

# سایت اختصاصی مهندسی کنترل



<https://controlengineers.ir>

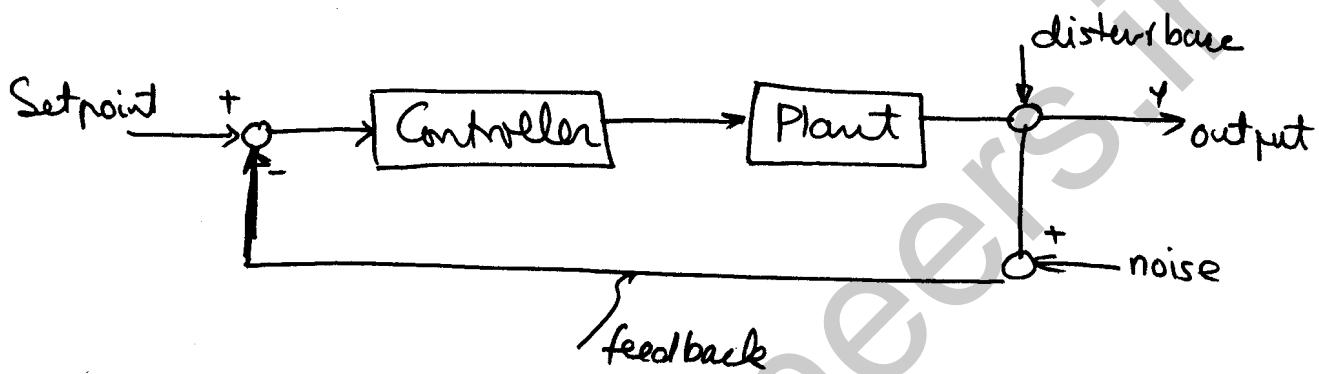


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## 1 Introduction

1-1 Modeling, uncertainty and Robustness  
 Control Engineers, attempt to regulate or track the important outputs of a system, by use of feedback. A schematic block diagram of a control problem is as follows:



In order to design a good Controller, meaning to have the following criteria:

- 1) Stability
- 1) Tracking error small
- 2) disturbance attenuation
- 3) noise rejection
- 4) Insensitivity to plant modeling errors \*

The first Step is to have a mathematical Model of a System. Several disciplines in Control Theory, approaches this point; Modeling from physical Laws, Modeling By Expertise

different Identification problems, ...

However, Always a model of System can only approximately represent the true behaviour of Systems.

We will define the notion of "uncertainty" for plant Models in which, this note will recap "by its best means" the discrepancy between true System behaviour & Model input-output relation. Now the criteria 4) is there to make sure that despite the presence of modeling errors: uncertainty in Model

The behavior of system (criteria 1) 2) & 3) are still satisfactory. Due to the importance of this issue in real implementation of controllers, Robust Control Theory has been introduced to control practitioners.

"Robustness" is exactly the notion defined above, for example robust stability means, that not only the closed loop system designed by the nominal Model is Stable, But also the stability is preserved in presence of modeling uncertainty.

Basically in this Subject we characterize the sys  $\Rightarrow$

1-3

we can perform  
Robust

Analysis

1).

: meaning having a controller known  
analyse the stability or performance of the sys in  
presence of uncertainty

2) Robust Synthesis: Design a controller  $\Rightarrow$  the  
robust stability and/or performance is achieved

Different Methods are developed for robust analysis &  
Synthesis, Depending the way we look at uncertainty  
and the nature of Model ( linear or nonlinear )  $\Rightarrow$

Nonlinear Robust Controllers : sliding Mode

Nonlinear  $H_\infty$  Controller

:

Linear Robust Controllers :  $\rightarrow$  Kharitonov approach  
for parametric uncertainty

We develop

These Methods



$\rightarrow H_\infty$  Controller : for unstructured

$\rightarrow \mu$  Synthesis : for structured &  
unstructured uncertainty

$\rightarrow$  QFT

: for unstructured uncertainty

1-4

let us introduce, for example, The Kharitonov analysis of robust Stability here

by parametric uncertainty we assume the parameters of a system are not accurately obtained, or can vary with time in a domain

$$\underline{a}_i \leq a_i \leq \bar{a}_i \quad (I)$$

$\underline{a}_i$  denotes min of  $a_i$  &  $\bar{a}_i$  is the maximum value  $a_i$  can get.

Now Kharitonov Theorem addresses the stability of characteristic equation

$$P(s, \underline{a}) = a_0 + a_1 s + \dots + a_m s^m \quad II$$

where all coefficients vary to the (I) eq. This is called as "interval polynomial family". Kharitonov has defined four "Kharitonov polynomials":

$$k_1(s) = \underline{a}_0 + \underline{a}_1 s + \bar{a}_2 s^2 + \bar{a}_3 s^3 + \underline{a}_4 s^4 + \underline{a}_5 s^5 + \bar{a}_6 s^6 + \dots$$

$$k_2(s) = \bar{a}_0 + \bar{a}_1 s + \underline{a}_2 s^2 + a_3 s^3 + \dots$$

$$k_3(s) = \bar{a}_0 + \underline{a}_1 s + \bar{a}_2 s^2 + \bar{a}_3 s^3 + \dots$$

$$k_4(s) = \underline{a}_0 + \bar{a}_1 s + \bar{a}_2 s^2 + \underline{a}_3 s^3 + \dots$$

Kharitonov Theorem: An interval Polynomial family  $P$ , with invariant degree ( $a_n = 0$ ) is robustly stable, iff, all four Kharitonov polynomials are stable.

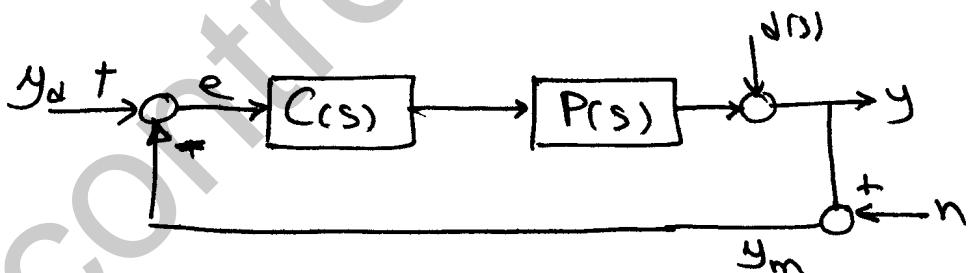
Many analysis & Synthesis methods are developed in this line, look at

What we develop here is the basics of H<sub>∞</sub> (linear) and μ-synthesis in both analytic & numeric approaches. Despite the Modern approach which is required to solve the robustness problems, the look at System and uncertainty in H<sub>∞</sub> approach is rather "Classical". Considering transfer functions, Nyquist, Bode, ...

However, instead of using  $L(s) = P(s) \cdot C(s)$  as the design approach, we rather use, Sensitivity function S(s) and its complementary function T(s) in design; which is best suited to the notion of objective 4) \*

## I-2] Sensitivity Transfer function S(s)

Consider the feedback design as before:



As in classical control the closed loop transfer function for the output  $y$  and the error  $e$  due to inputs  $y_d, d, n$

are :

$$y(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} y_d(s) + \frac{1}{1 + C(s)P(s)} d(s) - \frac{C(s)P(s)}{1 + C(s)P(s)} n(s)$$

$$y_d - y = e(s) = \frac{1}{1 + C(s)P(s)} y_d(s) - \frac{1}{1 + C(s)P(s)} d(s) + \frac{C(s)P(s)}{1 + C(s)P(s)} n(s)$$

Define: Sensitivity function & Complementary S.F

$$S(s) = \frac{1}{1 + C(s)P(s)} \quad \& \quad T(s) = \frac{C(s)P(s)}{1 + C(s)P(s)}$$

where :  $S(s) + T(s) = 1$  (or (I) in multivariable)

Now :

$y(s) = T(s)y_d(s) + S(s)d(s) - T(s)n(s)$	III
$e(s) = S(s)y_d(s) - S(s)d(s) + T(s)n(s)$	

Now, why we call  $S(s)$  Sensitivity? This is because of the important structures of feedback

Let's define : Sensitivity function as the normalized deviation of

The closed loop transfer function  $M(s) = \frac{y(s)}{y_d(s)}$  as a result of

~ normalized deviation of Plant Model  $P(s)$ , in other words

$$S_M^P = \frac{\partial M/M}{\partial P/P} = \frac{\partial M}{\partial P} \cdot \frac{P}{M}$$

For feedback control system

$$M(s) = \frac{y(s)}{y_d(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} \Rightarrow$$

$$S_M^P = \frac{C(s)(1 + P(s)C(s)) - C(s) \cdot P(s)C(s)}{(1 + P(s)C(s))^2} = \frac{P(s)(1 + P(s)C(s))}{P(s)C(s)}$$

$$S_M^D = \frac{C(s)(1+P(s)C(s))}{C(s)(1+P(s)C(s))^2} = \frac{1}{1+P(s)C(s)} = S(s)$$

11-7

The same as defined Sensitivity function, as predicted. Note that This is the magic of feedback structures that, remembering the design criteria:

1) Stability

1) tracking performance      }  $e_{rs} = S(s)y_{rs}(s) - S(s)d_{rs}(s) + T(s)n(s)$

2) disturbance reject

3) noise reject

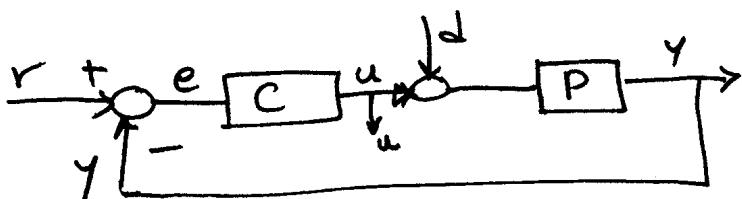
4) insensitivity <sup>to</sup> Model uncertainty

Reducing the tracking error, <sup>(1)</sup> attenuate of disturbance <sup>(2)</sup>, and getting robustness property <sup>(4)</sup> all relates to making  $S(s)$  small in the bandwidth of system.

In other hand we will show that robust stability <sup>(1)</sup> & noise reject <sup>(3)</sup> (as seen in the above transfer function  $e_{rs}/n(s)$ ) is dependent to minimization of  $T(s)$ , which is related to  $S(s)$  by  $T(s) = K(s)I - S(s)$

Hence in our Robust analysis & synthesis approach, we work on optimal minimization of quantities related to  $S(s)$  &  $T(s)$  respectively.

Example : Analyse internal stability of the following system 11-8  
 using  $S \in T$  as design parameters



internal stability means that all internal input-outputs of the system becomes bounded (not only  $y$ ), consider two inputs  $r+d$  & two outputs  $u, y$  which give all internal signals:

$$y(s) = \frac{CP}{1+CP} r(s) + \frac{P}{1+CP} d(s) = T(s)r(s) + P(s)S(s)d(s)$$

$$u(s) = \frac{C}{1+CP} y(s) + \frac{-CP}{1+CP} d(s) = P(s)^{-1}T(s)y(s) - T(s)d(s)$$

$$\rightarrow \begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} T(s) & P(s)S(s) \\ P(s)^{-1}T(s) & -T(s) \end{bmatrix} \begin{bmatrix} r(s) \\ d(s) \end{bmatrix}$$

For internal stability ~~all~~ all 4 T.F's must be stable meaning

$T(s), P(s), P(s)S(s), P(s)^{-1}T(s)$  must be stabilized by choosing appropriate  $C(s)$ .

This is called Interpolate Condition:

if  $P_0$  is an unstable pole of  $P(s)$  with multiplicity  $m$

$$\text{" } z_0 \text{ " zero } \sim \text{ PS stable} \rightarrow \left\{ \begin{array}{l} S(P_0) = \frac{dS(P_0)}{ds} = \dots = \frac{d^{m-1}S}{ds^{m-1}}(P_0) = 0 \\ T(P_0) = 1 + \frac{dT(P_0)}{ds} = \dots = \frac{d^{m-1}T}{ds^{m-1}}(P_0) = 0 \end{array} \right.$$

Interpolate Condition

$$\text{PT stable} \rightarrow \left\{ \begin{array}{l} T(z_0) = \frac{dT}{ds^i}(z_0) = 0 \quad i=1, \dots, m-1 \end{array} \right.$$

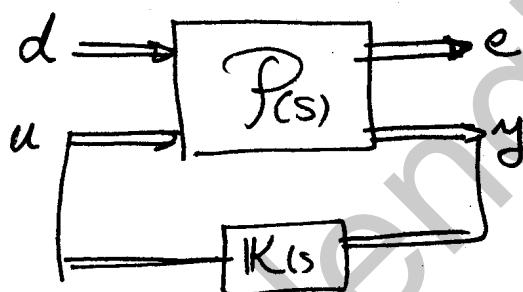
### L3 Outline of General Regulator Problem

11-9

To start with a mathematical framework, designing a controller  
⇒ design objectives 0) to 4) is satisfied, and introduce the notion  
and importance of infinity norm ( $H_\infty$ ), Let us give a general  
representation for these Objectives, (in MIMO Case).

[1-31] Plant with no uncertainty, or uncertainty removed

The general regulator problem states as follows:



in which  $P(s)$  is the generalized plant (MIMO System) whose  
input outputs are as follows:

- inputs {  
     $d(s)$  : exogenous inputs (inputs  $\Rightarrow$  disturbance, setpoint, ... which cannot be controlled directly)  
     $u(s)$  : controlled inputs
- {  
     $e(s)$  : regulated outputs (tracking errors, variables to be small)  
     $y(s)$  : measured outputs (used for controller input)

$K(s)$  is the feedback controller.

The design Objectives can be generally summarized

- 1) Given  $P(s)$ , design  $K(s) \Rightarrow$  the feedback system is internally stable.
- 2) The regulated outputs  $e(s)$  is 'small' for a class of exogenous inputs  $d(s)$

First we need a measure for size of signals, in order to small and large be meaningful. We explain in details norms of signals and systems in chapter two. But from before let's review def. of  $L_2$  norm of a vector valued signal as:

$$\|x\|_2 = \left[ \int_{-\infty}^{+\infty} \sum_{i=1}^n |x_i(t)|^2 dt \right]^{1/2}$$

which for Signal norms, from Parseval's Theorem this can be represent in frequency domain

$$\|x\|_2 = \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{i=1}^n |X(j\omega)|^2 d\omega \right]^{1/2}$$

Hence the Objective 2) can be rewrite, more rigorously as

2) Make  $\|e\|_2$  small.

On the other hand Since the system is linear, the output amplitude of the error and hence the  $\|e\|_2$  of the signal depends on the input amplitude  $d$ , hence it is sufficient to make normalized

output norm small,

to normalize consider the set  $d(s) : \{ d(s) \in \mathbb{L}_2 : \|d(s)\|_2 \leq 1 \}$

Hence the design objective can be rewrite into

Obj: Design  $K(s) \Rightarrow$

1) The system becomes internally stable

2) minimize  $\sup_{\substack{K(s) \\ d \in \mathbb{L}_2, \\ \|d\|_2 \leq 1}} \|e\|_2$

From here the notion of  $\infty$ -norm of system comes into picture  
we will define the transfer matrix  $T_{ed}$  as:

$$e(s) = T_{ed}(s) \cdot d(s)$$

$$\xrightarrow{d(s)} \boxed{T_{ed}} \xrightarrow{e(s)}$$

The induced norm of system  $T_{ed}$  or its  $\infty$ -norm is defined as

$$\|T_{ed}\|_\infty = \sup_{\substack{\text{all bounded} \\ d \in \mathbb{L}_2}} \frac{\|e(s)\|_2}{\|d(s)\|_2} = \sup_{d \in \mathbb{L}_2} \frac{\|T_{ed} \cdot d(s)\|_2}{\|d(s)\|_2}$$

$$\leq \sup_{\substack{d \in \mathbb{L}_2 \\ \|d(s)\|_2 \leq 1}} \|e(s)\|_2$$

Hence the objective is naturally converted to

Find  $K(s) \Rightarrow \min_{K(s)} \|T_{ed}(s)\|_\infty$   
Stabilizing

1-12

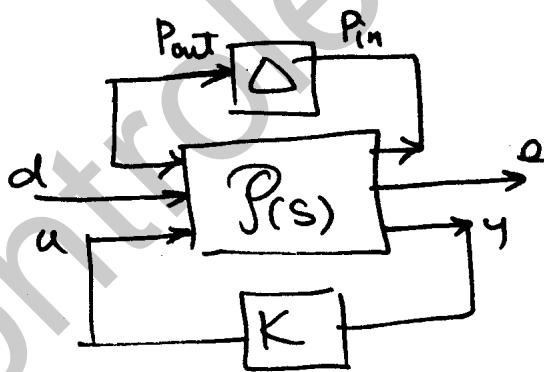
we will give detail definition and methods of determining

$$\|T_{\text{ad}}(s)\|_\infty \text{ for SISO & MIMO Case in Chapter 2.}$$

Hence, Many control problem can be recast into the general regulator problem, which is mathematically represented by an optimization of an  $\infty$ -norm of a transfer matrix, which we call  $H_\infty$  solution.

