \lambda_p[\mathbf{A}] \tag{A1.11}$$

holds.

Therefore, for an arbitrary (n,n) matrix $\mathbf{C}$, which may not be non-negative or positive,

$$\max_i |\lambda_i[\mathbf{C}]| \leqslant \lambda_p[|\mathbf{C}|] \tag{A1.12}$$

holds. For $\mathbf{0} \leqslant \mathbf{B} \leqslant \mathbf{A}$, Theorem A1.2 yields

$$\lambda_p[\mathbf{B}] \leqslant \lambda_p[\mathbf{A}]. \tag{A1.13}$$

M-matrices

Definition A1.1

An (n, n) matrix $\mathbf{P} = (p_{ij})$ is said to be an M-matrix if $p_{ij} \leqslant 0$ holds for all $i \neq j$ and all eigenvalues of $\mathbf{P}$ have positive real parts.

Theorem A1.3

An (n, n) matrix $\mathbf{P} = (p_{ij})$ with $p_{ij} \leqslant 0$ for all $i \neq j$ is an M-matrix if and only if one of the following equivalent conditions hold:

(i) All eigenvalues of $\mathbf{P}$ have positive real parts.
(ii) $\mathbf{P}$ is non-singular and $\mathbf{P}^{-1}$ is non-negative.
(iii) All the leading principal minors of $\mathbf{P}$ are strictly positive

$$\det\begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix} > 0 \qquad (k = 1, \ldots, n). \tag{A1.14}$$

(iv) All the principal minors of $\mathbf{P}$ are strictly positive.
(v) There exists some vector $\mathbf{x}$ such that all elements of the vector $\mathbf{Px}$ are strictly positive.
(vi) There exists some vector $\mathbf{y}'$ such that all elements of the vector $\mathbf{y}'\mathbf{P}$ are strictly positive.

Note that any of the conditions (i)–(vi) is necessary and sufficient for the matrix $\mathbf{P}$ to be an M-matrix.

Theorem A1.4

Assume that $\mathbf{P}$ is an M-matrix. Then the following assertions hold:

(i) If $\mathbf{P}_1 \geqslant \mathbf{P}$ holds and $\mathbf{P}_1$ satisfies the sign condition of Definition A1.1 then $\mathbf{P}_1$ is an M-matrix.
(ii) $\operatorname{diag} d_i\, \mathbf{P}$ and $\mathbf{P} \operatorname{diag} d_i$ with $d_i > 0$ are M-matrices.
(iii) There exists a diagonal matrix $\operatorname{diag} d_i$ with $d_i > 0$ such that $(\mathbf{P}' \operatorname{diag} d_i + \operatorname{diag} d_i \mathbf{P})$ is positive definite.

Hence, if $\mathbf{P}$ is symmetric with $p_{ij} < 0$ for all $i \neq j$ then it is an M-matrix if and only if it is positive definite.

Theorem A1.5

If $\mathbf{A}$ is a non-negative (n, n) matrix then $\mathbf{P} = \mu\mathbf{I} - \mathbf{A}$ is an M-matrix if and only if

$$\mu > \lambda_p[\mathbf{A}]. \tag{A1.15}$$

Theorem A1.6

Assume that $\mathbf{P}$ is an M-matrix. Then all the principal minors of $\mathbf{K} = \mathbf{P}^{-1}$ are positive.

(For a proof see Lunze (1988).) A result which is closely related to non-negative and M-matrices but does *not* refer only to these classes of matrices is stated in the following theorem.

Theorem A1.7

For the class of (n, n) matrices $\mathbf{A}$ the following equivalent conditions hold:

(i) All principal minors of $\mathbf{A}$ are positive.
(ii) Every real eigenvalue of $\mathbf{A}$ as well as of each principal minor of $\mathbf{A}$ is positive.
(iii) For every vector $\mathbf{x} \neq \mathbf{0}$ there exists a diagonal matrix $\mathbf{D} = \text{diag } d_i$ with positive diagonal elements d_i such that $\mathbf{x}'\mathbf{A}\mathbf{D}\mathbf{x} > 0$ holds.

Matrix Equations

The solution $\mathbf{x}$ of the linear equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

with (m, n) matrix $\mathbf{A}$ is described in the following theorem.