### 1-3-2/ System with uncertainty

The general regulator problem can be extended to incorporate uncertain or unknown modeling perturbations, as in the following



Several method will be classified in chapter 3 to represent modeling uncertainty or the " $\Delta$ " Block. in all cases we may extract the perturbation block as an extra component outside the generalized plant, in which the rest of the syst obeys the rules of preceding secti. Again we come across the  $\infty$ -norm in robustness analysis.

The most important theorem, which analyses the robust stability of a syst is "Small-gain" theorem, which states as follows

Small-Gain Theorem: For the uncertain system (not necessarily linear) interconnecting as follows

Robust Stability is conserved if  $G(s)$  is stable and strictly proper

By the condition that

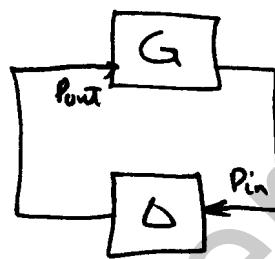
$$\|G\|_\infty \cdot \|\Delta\|_\infty < 1$$

usually  $\Delta$  blocks are also normalized and  $\|\Delta\|_\infty = 1$  hence the robust stability come down to finding  $K(s) \Rightarrow$

$$\|G(s)\|_\infty = \|T_{P_{in} P_{out}}(s)\|_\infty < 1$$

and the stability margin enlarges as the  $\infty$ -norm reduces.

These are the reasons why  $H_\infty$  theory is correlated with robustness analysis & synthesis so closely.



## 2| Norms

### 2-1 Norms for Signals (review)

One way to describe the performance of a system is in terms of the size of certain signals i.e. control effort, tracking error, ...

A Norm of a signal: a function mapping the  $(-\infty, \infty) \rightarrow \mathbb{R}$ ,  $\| \cdot \| : \mathbb{R} \rightarrow \mathbb{R}$  must have these properties

- i)  $\| u \| \geq 0$
- ii)  $\| u \| = 0 \iff u(t) = 0 \quad \forall t$
- iii)  $\| a u \| = |a| \cdot \| u \| \quad \forall a \in \mathbb{R}$
- iv)  $\| u + v \| \leq \| u \| + \| v \|$  Schwartz inequality  
triangle ..

define different type of Norms:

1-Norm: Integral of the Absolute (I<sub>A</sub>)

$$\| u \|_1 = \int_{-\infty}^{+\infty} |u(t)| dt$$

2-Norm: Energy based

$$\| u \|_2 = \left( \int_{-\infty}^{+\infty} u(t)^2 dt \right)^{1/2}$$

$\infty$ -Norm: Least upper bound

$$\| u \|_\infty = \sup_t |u(t)|$$

Note: Define the type of signal: if for a signal its  $\infty$ -norm is finite  $\rightarrow$  then the signal belongs to  $L_\infty$ .

for example:  $u(t) = \text{unit step}$

$$\|u\|_2 = \left[ \int_{-\infty}^{+\infty} u(t)^2 dt \right]^{1/2} = \left[ \int_0^{\infty} dt \right]^{1/2} \rightarrow \infty$$

$\Rightarrow$  unit step does not belong to  $l_2$  space.

define  $(l_{2e})$  is the variant of  $l_2$  space where the tie is not approaching  $\infty \Rightarrow$

unit step  $\in l_{2e}$  space

Power Signals:

The average power of  $u$  can be defined as:

$$\text{average power} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u(t)^2 dt$$

The signal  $u$  is called a power signal if this limit exists.

The square root of average power is called  $\text{pow}(u)$

$$\text{pow}(u) = \left[ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T u^2(t) dt \right]^{1/2}$$

Note: 1)  $\text{pow}(u)$  is not a norm :  $u(t) \neq 0 \Rightarrow \text{pow}(u) = 0$

2) if  $\|u^*(t)\|_2 < \infty \Rightarrow \text{pow}(u) = 0$

3) if  $\|u\|_2 \rightarrow \infty \Rightarrow \text{May be } \text{pow}(u) < \infty$

Example 1): If  $u$  is a power signal  $\Rightarrow \|u\|_{\infty} < \infty \Rightarrow \text{pow}(u) \leq \|u\|_{\infty}$

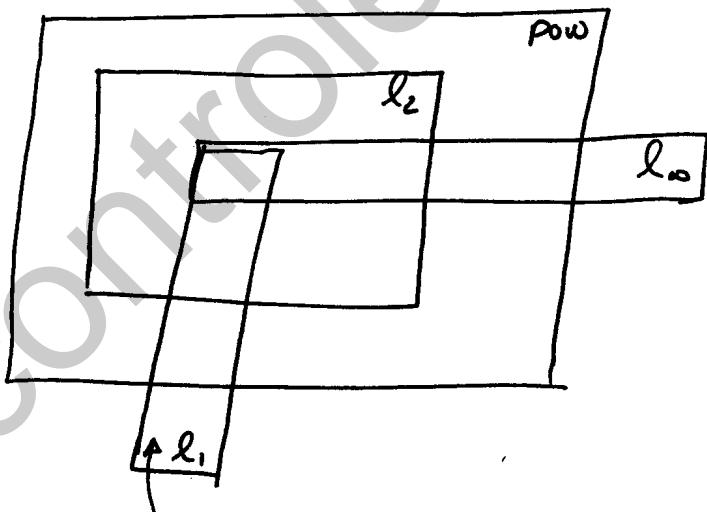
$$\text{pow}(u)^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} u^2(t) dt \leq \|u\|_{\infty}^2 \frac{1}{2T} \int_{-T}^{+T} dt = \|u\|_{\infty}^2$$

Example 2): if  $\|u\|_1 < \infty$   $\Rightarrow \|u\|_{\infty} < \infty \Rightarrow \|u\|_2 \leq (\|u\|_{\infty} \|u\|_1)^{1/2} < \infty$

or if  $u$  is in  $l_1$   $\Rightarrow$   $u$  is in  $l_2$  space as well.

$$\begin{aligned} \|u\|_2^2 &= \int_{-\infty}^{+\infty} u^2(t) dt = \int_{-\infty}^{+\infty} |u(t)| |u(t)| dt \leq \sup |u(t)| \int_{-\infty}^{+\infty} |u(t)| dt \\ &\leq \|u\|_{\infty} \cdot \|u\|_1 \end{aligned}$$

Relation of  $l_2$  spaces:



To illustrate:

$$u(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1/\sqrt{t} & 0 < t \leq 1 \\ 0 & t > 1 \end{cases}$$

$$\|u\|_1 = \int_0^1 \frac{1}{\sqrt{t}} dt = 2$$

$$\|u\|_2^2 = \int_0^1 \frac{1}{t} dt \rightarrow \infty$$

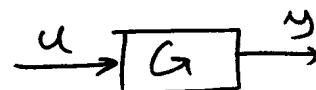
$$\text{pow}(u) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \frac{1}{t} dt \rightarrow \infty$$

## 2-2 Norms for Systems

12-4

Consider an LTI, causal and finite dimensional system  $G(s)$

in the domain



$$y(t) = G(t) * u(t) \quad \text{convolution integral}$$

$$= \int_{-\infty}^{+\infty} G(t-\tau)u(\tau) d\tau$$

in frequency domain

$$Y(s) = G(s) \cdot U(s)$$

To distinguish between impulse response  $G(t)$  & frequency response  $G(s)$  we use  $\hat{G}$  for frequency &  $G$  for time

$\hat{G}$  is stable if it is analytic in CRHP

$\hat{G}$  is proper if  $\hat{G}(j\infty) < \infty$  or degree of denom  $\geq$  degree of num

$\hat{G}$  is strictly proper if  $\hat{G}(j\infty) = 0$  degree  $>$  degree  $\sim$

2-Norm:

$$\|\hat{G}\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2}$$

if  $\hat{G}$  is stable & this norm exists, By Parseval Theorem

$$\|\hat{G}\|_2 = \|G\|_2 \text{ or}$$

$$\left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{G}(j\omega)|^2 d\omega \right)^{1/2} = \left( \int_{-\infty}^{+\infty} |G(t)|^2 dt \right)^{1/2}$$

$\infty$ -Norm

$$\|\hat{G}\|_\infty = \sup_{\omega} |\hat{G}(j\omega)|$$

or the Maximum value of Bode Response in Magnitude.

$\equiv$  furthest point from origin in Nyquist plot of  $\hat{G}$

- Property of  $\infty$  Norm : Multiplication prop.

$$\|\hat{G}\hat{H}\|_\infty \leq \|\hat{G}\|_\infty \cdot \|\hat{H}\|_\infty$$

proof by triangular property

- Lemma 1  $\|\hat{G}\|_2 < \infty \iff \begin{cases} \hat{G} \text{ is strictly proper} \\ \hat{G} \text{ has no poles on imaginary axis} \end{cases}$

$\|\hat{G}\|_\infty < \infty \iff \begin{cases} \hat{G} \text{ is proper} \\ \hat{G} \text{ has no poles on imaginary axis} \end{cases}$

- Computation By Residue Theorem  $\|\hat{G}\|_2$

$$\|\hat{G}\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{G}(j\omega)|^2 d\omega = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \hat{G}(-s) \hat{G}(s) ds$$

$$= \frac{1}{2\pi j} \oint \hat{G}(-s) G(s) ds = \sum_{\text{in the left half plane}} \text{Res} (G(s))$$

Ex:  $\hat{G}(s) = \frac{1}{1 + \tau s} \quad \tau > 0$

12-6

$$\hat{G}(-s) \cdot \hat{G}(s) = \frac{1}{-\tau s + 1} \cdot \frac{1}{\tau s + 1}$$

$$\|\hat{G}\|_2^2 = \sum_{\substack{\text{Res} \\ \text{L.H.P}}} \text{Residues} = \text{Res}(s = -1/\tau) = \lim_{s \rightarrow -1/\tau} \frac{1}{-\tau s + 1}$$

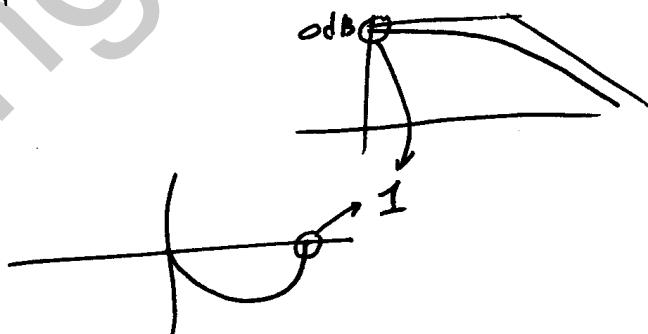
$$= \frac{1}{2\tau}$$

$$\Rightarrow \|\hat{G}\| \Leftarrow \frac{1}{\sqrt{2\tau}}$$

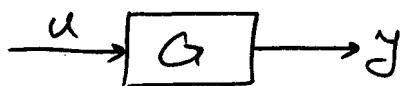
$$\|\hat{G}\|_\infty = \max_{\omega} \left| \frac{1}{\tau(j\omega) + 1} \right| = \lim_{\omega \rightarrow \infty} \left| \frac{1}{\tau(j\omega) + 1} \right| = 1$$

or from Bode

or from Nyquist



## 2-3] Relation between Signal Norms & System norms



What is the relation between  $\| \text{input} \| + \| \text{output} \|$  to  $\| \text{sys} \|$

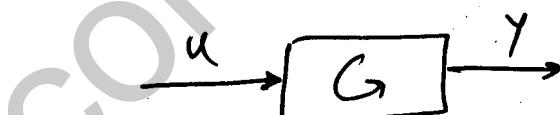
first consider particular inputs  $u(t) = \delta(t)$  OR  $u(t) = \sin \omega t$   
 Assume  $G(s)$  is stable & Strictly proper:

	$u(t) = \delta(t)$	$u(t) = \sin \omega t$	
$\  y \ _2$	$\  \hat{G} \ _2$	$\infty$	→ output in sinusoid $\notin l_2$
$\  y \ _\infty$	$\  \hat{G} \ _\infty$	$ \hat{G}(j\omega_0) $	

$\| \hat{G} \|_\infty \rightarrow \| \delta \|_2 \rightarrow \infty$   
 $\int f(t-\epsilon) \cdot f(\epsilon) d\epsilon = 1$

Now for a general input  $\in l_2$  or  $l_\infty$  we can define:

System Gain Or Induced Norm



$$\text{Syst Gain} = \sup_{u \neq 0} \frac{\| y \|_2}{\| u \|_2} = \text{Induced Norm}$$

$$= \sup_{\| u \| \leq 1} \| y \|_2$$

The magnitude of input  
does not change the syst gain

$$= \sup \| y \|_2$$

Hence,

$$\begin{aligned} \sup_{\|u\|=1} \|y\|_2 &= \sup_{\|u\|=1} \|\hat{G} \cdot u\|_2 \leq \sup \|\hat{G}\| \cdot \|u\|_2 \\ &\leq \|\hat{G}\|_\infty \cdot 1 \end{aligned}$$

OR Induced Norm of a Syst is  $\|\hat{G}\|_\infty$

$\infty$ -Norm of a syst has this physical meaning of Maxim System Gain  
and the facility of being mathematically used by 2-norm of signals

Generally we can show:

	$\ u\ _2$	$\ u\ _\infty$
$\ y\ _2$	$\ \hat{G}\ _\infty$	$\infty$
$\ y\ _\infty$	$\ \hat{G}\ _2$	$\ \hat{G}\ _1$

Proof (optional)

Entry (1,1) :

$$\begin{aligned} \|y\|_2^2 &= \|\hat{y}\|_2^2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{G}(j\omega)|^2 |\hat{u}(j\omega)|^2 d\omega \\ &\leq \|\hat{G}\|_\infty^2 \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{u}(j\omega)|^2 d\omega \\ &\leq \|\hat{G}\|_\infty^2 \|\hat{u}\|_2^2 \\ \Rightarrow \|y\|_2 &\leq \|\hat{G}\|_\infty \cdot \|\hat{u}\|_2 \end{aligned}$$

To show  $\|\hat{G}\|_\infty$  is the least upper bound,

choose  $\omega_*$  where  $|\hat{G}(j\omega_*)|$  is supremum  $|\hat{G}(j\omega_*)| = \|\hat{G}\|_\infty$

Now close to this frequency choose  $u$  to be

$$|\hat{u}(j\omega)| = \begin{cases} c & \text{if } |\omega - \omega_*| < \epsilon \text{ or } |\omega + \omega_*| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

where  $\epsilon$  is a small positive No.  $c$  is chosen such that  $\|u\|_2 = 1$

or  $c = \sqrt{\pi}/2\epsilon$  Then

$$\begin{aligned} \|u\|_2^2 &= \frac{1}{2\pi} \left[ |\hat{G}(-j\omega_*)|^2 \pi + |\hat{G}(j\omega_*)|^2 \pi \right] \\ &= |\hat{G}(j\omega_*)|^2 = \|\hat{G}\|_\infty^2 \end{aligned}$$

Entry (2,1)

$$|y(t)| = \left| \int_{-\infty}^{+\infty} G(t-u) u(c) du \right|$$

$$\leq \left( \int_{-\infty}^{+\infty} G(t-c)^2 dc \right)^{1/2} \left( \int_{-\infty}^{+\infty} |u(c)|^2 dc \right)^{1/2}$$

$$= \|G\|_2 \|u\|_2$$

$$= \|\hat{G}\|_2 \|u\|_2$$

$$\Rightarrow \|y\|_\infty \leq \|\hat{G}\|_2 \cdot \|u\|_2 \quad I$$

To show that  $\|\hat{G}\|_\infty$  is the least upper bound, apply the input

$$u(t) = G(-t)/\|G\|_2 \quad \text{Then } \|u\|_2 = 1$$

$$\text{and } |y(0)| = \|G\|_2 \quad \text{So } \|y\|_\infty \geq \|G\|_2 \quad II$$

## 2.4 - Computing $2 \rightarrow \infty$ Norms in State-Space

Consider SISO Case with S.Space Model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where  $G(s) = C(sI - A)^{-1}B$  can be calculated

To have finite norms, assume stable plant ( $\forall \lambda_i, \operatorname{Re}(\lambda_i) < 0$ )

define  $L := \int_0^\infty e^{At} B B' e^{A't} dt$  "Controllability Grammian"

a simple formula for  $\|\hat{G}\|_2 = (CLC')^{1/2}$

$$\begin{aligned} \text{for } \|\hat{G}\|_2^2 &= \|G\|_2^2 = \int_0^\infty (e^{At} B B' e^{A't}) C' dt \\ &= C \int_0^\infty e^{At} B B' e^{A't} dt \cdot C' \\ &= (CLC')' \end{aligned}$$

property of L :

$$\begin{aligned} \text{For } \frac{d}{dt} L &= \frac{d}{dt} \int_0^\infty (e^{At} B B' e^{A't}) dt = \int_0^\infty \frac{d}{dt} (e^{At} B B' e^{A't}) dt \\ &= \int_0^\infty (A e^{At} B B' e^{A't} + e^{At} B B' e^{A't} A') dt \end{aligned}$$

$$[e^{At} B B' e^{A't}]_0^\infty = AL + LA' \quad \text{since for stable } A \frac{e^{At}}{t \rightarrow \infty} \rightarrow 0$$

$$L e^{At} = I$$

Summary of procedure:

- ① Solve the Eq  $AL + LA' + BB' = 0$  for L
- ② Find the Norm  $\|\hat{G}\|_2 = ((LC')^{1/2})$

$\infty$ -Norm Calculation: Use the following theorem

Theorem:  $\|\hat{G}\|_\infty < 1$  iff  $H := \begin{bmatrix} A & BB' \\ -C'C & -A' \end{bmatrix}$  has no eigenvalues on the imaginary axis.

from this Theorem select a positive No.  $\gamma^* \geq \|\hat{g}^* G\|_\infty < 1$  or  $\|G\|_\infty < \gamma^*$  check the result in Theorem, if holds decrease  $\gamma^*$  and follow a

Search routine by bisection or any other means.

Example 1)  $G(s) = \frac{1}{s^2 + s + 1} \rightarrow \begin{array}{l} \dot{x} = Ax + bu \\ y = cx \end{array}$

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}; b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; c = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

use Matlab functions

$$\text{sys} = \text{tf}([1], [1, 1, 1])$$

$$\text{norm}(\text{sys}, 2) = \sqrt{2}/2$$

$$\text{norm}(\text{sys}, \infty) = 1.1547$$

$$S_{\text{sys}} = \text{ss}(a, b, c, d)$$

$\mu$ -Synthesis

$$\text{h2norm}(\text{sys}) = \sqrt{2}/2$$

$$\text{hinfnorm}(\text{sys}, \text{tol}) = 1.1547$$

for discrete  
→

$$\text{dhfnorm}(\text{sys})$$

To do the calculate by hand:  $\|G\|_2$

① Solve for Lyapunov factor

$$AL + LA' + BB' = 0$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{ORL} = \text{Lyap}(A, BB') = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\|\hat{G}\|_2 = \left[ [0 \ 1] \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

② For  $\|G\|_\infty$

Solve for

$$f = g_1$$

$$cc = c/c$$

$$h = \begin{bmatrix} A & BB' \\ -c'c & -A' \end{bmatrix}$$

$$\text{eig } h$$

$$g=10 \rightarrow h = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -0.01 & 1 & 0 \end{bmatrix}$$

$$\text{eig}(h) = \begin{cases} -0.4975 \pm j 0.8646 \\ 0.4975 \pm j 0.8646 \end{cases}$$

$$g=2 \rightarrow \text{eig}(h) = \begin{cases} -0.4278 \pm j 0.8267 \\ 0.4278 \pm j 0.8267 \end{cases}$$

$$g=1 \rightarrow \text{eig}(h) = 0, 0, 1, \pm j$$

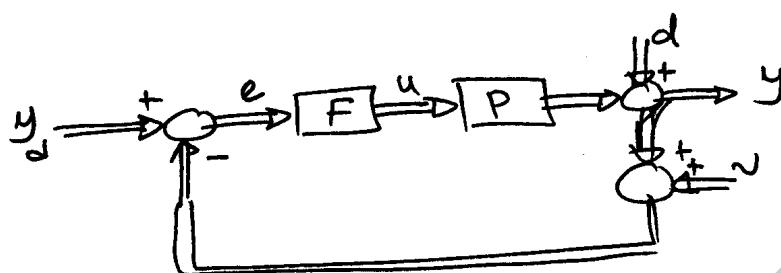
$$f=1.15 \rightarrow \sigma = \pm j 0.3812 \\ \pm j 0.473$$

$$g=1.15$$

$$\nu = \pm j 0.7072$$

## 2-5 | Multivariable Norms.

The purpose of this section is to present a few basic expressions in MIMO LTI system theory. Note the basic attention should be made for MIMO Systems in figure:



where  $F$  &  $P$  are transfer matrices &  $y_d, u, d, y, v$  are vector-valued inputs,  $[P]_{n \times r}$   $[F]_{r \times m}$   $n = \text{dim}(y)$ ,  $r = \text{dim}(u)$

"The division  $\rightarrow$  inversion  $\neq$  the order of multiplication matters"

$$\begin{aligned} y &= d + Pu \\ &= d + PF(y_d - y - v) \\ \Rightarrow (I + PF)y &= d + PF(y_d - v) \\ y &= (I + PF)^{-1}d + (I + PF)^{-1}PF(y_d - v) \end{aligned}$$

We may define Sensitivity Matrix & Complementary Sensitivity

as:

$$S = (I + PF)^{-1} \quad \text{where } I - S = T$$

$$T = (I + PF)^{-1}PF$$

very similar To SISO case

Hence  $y = Ty_d + Sd - Tv$

## 2-5-1] Singular Values

The  $\ell_2$  norm of a complex vector  $w$  is defined as

$$\|w\|_2 = (\sum |w_i|^2)^{1/2} = (w^* w)^{1/2}$$

where "\*" denotes conjugate transpose. This is  $\ell_2$  or Euclidean norm. To make the definition systematically related to the SVD norms, let's consider  $w$  the part of the output  $y$  that is due to disturbance  $d$

$$\|w\|^2 = (Sd)^*(Sd) = d^* S^* S d$$

Suppose  $\|d(jw)\|$  is given. Geometrically, we may think of  $d$  as a vector of given length, whose direction is unknown. For given  $\|d\|$ , the right-hand side (RHS)  $\|w\|$  has maximum & minimum values, given by

$$\text{Max } \|w\| = \sqrt{\lambda_{\max}(S^* S)} \|d\|$$

$$\text{min } \|w\| = \sqrt{\lambda_{\min}(S^* S)} \|d\|.$$

The matrix  $S^* S$  is Hermitian ( $=$  own conjugate transpose), and the eigenvalues of such a matrix are real and positive. The quantities  $\lambda_{\max}$  &  $\lambda_{\min}$  are the largest and smallest eigenvalues of  $S^* S$ .

The singular values of  $S$  are the square roots of the eigenvalues of  $S^* S$

To compute the singular value of any matrix  $S$  numerically,  $S$  can be written as

$$S = U \Sigma V^* \quad (\text{Svd in Matlab})$$

where  $U$  is an  $p \times n$  unitary matrix &  $\Sigma$  is an  $n \times n$  diagonal matrix whose

(note unitary matrix  $UU^* = I$ )

The largest and smallest singular values of  $S$  are denoted by

$\bar{\delta}(S)$  &  $\underline{\delta}(S)$  respectively, thus.

$$\bar{\delta}(S) = \sqrt{\lambda_{\max}(S^*S)}$$

$$\underline{\delta}(S) = \sqrt{\lambda_{\min}(S^*S)}$$

and the  $l_2$  norm of a vector valued signal  $w$  has the property:

$$\underline{\delta}(S) \|d\|_2 \leq \|w\|_2 \leq \bar{\delta}(S) \|d\|_2$$

The properties of Singular values are as following:

P I : if  $S^{-1}$  exists then

$$\underline{\delta}(S) = \frac{1}{\bar{\delta}(S^{-1})} + \bar{\delta}(S) = \frac{1}{\underline{\delta}(S^{-1})}$$

P II :  $\det S = \delta_1 \cdot \delta_2 \dots \delta_n$

P III :  $\bar{\delta}(A+B) \leq \bar{\delta}(A) + \bar{\delta}(B)$

P IV :  $\bar{\delta}(AB) \leq \bar{\delta}(A) \cdot \bar{\delta}(B)$

P V :  $\max [\bar{\delta}(A), \bar{\delta}(B)] \leq \bar{\delta}[A \ B] \leq \sqrt{2} \max [\bar{\delta}(A), \bar{\delta}(B)]$

→ Note that  $\bar{\delta}(S)$  describes the worst-case situation for a given  $\|d\|_2$ , as the effect of the output amplifiers. Here  $\bar{\delta}[S(j\omega)]$  is the key scalar quantity describing the maximum amplification of  $S(j\omega)$

and plays the role that was given to  $|S(j\omega)|$  in the SISO case.

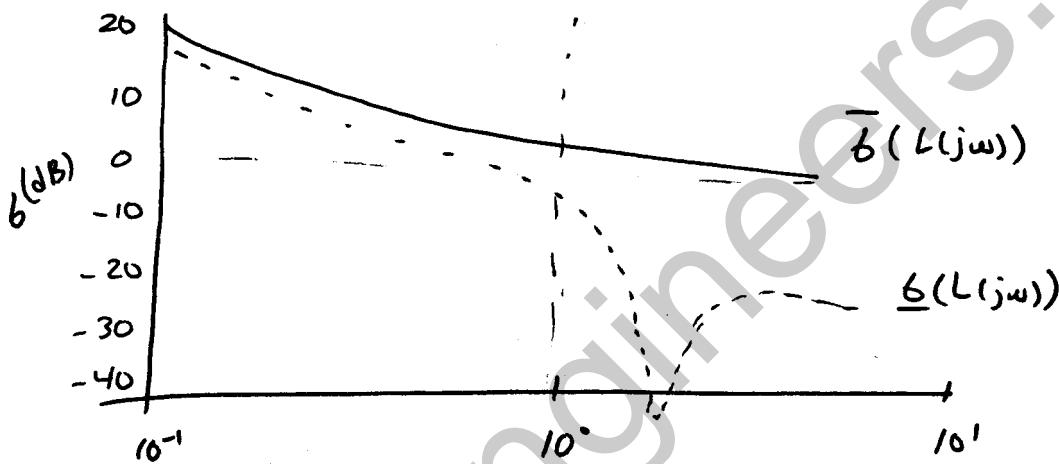
Example: For a 2input-2output System loop gain

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$$L(s) = P(s) \cdot F(s) = \begin{bmatrix} 1/s & -\frac{0.5}{s+1} \\ 1 & \frac{1}{s(s+1)} \end{bmatrix}$$

a) Compute and display  $\bar{b}(L(j\omega))$ ,  $\underline{b}(L(j\omega))$

→ use Sigma(sys) in matlab

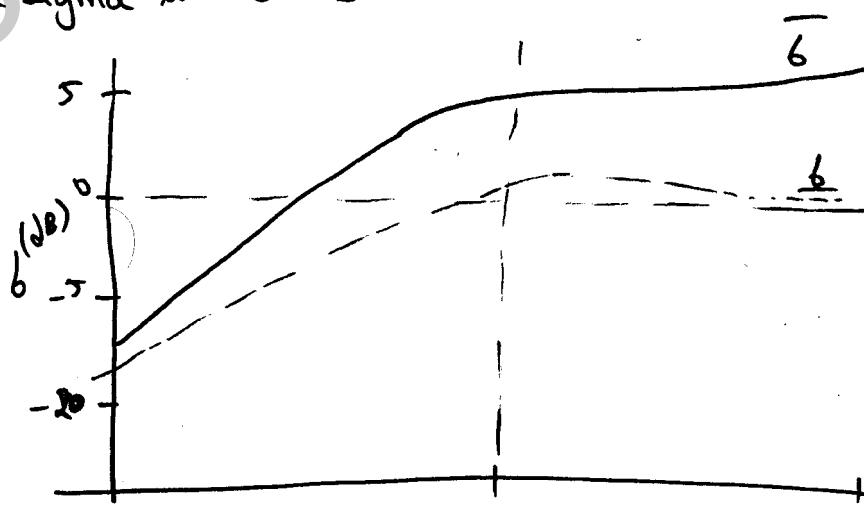


b) Calculate  $S(s)$  &  $T(s)$

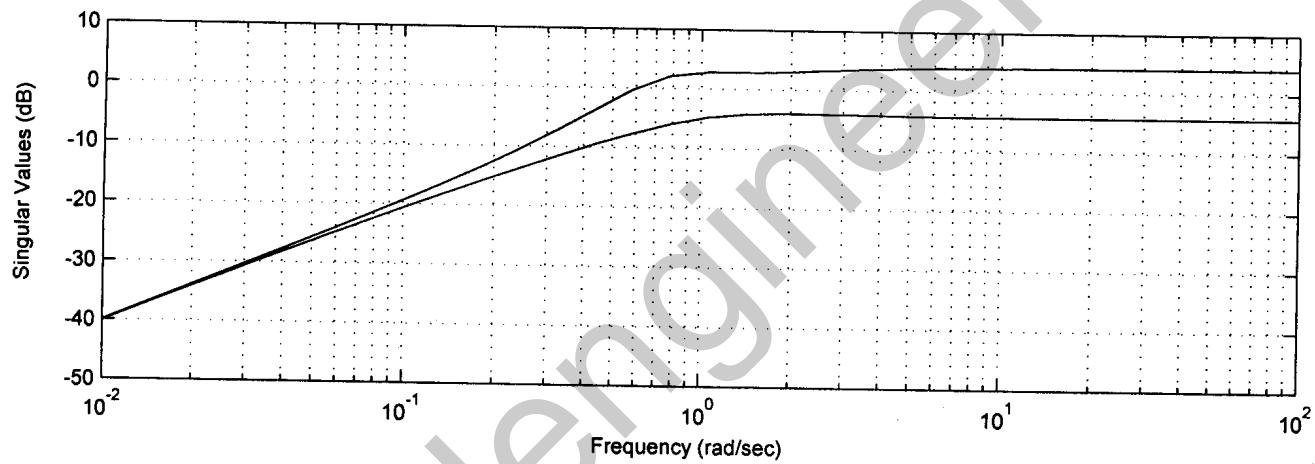
$$S(s) = [I + L(s)]^{-1} = \det[I - L(s)]^{-1} \cdot \begin{bmatrix} \frac{s^2 + s + 1}{s(s+1)} & \frac{0.5}{s+1} \\ -1 & \frac{s+1}{s} \end{bmatrix}$$

$$= \frac{s^2(s+1)}{s^3 + 2.5s^2 + 2s + 1} \begin{bmatrix} " \end{bmatrix} = \frac{1}{s^3 + 2.5s^2 + 2s + 1} \begin{bmatrix} s(s^2 + s + 1) & 0.5s^2 \\ -s^2(s+1) & s(s+1)^2 \end{bmatrix}$$