Theorem A1.8 (*Penrose theorem*)

A solution to the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ exists if and only if the relation

$$(\mathbf{A}\mathbf{A}^+ - \mathbf{I})\mathbf{b} = \mathbf{0}$$

holds. The solution is given by

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{A}^+\mathbf{A} - \mathbf{I})\mathbf{k} \tag{A1.16}$$

for an arbitrary n vector $\mathbf{k}$. $\mathbf{A}^+$ is the Moore–Penrose pseudoinverse of $\mathbf{A}$, which is given by

$$\begin{aligned} \mathbf{A}^+ &= \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1} \quad \text{for rank } \mathbf{A} = m \\ \mathbf{A}^+ &= (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}' \quad \text{for rank } \mathbf{A} = n. \end{aligned} \tag{A1.17}$$

If rank $\mathbf{A} = n \leqslant m$ holds and the solvability condition is not satisfied it is reasonable to determine such a vector $\mathbf{x}$ for which $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|$ is minimal. This $\mathbf{x}$ is given by

$$\mathbf{x} = \mathbf{A}^+\mathbf{b}.$$

The Lyapunov equation

$$\mathbf{A}'\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{A1.18}$$

with symmetric negative definite (n, n) matrix $\mathbf{Q}$ has a unique positive definite solution $\mathbf{P}$ if and only if all the eigenvalues of the (n, n) matrix $\mathbf{A}$ have negative real parts. Then the solution can be found analytically or by means of the following algorithm (Jamshidi 1983). For other algorithms see Müller (1977).

Algorithm A1.2

Given: (n, n) matrices $\mathbf{A}, \mathbf{Q}$, where $\mathbf{Q}$ is symmetric positive definite.

1. Determine the step size $s = 10^{-4}/(2\|\mathbf{A}\|)$ and the matrices $\mathbf{P}^0 = s\mathbf{Q}$ and

   $$\mathbf{E} = \left(\mathbf{I} - \frac{s}{2}\mathbf{A} + \frac{s^2}{12}\mathbf{A}^2\right)^{-1}\left(\mathbf{I} + \frac{s}{2}\mathbf{A} + \frac{s^2}{12}\mathbf{A}^2\right)^{-1}.$$

   Let $k = 1$.
2. Determinine $l = 2^k$ and

   $$\mathbf{P}^{k+1} = \mathbf{P}^k + (\mathbf{E}')^l\mathbf{P}^k\mathbf{E}^l.$$
3. If $\|\mathbf{P}^{k+1} - \mathbf{P}^k\| < \varepsilon$ holds for a prescribed threshold ε, stop; otherwise, let $k = k + 1$ and proceed with step 2.

Result: Solution $\mathbf{P}^k$ of the Lyapunov equation (A1.18).

Vector and Matrix Norms

Definition A1.2

A function $\|\mathbf{x}\|$: $\mathscr{R}^n \to \mathscr{R}$ is called vector norm if it has the following properties:

(i) $\|\mathbf{x}\| \geqslant 0$ for all $\mathbf{x}$
$\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$ (A1.19)

(ii) $\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$ for all n vectors $\mathbf{x}, \mathbf{y}$ (A1.20)

(iii) $\|\mathbf{a}\ \mathbf{x}\| = |a|\ \|\mathbf{x}\|$ for arbitrary real scalar a and n vector x. (A1.21)

In particular,

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{A1.22}$$

is the Euclidean vector norm of the n vector $\mathbf{x} = (x_1 \ \dots \ x_n)'$. Although nearly all considerations, including norms, hold for arbitrary vector norms, only the Euclidean norm is used throughout this book.

For two n vectors $\mathbf{x}$ and $\mathbf{y}$ the relation

$$|\mathbf{x}'\mathbf{y}| \leqslant \|\mathbf{x}\|\ \|\mathbf{y}\| \tag{A1.23}$$

holds. The equality sign is valid if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

The norm of an (n, n) matrix $\mathbf{A}$ can be defined similarly to Definition A1.2. Here, the spectral norm

$$\|\mathbf{A}\| = \sqrt{\lambda_{\max}[\mathbf{A}'\mathbf{A}]} \tag{A1.24}$$

is considered. For the spectral norm, the relation

$$\|\mathbf{A}\| = \sup_{\|\mathbf{x}\| \neq 0} \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|} \tag{A1.25}$$

holds where on the right-hand side $\|.\|$ denotes the Euclidean vector norm (A1.22) for the n vector $\mathbf{x}$ or the m vector $\mathbf{Ax}$, respectively. The matrix norm $\|\mathbf{A}\|$ is related to the largest eigenvalue by

$$\max_i |\lambda_i[\mathbf{A}]| \leqslant \|\mathbf{A}\|. \tag{A1.26}$$

In particular, eqns (A1.24) and (A1.26) yield for symmetric matrices

$$\|\mathbf{A}\| = \lambda_{\max}[\mathbf{A}]. \tag{A1.27}$$

For rectangular matrices the identity

$$\|\mathbf{A}\| = \|\mathbf{A}'\|$$

holds.