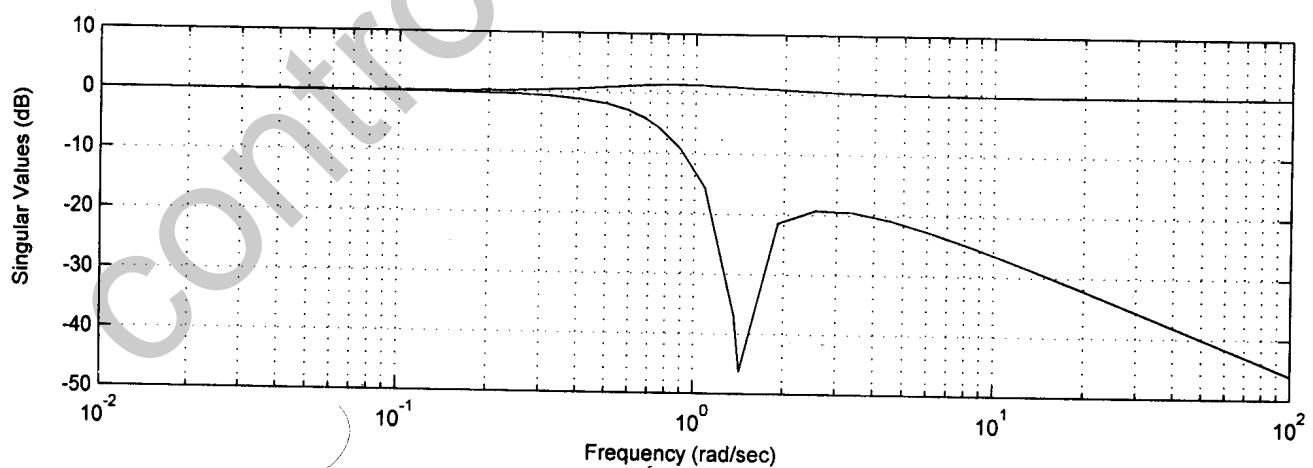
use Sigma in Matlab

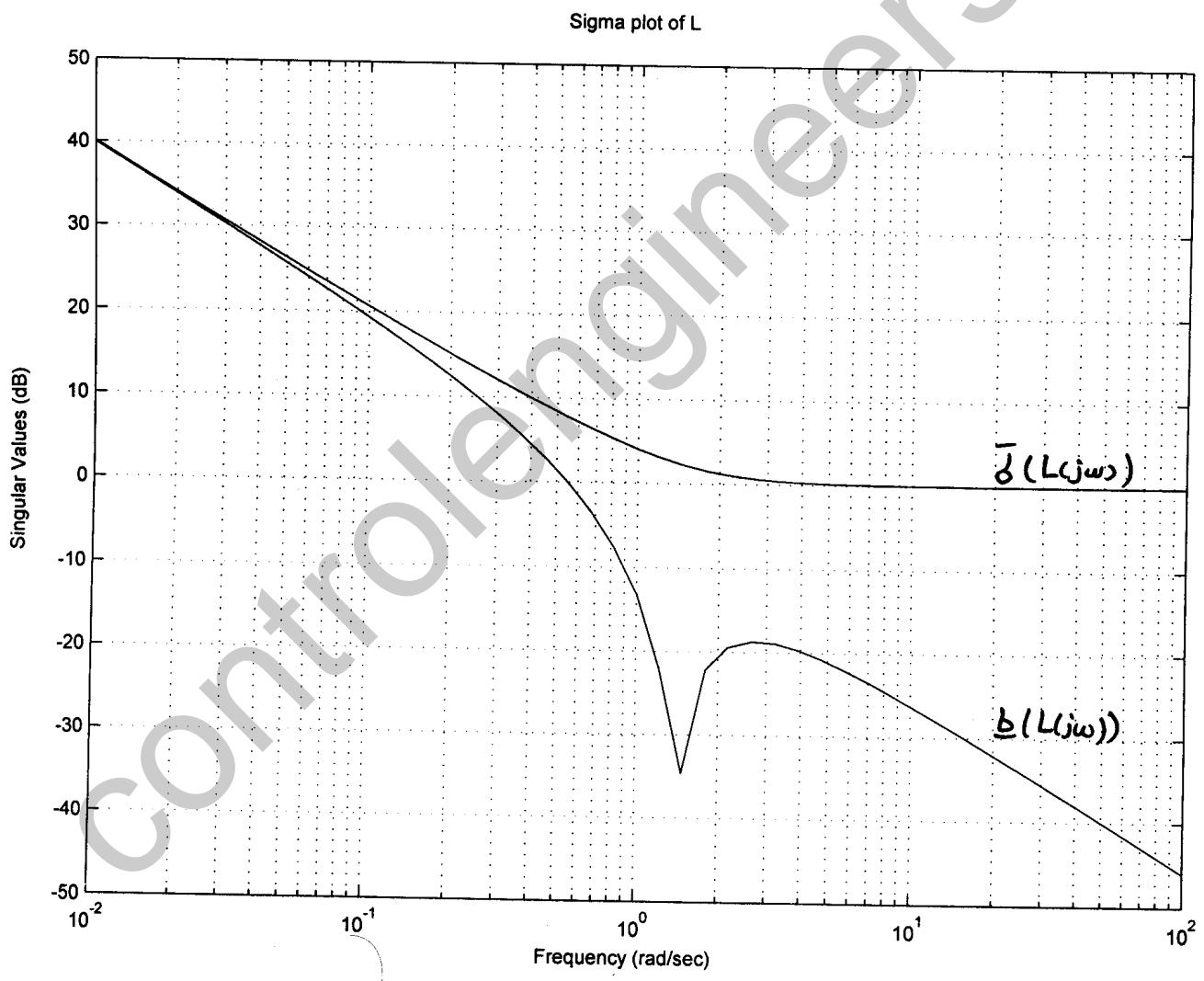


Sigma plot of S



Sigma plot of T





$$\text{Finally, } T(s) = I - S(s) = \frac{1}{s} \begin{bmatrix} 1.5s^2 + s + 1 & -0.5s^2 \\ s^2(s+1) & 0.5s^2 + s + 1 \end{bmatrix}$$

Note as in SISO case  $\overline{\delta}(S(j\omega))$  is small where  $\underline{\delta}(L(j\omega)) \gg 1$   
and is near 1 where  $\overline{\delta}(L(j\omega)) \leq 1$

### 2-5-2 different Norms definitions

The Euclidean, or  $l_2$ , norm of a vector  $x$  is defined as

$$\|x\|_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} = (x^T x)^{1/2}$$

for a vector-valued signal

$$\|x\|_2 = \left[ \int_{-\infty}^{+\infty} \underbrace{x^T(t)x(t)}_{\sim} dt \right]^{1/2} \quad \text{as before}$$

This norm is the square root of the sum of the energy in each component of the vector.

For an  $m \times r$  matrix, we can define "The Frobenious" Norm,

$$\|A\|_2 = \left( \sum_{i=1}^m \sum_{j=1}^r |A_{ij}|^2 \right)^{1/2} \quad \text{Similarly,}$$

It can be shown that

$$\|A\|_2 = \text{tr}(A^T A) = \text{tr}(AA^T)$$

$\text{tr} = \text{trace}$   
 $= \text{the sum of the diagonal elements.}$

a MIMO System is a generalization of a matrix, where ~~is~~ a matrix

where  $\mathbf{G}$  is a MIMO System is acting on a vector valued signal  $\underline{12-18}$

$\mathbf{u}(t)$  to produce another vector-valued signal. By analogy to

Frobenius norm, we define the  $L_2$  norm for an  $m \times r$  transformation

$$\|\mathbf{G}\|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{tr}(\mathbf{G}^T(j\omega) \mathbf{G}(j\omega)) d\omega \right)^{1/2}$$

As before,  $\|\mathbf{G}\|_2$  exists, iff, each elements of  $\mathbf{G}(s)$  is strictly proper and has no poles on the imaginary axis, hence  $\mathbf{G} \in L_2$

under some condition this can be evaluated as an integral in the

Complex plane  $\|\mathbf{G}\|_2^2 = \frac{1}{2\pi j} \int_{-\infty}^{j\infty} \text{tr}[\mathbf{G}^T(-s) \mathbf{G}(s)] ds$   
 $= \frac{1}{2\pi j} \oint \text{tr}[\mathbf{G}^T(-s) \mathbf{G}(s)] ds$

where the Residue Theorem can be applied

Ex: Calculate  $L_2$  norm of  $\mathbf{G}(s) = \frac{1}{s^3 + 3s^2 + 2} \begin{bmatrix} s+3 & -(s+2) \\ -2 & (s+2) \end{bmatrix}$

Solution: We compute

$$\text{tr } \mathbf{G}^T(-s) \mathbf{G}(s) = \dots = \frac{-3s^2 + 21}{(s+1)(s+2)(-s+1)(-s+2)}$$

Use Residue theorem about a contour enclosing the LHP

$$\|\mathbf{G}\|_2^2 = \frac{-3(-1)^2 + 21}{(-1+2)(-1+1)(+1+2)} + \frac{-3(-2)^2 + 21}{(-2+1)(+2+1)(+2+2)}$$

As to the induced Norm, for matrix, the induced Euclidean norm is

$$\|A\|_{2i} = \max_{\|d\|=1} \|Ad\|_2$$

$$= \bar{\sigma}(A) \quad \text{As defined before.}$$

To generalize for a MIMO LTI System, Consider first a stable, strictly proper SISO syst

$$\begin{aligned}\|y\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(j\omega)|^2 |u(j\omega)|^2 d\omega \\ &\leq \sup_{\omega} |G(j\omega)|^2 \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} |u(j\omega)|^2 d\omega \\ &\leq \sup_{\omega} |G(j\omega)|^2 \cdot \|u\|_2^2\end{aligned}$$

Hence

$$\sup_{\|u\|_2=1} \|y\|_2 = \sup_{\omega} |G(j\omega)| = \|G\|_\infty$$

which was defined as the  $\infty$  norm of  $G$ , and exists iff, no poles on  $j\text{-axis}$ .

If this norm exists then  $G \in \mathcal{L}_\infty$ . If in addition  $G$  is stable then  $G \in H_\infty$

Now for Multivariable Systems

$$\|y\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|G(j\omega) u(j\omega)\|_2^2 d\omega$$

$$= \left( \int_{-\infty}^{+\infty} \|G(j\omega) u(j\omega)\|^2 d\omega \right)^{1/2}$$

$$\|y\|_2^2 \leq \sup_{\omega} [\bar{\sigma}(G(j\omega))]^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \|u(\omega)\|^2 d\omega$$

hence

$$\Rightarrow \|y\|_2^2 \leq \left[ \sup_{\omega} \bar{\sigma}[G(j\omega)] \right]^2 \|u\|_2^2$$

Hence

$$\sup_{\substack{\|u\|_2=1}} \|y\|_2 = \sup_{\omega} \bar{\sigma}[G(j\omega)] = \|G\|_\infty$$

The induced Norm or  $\|G\|_\infty$  of  $G$  is the supremum of the largest singular value of  $G(j\omega)$  w.r.t  $\omega$ .

## 1) Hilbert Space

Recall the inner product of vectors defined on Euclidean space  $\mathbb{R}^n$

$$\langle x, y \rangle := x^* y = \sum_{i=1}^n \bar{x}_i y_i \quad x, y \in \mathbb{C}^n$$

many metric notions and geometrical properties can be deduced from this inner product  $\Rightarrow$

$$\|x\| := \sqrt{\langle x, x \rangle} \quad \text{length} \quad \cos \angle(x, y) = \frac{\langle x, y \rangle}{\|x\| \|y\|}; \quad \angle(x, y) \in [0, \pi]$$

orthogonality if  $\angle(x, y) = \frac{\pi}{2}$  or  $\langle x, y \rangle = 0$

now let us generalize the inner product for more general vector spaces

Def 1: let  $V$  be a vector space over  $\mathbb{C}$ . An inner product

on  $V$  is a complex-valued function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$

such that  $\forall x, y, z \in V \in \alpha, \beta \in \mathbb{C}$

$$(i) \quad \langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$$

$$(ii) \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

$$(iii) \quad \langle x, x \rangle > 0 \text{ if } x \neq 0$$

A vector space  $V$  with an inner product is called an inner product

space, and it induces a norm  $\|x\| := \sqrt{\langle x, x \rangle}$  and a distance

between the vectors  $d(x, y) = \|x - y\|$

"... and if two vectors in a vector space  $V$  are orthogonal"

Theorem: Let  $V$  be an inner product space and let  $x, y \in V$ . Then

$$i) |\langle x, y \rangle| \leq \|x\| \|y\| \quad (\text{Cauchy-Schwarz inequality})$$

moreover equality holds iff,  $x = \alpha y$  for some const  $\alpha$  or  $y = 0$

$$\text{ii) } \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad (\text{Parallelogram Law})$$

$$iii) \|x+y\|^2 = \|x\|^2 + \|y\|^2 \text{ if } x \perp y$$

Def: A Hilbert Space is a complete inner product space with the norm induced by its inner product.

Ex 1:  $\mathbb{C}^n$  with usual inner product  $\langle x, y \rangle = x^* y$ ,  $\|x\| = \sqrt{\langle x, x \rangle}$

Ex 2:  $\mathbb{C}^{n \times m}$  with  $\langle A, B \rangle = \text{trace } A^* B$  if  $A, B \in \mathbb{C}^{n \times m}$

$\mathcal{L}_2[a, b]$ : All square integrable and Lebesgue measurable functions defined on an interval  $[a, b]$  with  $\langle f, g \rangle = \int_a^b f(t)^* g(t) dt$

Ex 4: for vector valued functions  $\langle f, g \rangle = \int_a^b \text{trace} [f(t)^* g(t)] dt$

Some common intervals are

$$\mathcal{L}_2 = \mathcal{L}_2(-\infty, \infty) \Rightarrow \langle f, g \rangle := \int_{-\infty}^{+\infty} \text{trace}[f^*(t) g(t)] dt$$

$L_{2+} = L_2(0, \infty)$ : subspace of  $L_2$  with functions zero for  $t < 0$

$$\mathcal{L}_2^- = \mathcal{L}_2(-\infty, 0) : \quad t > 0$$

## 2) $\ell_2$ & $\ell_\infty$ Spaces

Let  $S \subset \mathbb{C}$  be an open set, and let  $f(s)$  be a complex-valued function defined on  $S$ :  $f(s) : S \rightarrow \mathbb{C}$

$f(z)$  is analytic at point  $z_0$  in  $S$  if it is differentiable at  $z_0$ .  
 If  $\{z\}$  is a neighborhood of  $z_0$

→ A function  $f(s)$  is said to be analytic in  $S$  if it is analytic at each point of  $S$

→ A vector valued function  $f(s)$  is analytic if every element of the matrix is analytic in  $S$ .

Ex 1: All-real-rational stable transfer functions are analytic in the right half plane

Ex 2:  $e^{-s}$  is analytic everywhere.

$L_2(jR)$  space:

$L_2$  is a Hilbert space of matrix valued functions on  $jR \rightarrow$  the following integral is bounded  $\int_{-\infty}^{+\infty} \text{trace}[F^*(jw) F(jw)] dw < \infty$

The inner product for this Hilbert Space is defined as

$$\langle F, G \rangle := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace}[F^*(jw) G(jw)] dw$$

and the  $L_2$  norm is defined as

$$- H_2 \text{ Space } \|F\|_2 = \sqrt{\langle F, F \rangle}$$

$H_2$  is a closed subspace of  $L_2(jR)$  with matrix functions  $F(s)$  analytic in  $\text{Re}(s) > 0$  (ORHP). The corresponding norm is defined as

$$\|F\|_2^2 := \sup_{\delta > 0} \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace}[F^*(\delta + jw) F(\delta + jw)] dw \right\}.$$

$$\text{it can be shown} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace}[F^*(jw) F(jw)] dw.$$

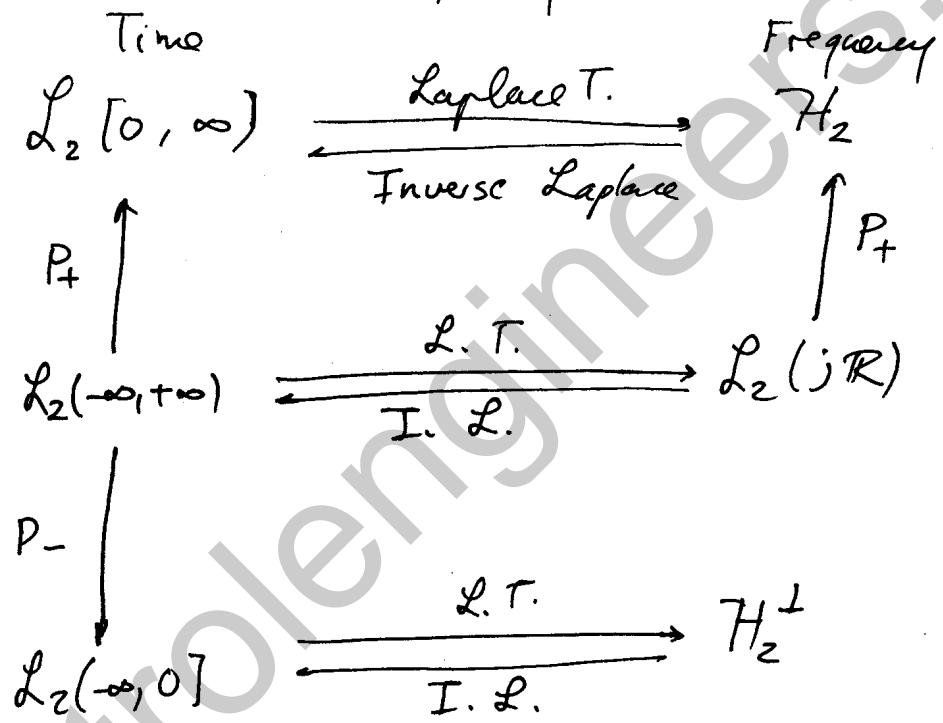
Just as we calculate  $L_2$  norm.

The Real rational subspace of  $H_2$ , which consists of all strictly

## $H_2^\perp$ Space

$H_2^\perp$  is the orthogonal complement of  $H_2$  in  $L_2$ ; that is, the closed subspace of functions in  $L_2$  that are analytic in the OHP similarly  $RH_2^\perp$  is defined respectively.

Relation between time and frequency domain Spaces:



→ By Parseval's Theorem, we can say if  $g(t) \in L_2(-\infty, \infty)$   
its Bilateral Laplace transform is  $G(s) \in L_2(jR)$  Then

$$\|G\|_2 = \|g\|_2$$

→ Define Orthogonal Project  $P_+: L_2(-\infty, \infty) \mapsto L_2[0, \infty)$

$\Rightarrow \forall f(t) \in L_2(-\infty, \infty), \exists g(t) = P_+ f(t)$  with

$$g(t) = \begin{cases} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

\* in the domain  $P_+ = u_+(t)$  step function.

## $L_\infty(jR)$ Space

$L_\infty$  is a Banach space of matrix-valued functions that are (essentially) bounded on  $j\mathbb{R}$  with norm

$$\|F\|_\infty := \sup_{\omega \in \mathbb{R}} \overline{\delta}[F(j\omega)]$$

The rational subspace of  $L_\infty$ , denoted by  $R L_\infty(j\mathbb{R})$  consists of all proper and real rational transfermatrices with no poles on the imaginary axis.

## $H_\infty$ Space

$H_\infty$  space is a (closed) subspace of  $L_\infty$  with functions that are analytic and bounded in the ORHP. The  $H_\infty$  norm is defined as

$$\|F\|_\infty := \sup_{\operatorname{Re}(s) > 0} \overline{\delta}[F(s)] = \sup_{\omega \in \mathbb{R}} \overline{\delta}[F(j\omega)]$$

→ The second equality is a generalization of Maximum Modulus Theorem.

$RH_\infty$  is a real rational subspace of  $H_\infty$ , which consists of all proper and real rational stable transfer functions.

## $\bar{H}_\infty$ Space

$\bar{H}_\infty$  is a closed subspace of  $L_\infty$  with functions that are analytic and bounded in the OLHP. The  $\bar{H}_\infty$  is defined as

$$\|F\|_\infty := \sup_{\operatorname{Re}(s) < 0} \overline{\delta}[F(s)] = \sup_{\omega \in \mathbb{R}} \overline{\delta}[F(j\omega)]$$

### 3 Robustness Analysis

No mathematical Model can exactly represent a physical System.

Robust analysis, is a mathematical framework to analyse the adverse effect modelling error can have on the system characteristics such as stability, performance, ...

The key idea, and the branches of this analysis get back to representation of modelling errors in terms of "uncertainty".

Different look at plant uncertainties provides different tools of Robust analysis in Control Theory.

Uncertainty as parametric unc.  $\rightarrow$   $\mu$  analysis

Uncertainty as unstructured unc.  $\rightarrow$   $H_\infty$  analysis

Uncertainty as bounded norms  $\rightarrow$  Sliding Control  
Nonlinear analysis

In this chapter we first look at Uncertainty representation and their mathematical background, then provide Robust Stability & Performance Notn, and the  $H_\infty$  view of This subject.

## 3-1/ Plant Uncertainty

The Basic Technique is to model the plant as belonging to a set

P. This Set can be structured or unstructured.

### 3-1/1 Structured Uncertainty:

as an example consider a man-damper-spring Syst or an RLC circuit which may represent quite accurately with a Second order Syst

$$P(s) = \frac{1}{s^2 + as + 1}$$

Now the value of damping coefficient may not identified accurately or may vary with time in some interval  $a \in [a_{\min}, a_{\max}]$

Then the plant belongs to the structured set

$$\mathcal{P} = \left\{ \frac{1}{s^2 + as + 1} : a_{\min} \leq a \leq a_{\max} \right\}$$

A mathematical representation of this kind of uncertainty which best suits to the Robust analysis ( $M$ -analysis) is

done by linear Fractional Transformation suppose the uncertainty is given as a percentage variation over the mean

value:  $a = a_{\text{mean}} + a_{\text{perc}} \cdot \delta_a$  where  $\delta_a \in [-1, 1]$

## Linear Fractional Transformations

A better method to represent uncertainty (parametric) is to extract it from the equations, or pull out it as a separate block. To illustrate

suppose we have an uncertain real parameter

$$a \in [\underline{a}, \bar{a}] \quad \text{OR equivalently} \quad a = a_m (1 + \gamma \cdot \delta a)$$

where  $\delta a \in [-1, 1]$

these representations are usual form used for an uncertain real parameter and can be transformed into another

$$a_m = \frac{\underline{a} + \bar{a}}{2} \quad (\text{mean value of parameter } a)$$

$$\gamma = \frac{\bar{a} - \underline{a}}{2a_m} \quad (\text{percentage of parameter variation } 0 < \gamma < 1 \text{ if variation less than } 100\%)$$

in Matlab we use function 'ureal' to define an uncertain real parameter. ureal is the simplest uncertain objects used in Matlab to generate uncertain systems, have properties that can be extracted by 'get' and 'set' functions (as other structure formats in Matlab)  
the Syntax is:

$$p1 = \text{ureal}(\text{name}, \text{NominalValue}, \text{Prop1}, \text{val1}, \text{Prop2}, \text{val2}, \dots)$$

where the properties are given in the following table.

## Ureal Properties and descriptions

Properties	Meaning	Class
Name	Internal Name	Char
NominalValue	nominal value of atom	double
Mode	Signifies the mode of description: 'PlusMinus', 'Range' or 'Percentage'	char
PlusMinus	Additive variation	1x2double
Range	Numerical range	"
Percentage	% of absolute value of additive variation w.r.t. the nominal value	1x2double

\* the default value of variate if not specified is  $\pm 1$

Example : ①  $a = \text{ureal}('a', 3)$ ; means

$$\begin{array}{lll} a.\text{Name} = 'a' & a.\text{Mode} = \text{PlusMinus} & a.\text{Range} = [2 \ 4] \\ a.\text{NominalValue} = 3 & a.\text{PlusMinus} = [-1 \ 1] & a.\text{Percentage} = [-33.3 \ 33.3] \end{array}$$

②  $b = \text{ureal}('b', 2, 'Percentage', 20)$ ; means

$$b.\text{Mode} = \text{'Percentage'}; \ b.\text{Range} = [1.6 \ 2.4] \quad b.\text{PlusMinus} = [-0.4 \ +0.4]$$

$b.\text{Range} = [1.9 \ 2.3]$  (only changes the range & not the Nom. value)

$$b.\text{Mode} = \text{'Percentage'}; b.\text{Percentage} = [-5.0 \ +15.0]; b.\text{PlusMinus} = [-0.1 \ 0.3]$$

$b.\text{NominalValue} = 2.2$  (keeps the previous mode & changes the values)

$$b.\text{Mode} = \text{'Percentage'}; b.\text{Percentage} = [-5.0 \ +15.0]; b.\text{NominalValue} = 2.2$$

$$b.\text{Range} = [2.09 \ 2.53] \quad b.\text{PlusMinus} = [-0.11 \ +0.33]$$

other examples :

```
c = ureal('c', -5, 'per', [-20 30]);
```

```
d = ureal('d', -1, 'mode', 'range', 'perc', [-40 60]);
```

```
e = ureal('e', 10, 'plusminus', [-2 3], 'mode', 'perc', ...)
```

We can use 'usample' function to give any number of Monte Carlo random point for the uncertain parameter

`aSample = usample(a, 5)` 'gives 5 random sample within Range'

In this form the uncertainty is embedded into the structure of the parameter LFT (Linear Fractional Transformation) can be used to pull out the uncertainty out of the parameter structure ; Let's define LFT manipulations

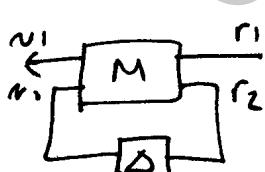


where  $\begin{cases} v_1 = M_{11}r_1 + M_{12}r_2 \\ v_2 = M_{21}r_1 + M_{22}r_2 \end{cases}$

(Note the arrows are left-sided)  
to ease up Matrix manipulation

Now if an uncertain block is connected to the above we have the following two forms

Lower loop



Upper loop



$$\left\{ \begin{array}{l} r_2 = \Delta v_2 \\ v_1 = [M_{11} + M_{12} \Delta (I - M_{22} \Delta)^{-1} M_{21}] r_1 \\ v_1 \triangleq F_L(M, \Delta) \end{array} \right. \quad (\text{lower loop LFT})$$

$$\left\{ \begin{array}{l} r_1 = \Sigma v_1 \\ v_2 = [M_{22} + M_{21} \Sigma (I - M_{11} \Sigma)^{-1} M_{12}] r_2 \\ v_2 \triangleq F_U(M, \Sigma) \end{array} \right. \quad (\text{upper loop LFT})$$

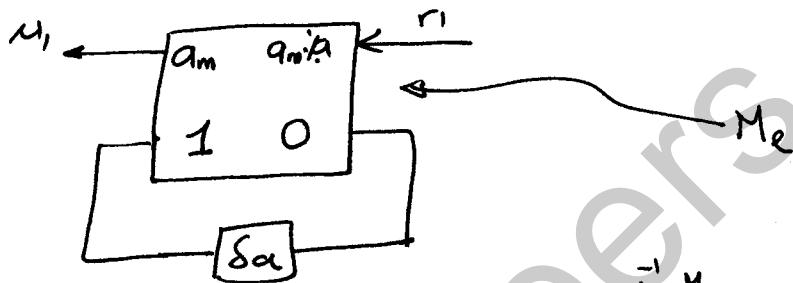
The LFT formulation is a general form that can be used for any structure of uncertainty, for the simple case

$$a = a_m (1 + \gamma_a \cdot \delta_a), \quad \delta_a \in [-1, 1]$$

$$M_{11} + M_{12} \triangleq (I - H_{22}\Delta)^{-1} H_{21}$$

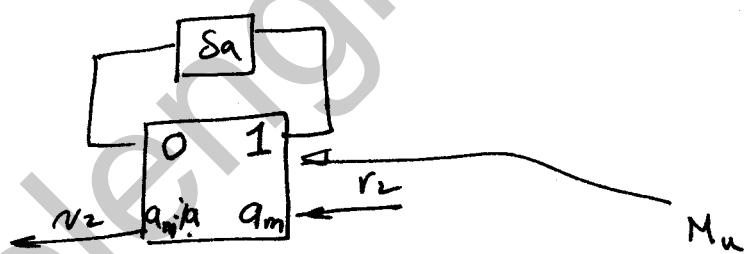
For Lower loop  $a = [a_m + a_m \cdot \gamma_a \cdot \delta_a (I - O) \cdot 1]$

OR



For Upper Loop

$$a = [a_m + a_m \cdot \gamma_a \cdot \Delta \cdot (I - M_{11}\Delta)^{-1} M_{12}]$$

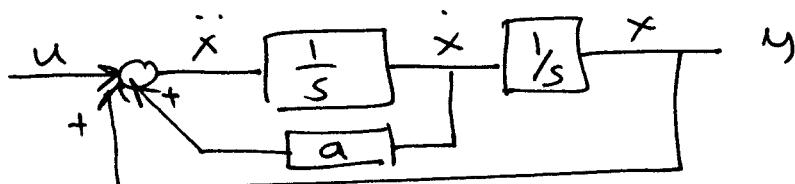


Therefore the uncertainty block is normalized and pull out of the structure

For example Consider the System :

$$G(s) = \frac{y}{u} = \frac{1}{s^2 + 2s + 1}$$

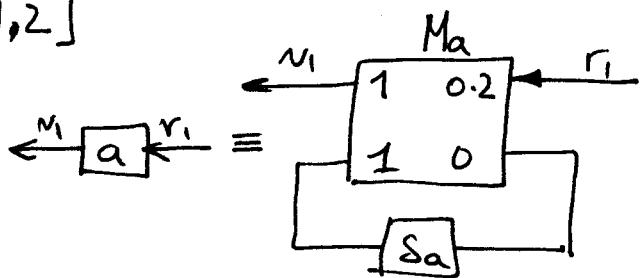
We put it into Block diagram



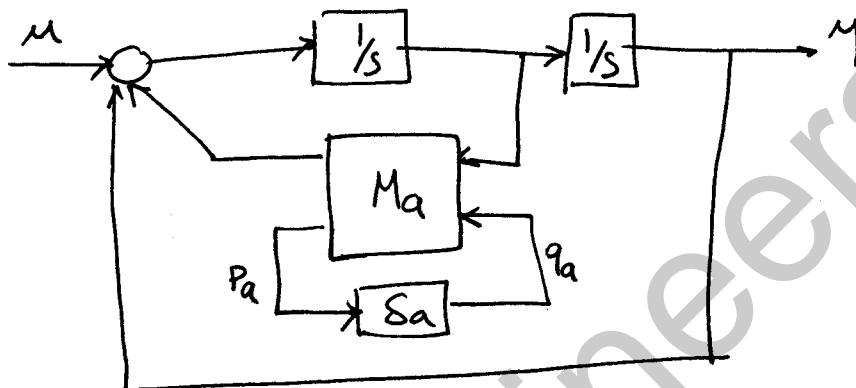
Suppose  $a = \text{ureal}('a', 1, 'perc', [-20 20])$

meaning that  $a \in [0.8, 1.2]$

we may use lower loop LFT :

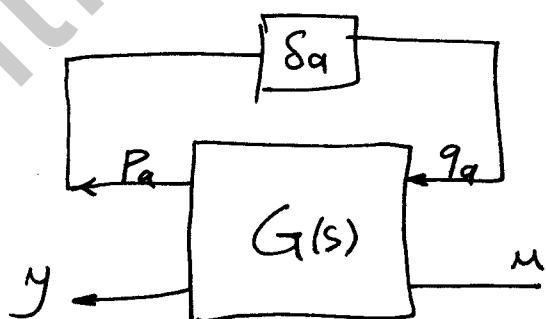


Put into the main Block diagram



Using Block diagram Simplification rules , (OR sysic, iconnect)

we can reduce the problem into a General LFT form with one uncertainty Block



To complete the LFT for this discussion, consider a parameter appearing in reverse form in the block-diagram :

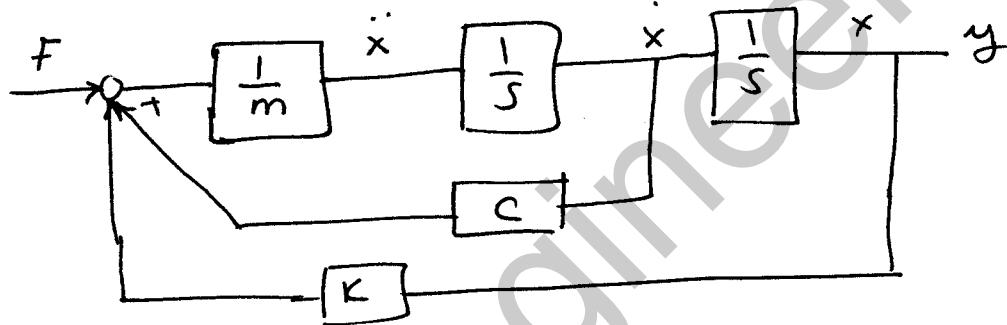
For example  $m\ddot{x} + c\dot{x} + Kx = F$

all three parameters  $m, c, K$  are uncertain

$$m = \bar{m}_m(1 + \pm m \delta_m) ; c = \bar{c}(1 + \pm c \delta_c) ; K = \bar{K}(1 + \pm K \delta_K)$$

$$\delta_m, \delta_c, \delta_K \in [-1, 1]$$

Note that in Block Diagram  $\frac{1}{m}$  appears not  $m$



The power of LFT is that in this case also the uncertainty can be pull out

$$\frac{1}{m} = \frac{1}{m_m(1 + \pm m \delta_m)} = \frac{1}{m_m} - \frac{\pm m}{m_m} \delta_m (1 + \pm m \delta_m)^{-1} \left[ = \frac{(1 + \pm m \delta_m) - \pm m \delta_m}{m_m(1 + \pm m \delta_m)} \right] \\ (M_{11} + M_{12} \Delta (I - M_{22} \Delta)^{-1} M_{21}) \leftarrow F_0$$

$$(M_{m-i})_l = \begin{bmatrix} \frac{1}{m_m} & -\frac{\pm m}{m_m} \\ 1 & -\pm m \end{bmatrix} \quad \text{Lower-loop LFT}$$

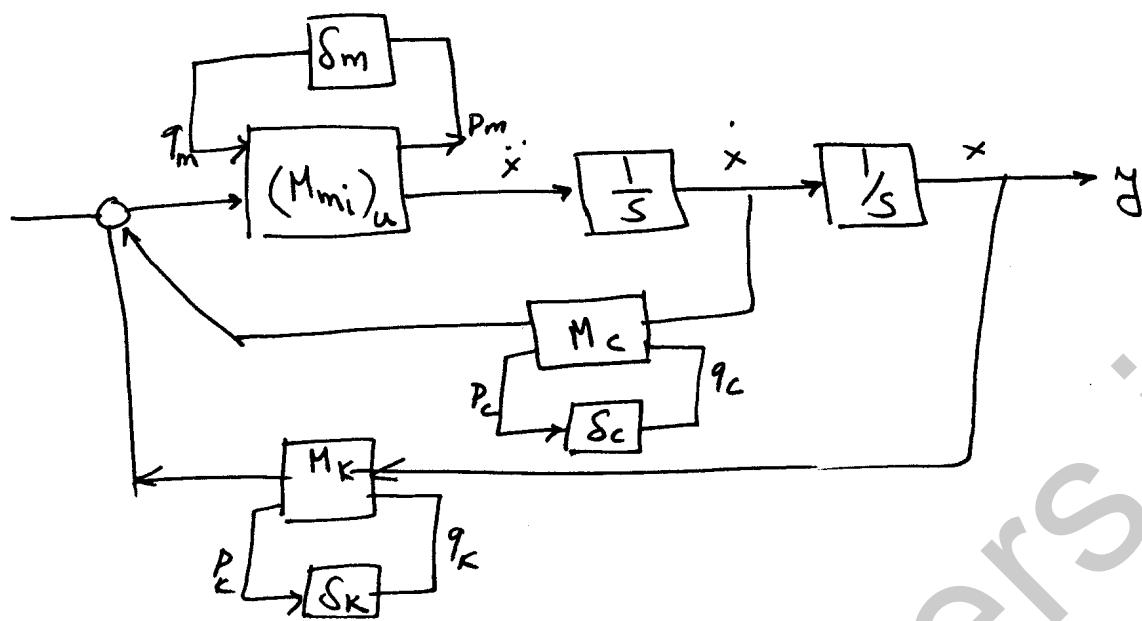
OR in case of upper loop LFT we have

$$(M_{22} + M_{21} \Delta (I - M_{11} \Delta)^{-1} M_{12})$$

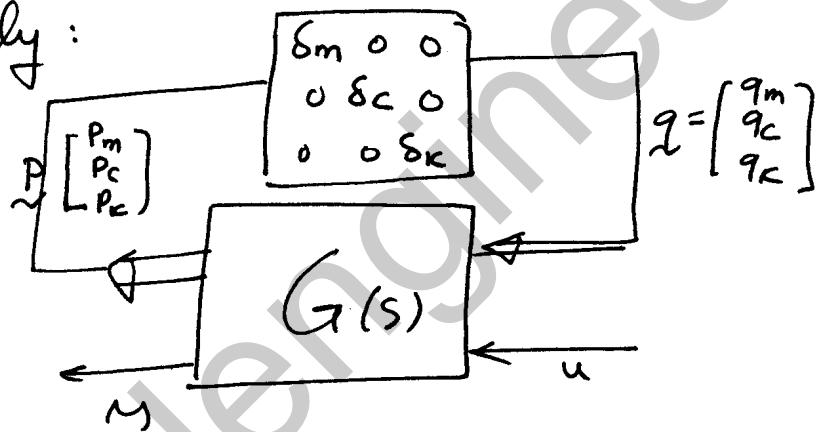
$$(M_{m-i})_u = \begin{bmatrix} -\pm m & 1 \\ -\frac{\pm m}{m_m} & \frac{1}{m_m} \end{bmatrix}$$

Hence for the Structure given in the example :

3-1-7



or Equivalently :



wherein  $G(s)$  everything is determined from  $M_{mi}$ ,  $M_c$ ,  $M_k$ , other System parameters

To be able to complete the above manipulation by Matlab  
it is required that Sysic & OR iConnect Commands are  
read Carefully.