A corollary of eqn (A1.25) is that for $\mathbf{y} = \mathbf{Ax}$ the relation

$$\|\mathbf{y}\| = \|\mathbf{Ax}\| \leqslant \|\mathbf{A}\|\ \|\mathbf{x}\| \tag{A1.28}$$

holds. A symmetric form $\mathbf{x}'\mathbf{Q}\mathbf{x}$ with symmetric matrix $\mathbf{Q}$ can be estimated by

$$\lambda_{\min}[\mathbf{Q}]\,\|\mathbf{x}\|^2 \leqslant \mathbf{x}'\mathbf{Q}\mathbf{x} \leqslant \lambda_{\max}[\mathbf{Q}]\,\|\mathbf{x}\|^2. \tag{A1.29}$$

APPENDIX 2: DIRECTED GRAPHS

A graph $G(\mathscr{V}, \mathscr{E})$ is described by a set $\mathscr{V} = \{v_1, v_2, \ldots\}$ of vertices and a set $\mathscr{E} = \{e_1, e_2, \ldots\}$ of edges. The edges can be represented by their end points as $e_i = (v_k, v_l)$, which means that the edge e_i connects the vertices v_k and v_l and is directed from v_k to v_l. The following considerations concern graphs in which for any pair v_k, v_l their is at most one edge $(v_k\ v_l)$ and one edge $(v_l\ v_k)$.

The (n, n) adjacency matrix $\mathbf{A} = (a_{ij})$ with n being the number of vertices of the graph signifies which vertices of the graph are connected by an edge

$$a_{ij} = \begin{cases} * & \text{if there exists an edge } (v_j\ v_i) \\ 0 & \text{otherwise} \end{cases}$$

$G(\mathscr{V}, \mathscr{E})$ is completely described by the matrix $\mathbf{A}$ (Figure A2.1).

A path is a sequence of edges $\{(v_{i1}\ v_{i2})\ (v_{i2}\ v_{i3})\ \ldots\ v_k\ v_l)\}$ such that the final vertex and the initial vertex of succeeding edges are the same. For example, $\{(v_3\ v_1)\ (v_1\ v_2)\ (v_2\ v_4)\}$ is a path from v_3 to v_4 in the graph of Figure A2.1, but there is no path from v_5 to v_1.

Definition A2.1

Two vertices v_k, v_l are said to be *strongly connected* if there is a path from v_k to v_l as well as a path from v_l to v_k. The graph is called *strongly connected* if every pair of vertices v_k, v_l is strongly connected.

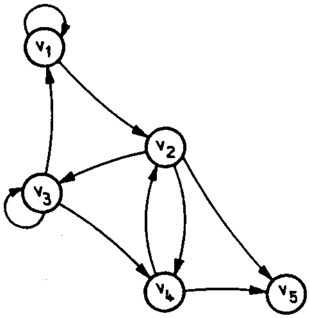
In the graph $G(\mathscr{V}, \mathscr{E})$ the subset of vertices that are strongly connected to a given vertex v_i forms an equivalence class $\mathscr{K}(v_i)$ within the set $\mathscr{V}$, for example $\mathscr{K}(v_1) = \{v_1, v_2, v_3, v_4\} \subset \mathscr{V}$ in Figure A2.1.

The reachability matrix $\mathbf{R} = (r_{ij})$ describes which pairs of vertices are connected by a path

$$r_{ij} = \begin{cases} * & \text{if there exists a path from } v_j \text{ to } v_i \\ 0 & \text{otherwise.} \end{cases}$$

The reachability matrix $\mathbf{R}$ and the adjacency matrix $\mathbf{A}$ are related by

$$\mathbf{R} = \sum_{i=1}^{n-1} \mathbf{A}^i \tag{A2.1}$$



$$A = \begin{pmatrix} * & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & 0 & * & 0 \end{pmatrix}$$

Figure A2.1 Directed graph

where the multiplication and addition of the elements of **A** are carried out according to the rules

$$a_{ij}a_{jk} = \begin{cases} * & \text{if } a_{ij} = * \text{ and } a_{jk} = * \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ij} + a_{kl} = \begin{cases} 0 & \text{if } a_{ij} = 0 \text{ and } a_{kl} = 0 \\ * & \text{otherwise} \end{cases}$$

A cycle is a path with identical initial and final vertices; $\{(v_3\ v_1)\ (v_1\ v_2)\ (v_2\ v_3)\}$ in Figure A2.1 is an example of a cycle. A set of vertex-disjoint cycles is said to be a cycle family. The cycle mentioned above represents a cycle family with only a single cycle, but the graph in Figure A2.1 also has the cycle family consisting of the cycles $\{(v_3\ v_4)\ (v_4\ v_2)\ (v_2\ v_3)\}$ and the self-cycle $\{(v_1\ v_1)\}$.

Graph search algorithms for determining paths, cycles, reachability matrices, etc. can be found in the books by Evan (1979) or Walther and Nägler (1987).

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