## Parameter-Dependent Systems (P-systems)

There is another way to describe parametric uncertain syst in Matlab which is based on LMI description of System. First Consider a descriptor syst as

$$\begin{cases} E\dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \text{plant} = \text{dss}(A, B, C, D, E);$$

There are two ways to include parametric uncertainty into the Syst description: The first approach is 'Polytopic Models'

Call a polytopic model, a linear-time-varying system

$$E(t) \dot{x} = A(t)x(t) + B(t)u(t)$$

$$y = C(t)x(t) + D(t)u(t)$$

where Syst matrix  $S(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$

varies within a fixed polytope of matrices

$$S(t) \in \text{Co}\{S_1, \dots, S_k\} := \left\{ \sum_{i=1}^k \alpha_i S_i : \alpha_i \geq 0; \sum_{i=1}^k \alpha_i = 1 \right\}$$

where  $S_i$  are given vertex systems  $S_i = \begin{bmatrix} A_i + t\bar{\epsilon}_i & B_i \\ C_i & D_i \end{bmatrix}$

In other words  $S(t)$  is a convex combinati of the syst matrices  $S_i$ ; the nonnegative numbers  $\alpha_1, \dots, \alpha_k$  are called the polytopic coordinates of  $S$ .

Such models are also called "polytopic linear differential inclusions" in the literature, and arises from many practical situations:

- Multimodel representation of a system, each model being derived around particular operating point.
- Nonlinear Systems of the form

$$\dot{x} = A(x)x + B(x)u$$

$$y = C(x)x + D(x)u$$

- State-spaces depending affinely on time-varying parameters

To generate such polytopic Systems use "pssys" as

$$\text{polysys} = \text{pssys}([S_1 \ S_2 \ S_3]);$$

The second and more important representation for parametric uncertainty is "Affine Parameter-Dependent Models". In this representation the descriptor state-space representation involves uncertain or time-varying coefficients. When the system is linear, this naturally gives rise to (PDS) of the form

$$E(p)\dot{x} = A(p)x + B(p)u$$

$$y = C(p)x + D(p)u$$

where system matrices are functions of some parameter vector  $\underline{P} = (p_1, \dots, p_n)$

The Robust Control toolbox offers various tools to analyse PDS & convert it into LFT representation.

3-1-10

Consider affine representations of Syst State-space w.r.t. parameters

$$A(p) = A_0 + p_1 A_1 + \dots + p_n A_n$$

(extract each parameter from)

$$B(p) = B_0 + p_1 B_1 + \dots + p_n B_n$$

system matrices

:

with Syst notation

$$S(p) = \begin{bmatrix} A(p) + jE(p) & B(p) \\ C(p) & D(p) \end{bmatrix}; S_i = \begin{bmatrix} A_i + jE_i & B_i \\ C_i & D_i \end{bmatrix}$$

The affine dependence on  $p$  is written more compactly in Syst matrix form as

$$S(p) = S_0 + p_1 S_1 + \dots + p_n S_n$$

The Syst coefficients  $S_i$ 's fully characterizes the dependence on the uncertain Syst; for instance the syst

$$S(p) = S_0 + p_1 S_1 + p_2 S_2 \quad (\text{two parameters})$$

is defined by

$$S_0 = dss(a_0, b_0, c_0, d_0, e_0);$$

$$S_1 = dss(a_{11}, \dots, \dots);$$

$$S_2 = dss(a_{21}, \dots, \dots);$$

$$\text{affsys} = \text{psys}(\text{pv}, [S_0 \ S_1 \ S_2])$$

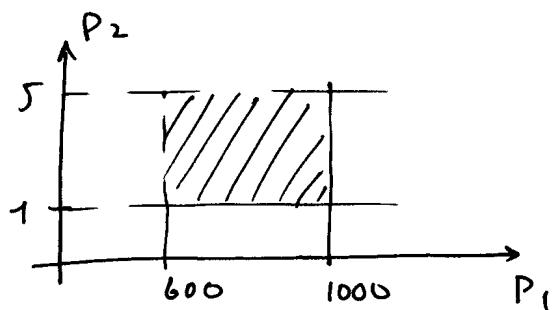
where  $\text{pv}$  is the pvector generated by  $\text{pvec}$

The pvec command can generate different shape of parameter variations for example: [3-1-11]

Ex1: for two parameters

$$P_1 \in [600 \quad 1000]$$

$$P_2 \in [1 \quad 5]$$



$p = \text{pvec}('box', \text{range})$

$\text{range} = [600 \quad 1000; 1 \quad 5]$

if there is constraints on ( $\dot{P}$ ) this can be given as

$$0.1 \leq \dot{P}_1(t) \leq 1 \quad ; \quad |\dot{P}_2(t)| \leq 0.001$$

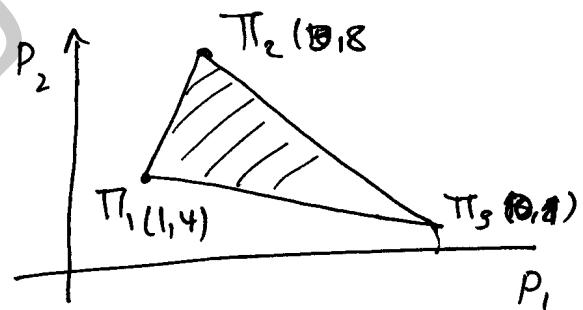
$$\text{rate} = [0.1 \quad 1; -0.001 \quad 0.001];$$

$p = \text{pvec}('box', \text{range}, \text{rate})$

Ex2: Polytopic variations

$$pi1 = [1, 4]; pi2 = [3, 8]; pi3 = [10, 1]$$

$p = \text{pvec}('pol', [pi1, pi2, pi3]);$



See more details in the Help.

for Simulation of Parameter Dependent Systems

use "pdSimul" (pds, 'traj')

To convert affine  $\rightarrow$  Polytopic use

"aff2pol"

polsys = aff2pol(affsys)

To Convert affine Parameters to lft representation

$$[P, \text{delta}] = \text{aff2lft}(pds)$$

Example: Consider a simple 2<sup>nd</sup> order RLC circuit with 2 equations

$$L \frac{di^2}{dt} + R \frac{di}{dt} + Ci = V$$

whose physical parameters ranging in

$$L \in [10 \ 20]; \quad R \in [1 \ 2] \quad C \in [100 \ 150];$$

A parameter dependent state-space representation can be formed

by  $E(P)\dot{x} = A(P)x$        $P = (L, R, C)$

$$A(P) = \begin{bmatrix} 0 & 1 \\ -R & -C \end{bmatrix} \rightarrow A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad A_L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_R = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}; \quad A_C = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A(P) = A_0 + L A_L + R A_R + C \cdot A_C$$

Similarly

$$E(P) = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + L \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + R \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + C \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

hence the system can be generated in matlab using

$$a_0 = [0 \ 1; 0 \ 0]; \quad e_0 = [1 \ 0; 0 \ 0];$$

$$a_L = [0 \ 0; 0 \ 0]; \quad e_L = [0 \ 0; 0 \ 1];$$

$$a_R = [0 \ 0; 1 \ 0]; \quad e_R = [0 \ 0; 0 \ 0];$$

$$a_C = [0 \ 0; 0 \ -1]; \quad e_C = [0 \ 0; 0 \ 0];$$

for all Systns  $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ;  $c = \begin{bmatrix} 1 & 0 \end{bmatrix}$ ,  $d = [0]$  3-1-13

hence the  $S_i$ 's are generated by

$$S_0 = dss(a_0, b, c, d, e_0);$$

$$S_L = dss(a_L, \dots, e_L);$$

$$S_R = dss(a_R, \dots, e_R);$$

$$S_C = dss(a_C, \dots, e_C);$$

$$pv = pvec('box'; [10 20; 1 2; 100 150]);$$

$$pds = psys(pv, [S_0 S_L S_R S_C]);$$

To convert it to lft form

$$[P_{nom}, \Delta] = aff2lft(pds);$$

### 3-1-2] Unstructured Uncertainty

(3-9)

As explained before, parametric uncertainty is a special case, where the model is assumed to be well-known, but the parameters not accurately identified. However, unstructured uncertainties are more important, since, there exists <sup>un</sup> Modelled dynamics, particularly at high frequencies. U.U. usually represents frequency dependent elements such as actuator saturations, and unmodelled structural modes at high frequencies, or flat disturbances at the low frequencies.

There are various representations for this kind of uncertainty. Let us first consider disk-type, or multiplicative uncertainty of the form

$$\tilde{P} = (1 + \Delta W) P_0 \quad \forall P \in \mathcal{P}$$

where  $P_0$  is the nominal Model and  $\frac{P}{P_0} - 1 = \Delta W$

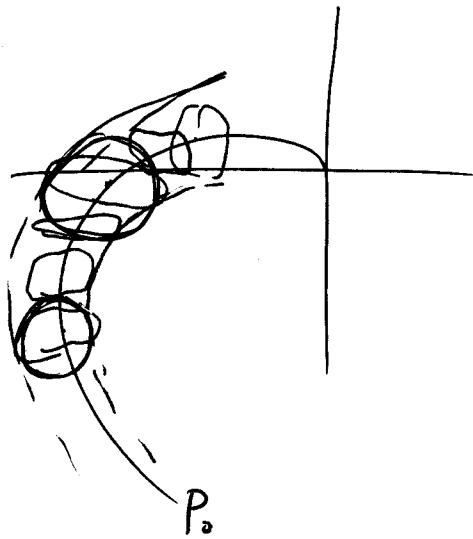
$\Delta W$  is the normalized plant deviation from 1, and,  $W$  is a fixed stable transfer function, the weight, which gives the frequency content of the uncertainty &  $\Delta$  is a variable but stable transfer function where  $\|\Delta\|_{\infty} \leq 1$ . Hence all unstable poles of  $P$  is represented in  $P_0$  and other poles & zeros which are not seen in the nominal Model  $P_0$  are all stable.

Hence if  $\|\Delta\|_{\infty} \leq 1$

$$\left| \frac{P(j\omega)}{P_0(j\omega)} - 1 \right| \leq |W(j\omega)| \quad \forall \omega$$

$W$  gives the uncertainty profile with respect to frequency

A graphical representation of this kind of uncertainty is shown below 3-10



At each frequency  $P_0 \neq P$  differs, the radius  $W(j\omega)$  gives the maximum discrepancy which is normalized  $\left| \frac{P - P_0}{P_0} \right| < |W(j\omega)|$ .  
 $W(j\omega)$  is an increasing function w.r.t. frequency. Since this uncertainty represents unmodeled dynamics, which are usually higher at high frequencies. Therefore  $W$  accounts for the magnitude discrepancies while  $\Delta$  is used for phase discrepancy &  $\|\Delta\|_\infty < 1$  is set for further simplification for stability analysis.

The key point in this representation is to verify the unmodeled dynamics as a bounded transfer function  $W$ .

See the following examples:

## Uncertain LTI Dynamics Atoms

Similar to real uncertain parameter assignments in Matlab RCT Toolbox, 'ultidyn' function can be used to represent a Full Block  $\Delta(s)$ . In below, 'make weight' is used to represent  $W(s)$  'ultidyn' to generate  $\Delta(s)$ :

```
Gnom = H;  
W = makeweight(.05,9,10);  
Delta = ultidyn('Delta',[1 1]);  
G = Gnom*(1+W*Delta)  
USS: 2 States, 1 Output, 1 Input, Continuous System  
Delta: 1x1 LTI, max. gain = 1, 1 occurrence  
bw: real, nominal = 5, variability = [-10 10]%, 1 occurrence
```

```
G = makeweight(DC,Cronw, HF)  
G(0) = DC; G(jCronw) = HF  
H = ultidyn('Name',iosize)
```

To create a  $\Delta_{2 \times 3}(s)$  Full Block we can use

```
Delta = ultidyn('Delta',[2 3])
```

here make weight ( Dgain, cron over frequency, high frequency gain ) function is used to fit a 1<sup>st</sup> order transfer function to the above required characteristics.

As it is seen in the above Example, usual matrix product matrix substitution & ... can be applied on uncertain transfer functions, and state-space representation. USS is formed by an uncertain 'ultidyn' Syst multiplied by some known T.F. and has build up an uncertain State-Space Syst USS.

Ex 1: Nonlinear Systems  $\rightarrow$  Linear Model + uncertainty

Suppose for a stable (Nonlinear) Syst by means of experiments a frequency response estimates are obtained, i.e. magnitude and phase of a number of inputs with frequencies  $\omega_i = i=1, \dots, m$  is obtained and this experiment is repeated several times, say  $N$  times, with different amplitudes. (Since Syst is N.L. different amplitudes give diffret results)

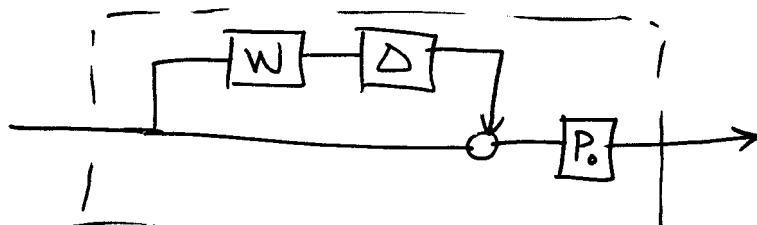
Now, denote  $(M_{ik}, \phi_{ik})$  be the magnitude-phase measured at frequency  $\omega_i$  and experiment  $k$ .

Select the best fit through these data as nominal Model by

$(M_i, \phi_i)$  for  $\omega_i, i=1, \dots, m$ , and fit a nominal transfer function through them  $P_0(s)$ , and get  $W_2$  to be the upper bound of this discrepancies

$$\left| \frac{M_{ik} e^{j\phi_{ik}}}{M_i e^{j\phi_i}} - 1 \right| < |W(j\omega_i)| \quad \begin{array}{l} i=1, \dots, m \text{ all freqs} \\ k=1, \dots, n \text{ all exps} \end{array}$$

Then The Nonlinear Syst is represented by a linear Model  $P_0$  with an uncertainty profile  $W$

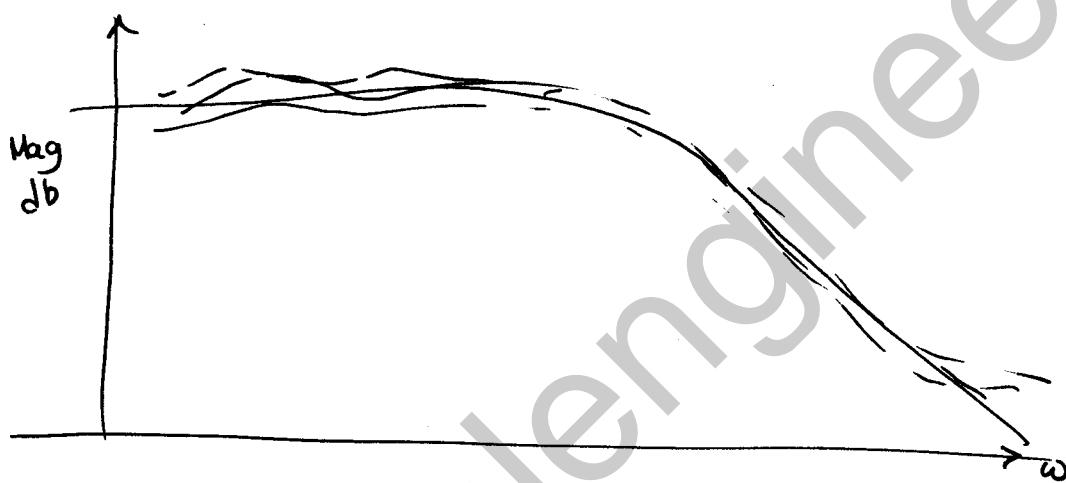


A Realization of this technique is done on the [1, 2]

(3-12)

Harmonic Drive available in the Lab, in two different Oper. Cond.

Using SigLab Hardware, different frequency estimate response are extracted by sine-sweep & random inputs, with different input amplitudes. and plotted as in Fig 9, Paper 1 & Fig 3 & 4 Paper 2; typically



~ a Nominal Model is fit into the data using invfreqs in matlab

using a simple 3<sup>rd</sup> order System for both cases. :  $P_0(s) = \frac{1.0755 \times 10^6}{s^3 + a_2 s^2 + a_1 s + a_0}$

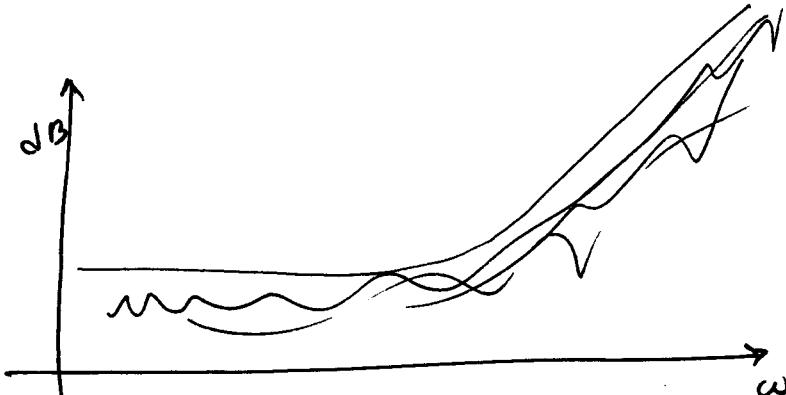
then

$$\left| \frac{P(j\omega)}{P_0(j\omega)} - 1 \right|$$
 is plotted

and an upper bound is found for the uncer. as:

$$W(s) = \frac{s+100}{200}$$

A High pass filter



Example 2: Time delay  $\rightarrow$  linear system + uncertainty

3-13

Consider a variable time delay syst

$$P(s) = \frac{e^{-cs}}{s(s+1)} \quad \text{where } 0.1 \leq c \leq 0.2$$

we would like to Model this syst as nominal Model

$$P_0(s) = \frac{1}{s(s+1)}$$

+ a multiplicative uncertainty  $\{ P = P_0(1+\Delta\omega), |\Delta\omega| \leq 1 \}$

to find the uncertainty upper bound calculate - plot

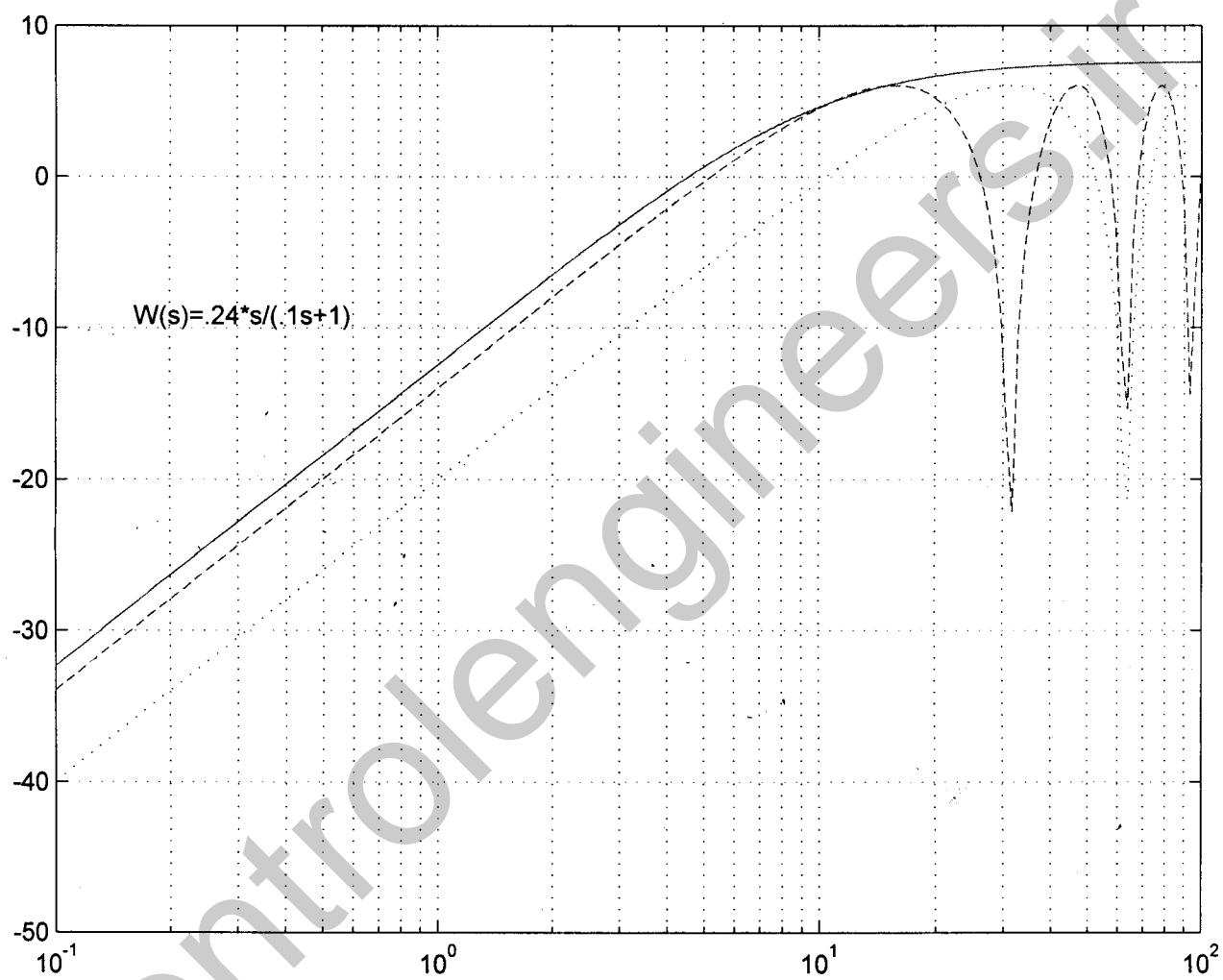
$$\left| \frac{\frac{e^{-j\omega c}}{j\omega(j\omega+1)} - 1}{\frac{1}{j\omega(j\omega+1)}} \right| \quad \text{for } c=0.1 \text{ & } c=0.2$$

The magnitude plot is given in the next page.

The uncertainty bound can be found by  $\frac{K \cdot s}{\tau_{au} s + 1}$

where  $\tau_{au}$  is the  $\approx \frac{1}{\omega_{crown}} \approx \frac{1}{10}$  &  $K$  can be found by trial & error

to be  $W(s) = \frac{0.245 s}{0.1 s + 1}$  (again  $W$  is high pass!)



Ex 3 | parametric uncertainty  $\rightarrow$  unstructured unc.

13-15

Suppose  $P(s) = \frac{K}{s-2}$  &  $0.1 \leq K \leq 10$  may vary

Hence  $P_r(s) = \frac{K_0}{s-2}$   $K_0 = \text{mean}(0.1, 10) = 5.05$

W must satisfy

$$\left| \frac{\frac{K}{s-2} - 1}{\frac{K_0}{s-2}} \right| \leq |W(j\omega)|$$

$$\max \left| \frac{\frac{K}{s-2} - 1}{\frac{K_0}{s-2}} \right| \leq |W(j\omega)|$$

LHS is minimized by  $K_0 = 5.05$  to be a const. as  $\left| \frac{10}{5.05} - 1 \right| = 0.9802$

$W(s) = 1$  is perfectly the upper bound (not varying with freq)

Ex 4 | Sometimes, Multiplicative Unc. is not a good choice

$$G(s) = \frac{1}{s^2 + \alpha s + 1} \quad 0.4 \leq \alpha \leq 0.8$$

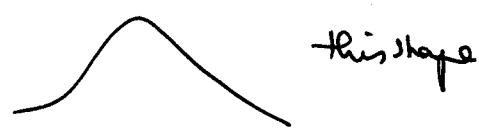
$$\text{use average for } \bar{\alpha} = 0.6 \quad P_o = \frac{1}{s^2 + 0.6s + 1}$$

plot the frequency response as in the next page

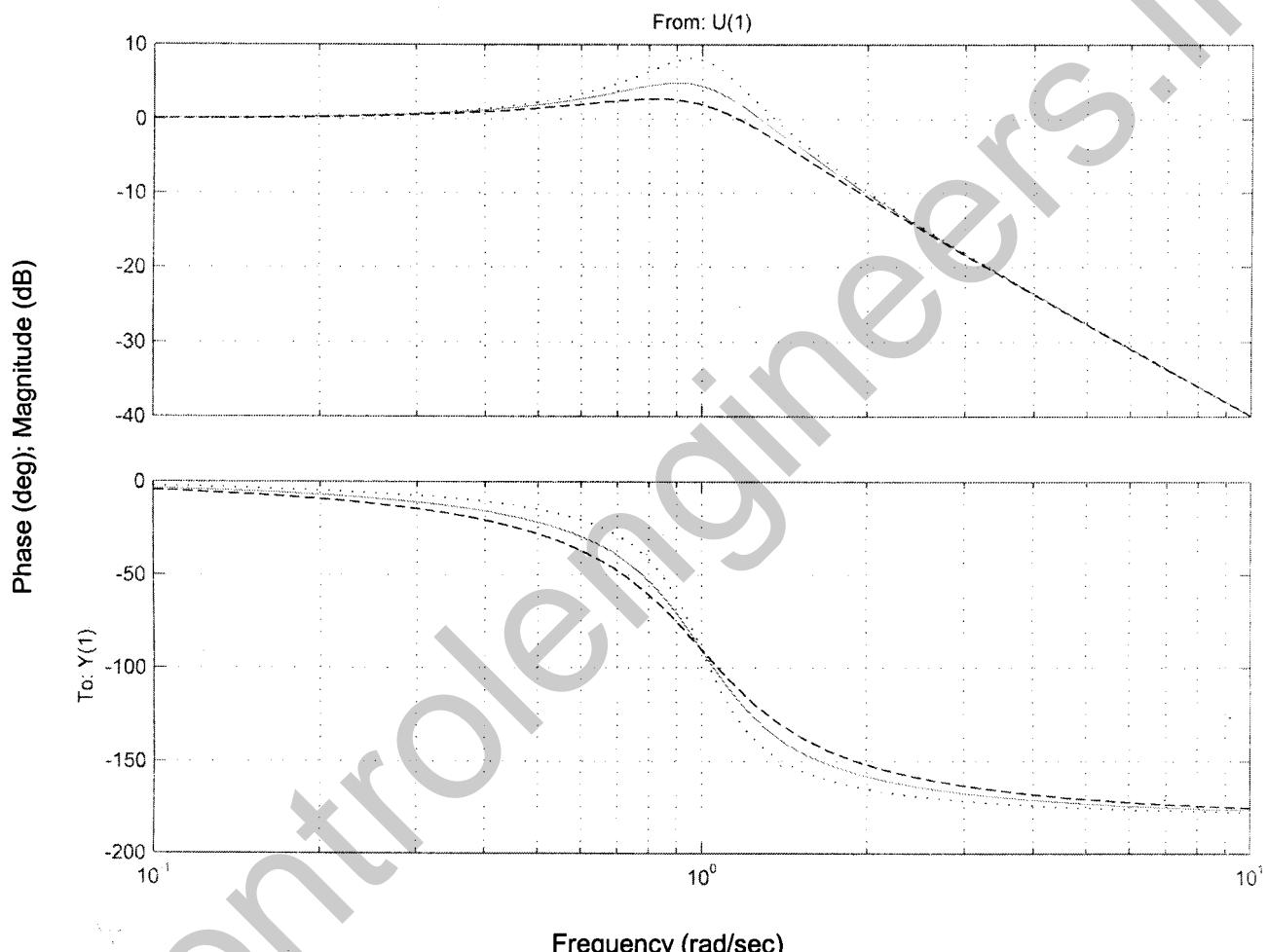
and the uncertainty

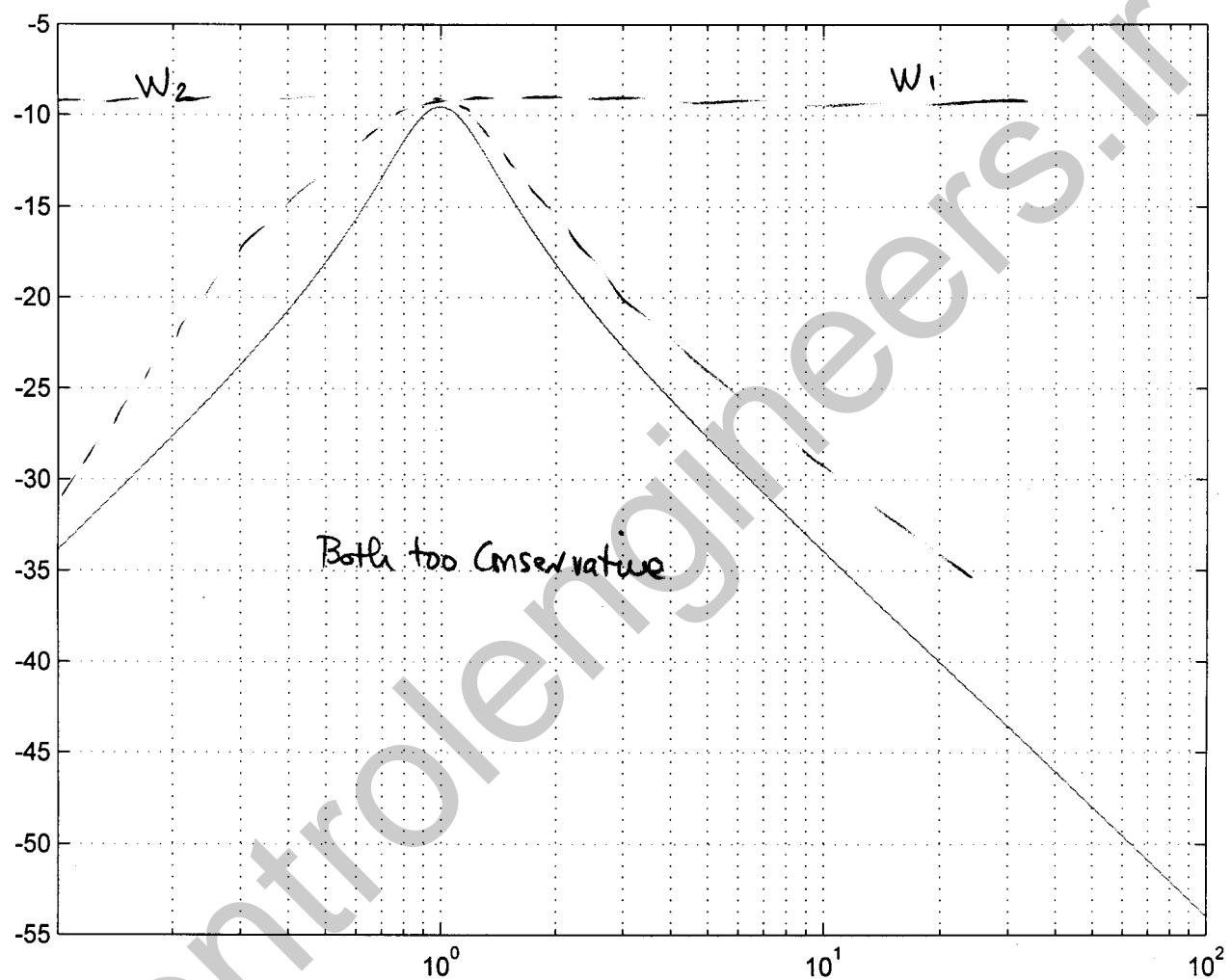
$$\left| \frac{G(j\omega)}{P_o(j\omega)} - 1 \right| \quad \text{for } \alpha = 0.4 \text{ or } 0.8$$

give rise to an uncertain bond



## Bode Diagrams





which is not simple to introduce a single 1<sup>st</sup> or 2<sup>nd</sup> order bound for it [3-18]

But if we consider this type of uncertainty

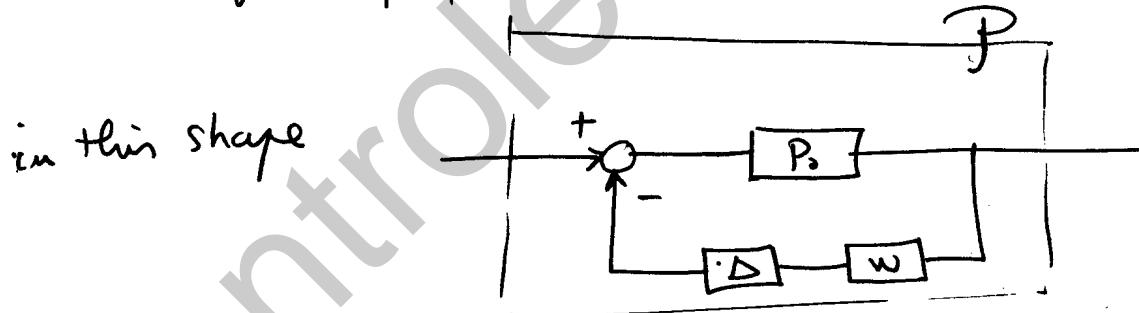
$$P(s) = \frac{1}{s^2 + \alpha s + 1} \quad 0.4 < \alpha < 0.8 \quad \alpha = 0.6 + 0.2\delta \quad -1 \leq \delta \leq 1$$

The family can be expressed

$$\left\{ \begin{array}{l} \frac{P_0(s)}{1 + \Delta W(s) P_0(s)} \quad -1 \leq \delta \leq 1 \\ P_0(s) = \frac{1}{s^2 + 0.6s + 1} \quad W(s) = 0.2s \end{array} \right.$$

(in which  $P(s) = \frac{1}{s^2 + 0.6s + 1 \pm 0.2s}$  put it into feedback-multiplication)

Then  $W(s)$  get simply By  $W(s) = 0.2s$



in this shape

$$P = \frac{P_0}{1 + \Delta W(s) P_0(s)}$$

Feed back - multiplicative

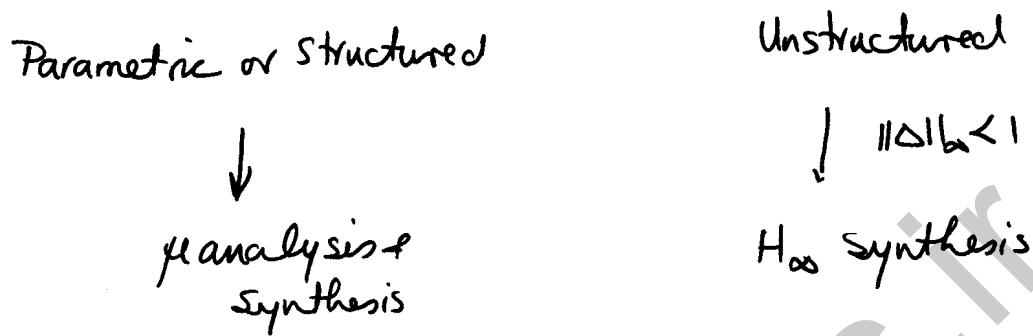
Other types of uncertainty are

$$P = P_0 + \Delta W(s) \quad \text{additive}$$

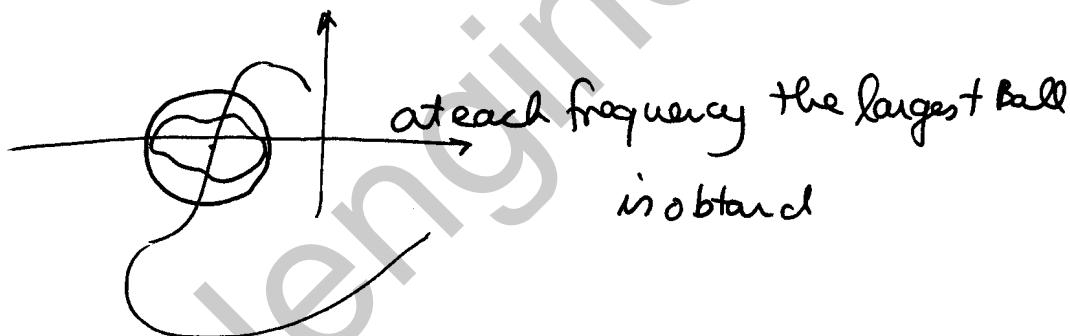
$$P = \frac{P_0}{1 + \Delta W(s)} \quad \text{additive-multiplicative}$$

Summary :

Two looks at uncertainty representation is introduced



- (A) Unstructured uncertainties are more common, But their Multiplicative representations are conservative



- (B) Physical Realization of uncertainty is the most difficult part of Controller analysis, The smaller the W the Better our knowledge  $\Rightarrow$  The better the controller can be.

when the plant uncertainty is possible to <sup>be</sup> modeled

Robust analysis & Synthesis is

possible to accomplish

### 3-2/ Robust Stability

Notion of Robustness: Suppose the plant transfer function  $\in \mathcal{P}$

Consider some characteristics of the feedback syst., for example internal stability.

A controller  $C$  is "Robust" w.r.t. this characteristics if that holds for all  $P \in \mathcal{P}$  (not only  $P_0$ )

Now for stability for example the syst. is Normally stable if it is stable for  $P_0$ , But it is Robust Stable, if  $\forall P \in \mathcal{P}$  the feedback Syst. is Stable.

From Now Consider that Multiplicative uncertainty is our base unc. representation. we may Graphically See the robust stability by the following picture

For robust stability  $\forall \omega$

$$|a| > |r|$$

No Ball contains  $-1$  for stability

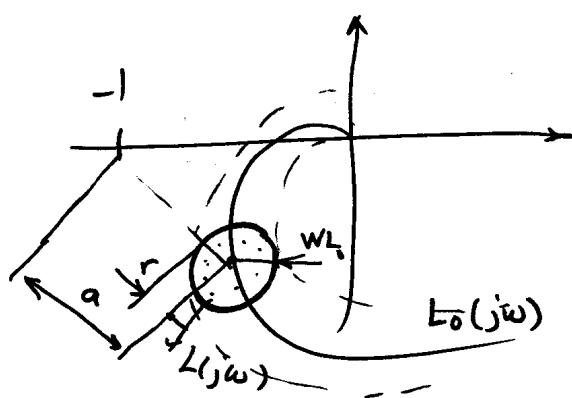
$$|-1 - L_o(j\omega)| > |WL_o(j\omega)| \leftarrow \text{disk type uncertainty}$$

$$\left| \frac{WL_o(j\omega)}{1 + L_o(j\omega)} \right| < 1$$

$$\text{where } S(j\omega) = \frac{1}{1 + L_o(j\omega)}$$

$$|WT(j\omega)| < 1 \quad \forall j\omega$$

$$T(j\omega) = \frac{L_o(j\omega)}{1 + L_o(j\omega)}$$



Theorem 1: For multiplicative uncertainty Model, C provides 3-20

Robust stability, iff,  $\|WT\|_\infty < 1$

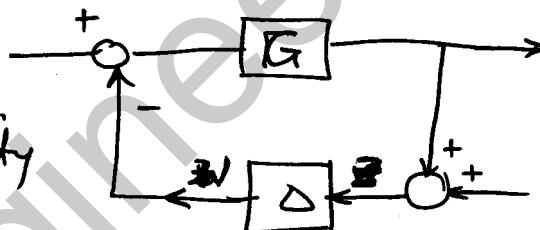
proof in the Book.

Let us prove this Theorem (one part of it) from one Basic Theorem

Called "Small Gain Theorem" (Zames, 67)

Small Gain Theorem: (Zames, 67)

Consider an uncertain System

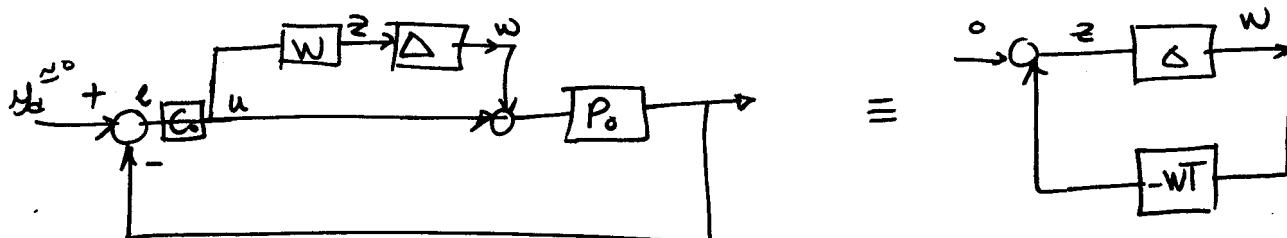


The sufficient Condition for stability  
of the feedback System is

- 1)  $G(s)$  is stable
- 2)  $G(s)$  is strictly proper
- 3.)  $\|G\|_\infty \|\Delta\|_\infty < 1$ .

Now if we had  $\|\Delta\|_\infty < 1 \Rightarrow \|G\|_\infty < 1$  For Robust Stability

Now Consider our System



Since  $y = \frac{P_0 C}{1 + P_0 C} y_d + \frac{P_0}{1 + P_0 C} w$

$$z = W \cdot u = W \cdot C \cdot (y_d - y)$$
$$\rightarrow z = \frac{W C}{1 + P_0 C} y_d - W \cdot \frac{C P_0}{1 + P_0 C} \cdot w$$
$$\rightarrow \frac{z}{w} = -W \cdot \frac{C P_0}{1 + P_0 C} = -WT$$

Hence from S.G.T., The system is Robust stable

✓ admissible  $\Delta$ , if

$$\|WT\|_{\infty} < 1 \quad \text{OR}$$

$$\boxed{\|WT\|_{\infty} < 1}$$

Similarly for other type of perturbation we can conclude

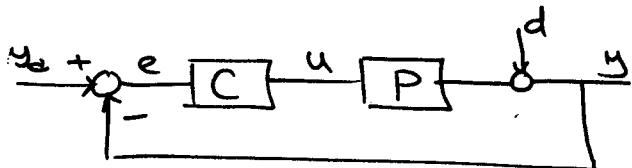
Uncertainty	R.S. Condition
multiplicative $P = (1 + \Delta W)P_0$	$\ WT\ _{\infty} < 1$
additive $P = (P_0 + \Delta W)$	$\ WCS\ _{\infty} < 1$
mult. feedback $P = \frac{P_0}{1 + \Delta WP_0}$	$\ WPS\ _{\infty} < 1$
add. feedback $P = \frac{P_0}{1 + \Delta W}$	$\ WS\ _{\infty} < 1$

all can be concluded from S.G. Theorem similarly

### 3-3 Robust Performance

The characteristics of robust analysis is to make the design base on  $S$  and  $T$ , rather than the  $L$ : loop gain in classical control.

For performance, as well, we need tracking error or disturbance reject as the criteria. Let us first consider unperturbed Syst :



$$y(s) = \frac{C(s)P(s)}{1+C(s)P(s)} y_d(s) + \frac{1}{1+C(s)P(s)} d(s)$$

$$\begin{aligned} e(s) &= y_d - y = \frac{1}{1+C(s)P(s)} y_d(s) - \frac{1}{1+C(s)P(s)} d(s) \\ &= S(s) y_d(s) - S(s)d(s) \end{aligned}$$

Hence Both objectives of tracking error & dist reject can be done by making  $S(s)$  small.  $|S(j\omega)| \ll 1$

But as we know this should be done for the controller bandwith to make this criteria weighted in frequency domain we can write

$$\text{Nominal Performance : } \|W_s S\|_\infty < 1$$

The Magnitude of  $W_s$  makes how small the tracking error should be and the frequency range gives the bandwith information

→ Note on Selection of  $W_s$

Note the design for desired performance in a Robust synthesis is reflected into the Selection of  $W_s$ , since

$$\|W_s S\|_\infty \leq 1 \rightarrow |S(j\omega)| < \frac{1}{|W_s(j\omega)|} \quad \forall \omega$$

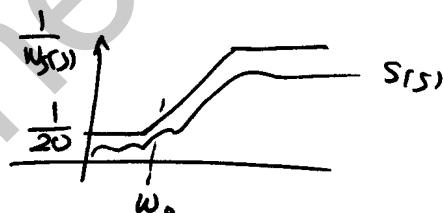
hence the inverse of  $W_s$  shapes the Sensitivity funct. and  $\Rightarrow$   
the performance : for example Suppose

$$W_s(s) = \frac{s+100}{5(s+1)} \rightarrow W_s^{-1} = \frac{1}{W_s(s)} = \frac{5(s+1)}{s+100}$$

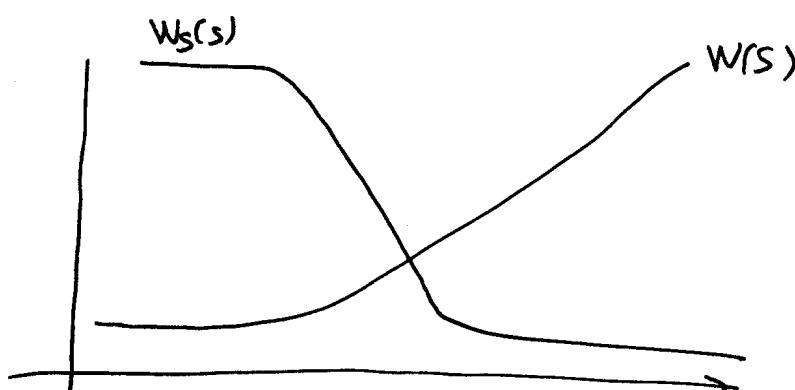
$$\underset{s \rightarrow 0}{\text{L}} W_s^{-1}(s) = \frac{5}{100} = 1/5$$

and the frequency shape is

cutofffrequency  $s = 1 \text{ rad/sec}$



This Selection guarantees a 1.5 steady state error due to a unit step or disturbance attenuation at low frequency with a scale of  $\frac{1}{20}$ , and provides a bandwidth for the closed Loop system to be  $\omega_{bw} = \omega_c = 1 \text{ rad/sec}$

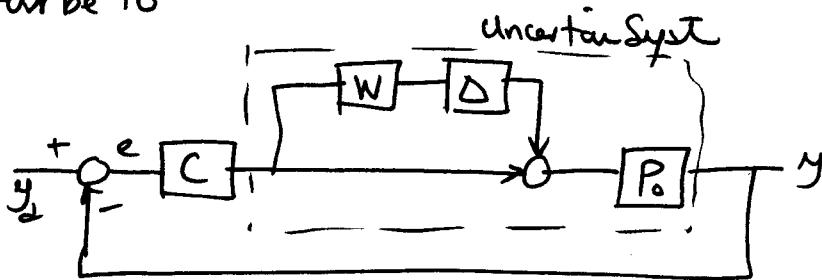


The bandwidth  
Selection depends  
on the uncertainty  
Profile  $W(s)$   
as we will see later.

Now in presence of uncertainty  $W(s)$

Robust Stability :  $\|WT\|_\infty < 1$  (we need stability for min performance)

$S$  is perturbed to



$$S_p = \frac{Y(s)}{E(s)} = \dots = \frac{1}{1 + (1 + \Delta W)L} = \frac{1}{1 + L + \Delta WL} = \frac{\frac{1}{1 + L}}{\frac{1 + L + \Delta WL}{1 + L}}$$

$$S_p = \frac{S_p}{1 + \Delta WT}$$

Hence for Robust performance we must have

$$\|WT\|_\infty < 1, \quad \|W_s S_p\| < 1$$

OR  $\|WT\|_\infty < 1$  and  $\left\| \frac{W_s S}{1 + \Delta WT} \right\|_\infty < 1$   $\Delta$  admissible

We can write these two condition in the form of an norm as:

Theorem 2: (Robust Performance)

A necessary & sufficient condition for Robust Performance of multiplicative uncertainty model is

$$\|W_s S + WT\|_\infty < 1$$

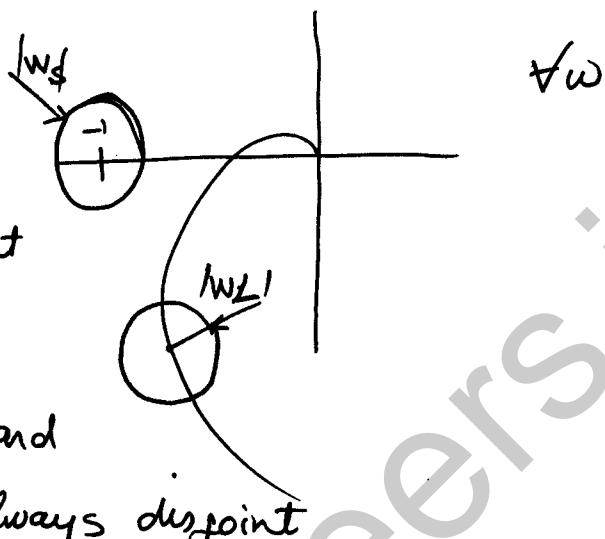
This problem proved to be very hard to solve is called

Two-disk problem, because of the graphical representation:

for Robust Perf.

Condition to be hold

two disks with radius  $|W_S|$  at each frequency  $\omega$  center -1



and disk with radius  $|WL|$  and center  $\{L\}$ , may ~~never~~ be always disjoint

proof: (in the book) (N.N.C.)

$$(\Leftarrow) \quad \|W_{SS} + W_T\|_\alpha < 1$$

$$\|W_{SS} + W_T\|_\infty \leq \|W_{SS}\|_\infty + \|W_T\|_\infty < 1$$

Note If  $\left\| \frac{W_{SS}}{1-W_T} \right\|_\alpha < 1 \quad \text{and} \quad \|W_T\|_\infty < 1$

$$\|W_{SS}\|_\infty < 1 - \|W_T\|_\infty$$

$$\|W_{SS}\|_\infty < 1 - \|W_T\|_\alpha$$

$$\|W_{SS}\|_\alpha + \|W_T\|_\alpha < 1$$

$$\text{then } \|W_{SS} + W_T\|_\alpha < 1$$

But

$$1 = \|1 + \Delta W_T - \Delta W_T\| \leq \|1 + \Delta W_T\| + \|\Delta W_T\|$$

$$\leq \|1 + \Delta W_T\| + \|\Delta W_T\|$$

$$\leq \|1 + \Delta W_T\| + \|W_T\|$$

$$\Rightarrow 1 - \|W_T\| \leq \|1 + \Delta W_T\|$$

$$\Rightarrow \left\| \frac{W_{SS}}{1 - W_T} \right\|_\infty \geq \left\| \frac{W_{SS}}{1 + \Delta W_T} \right\|_\infty$$

$$1 > " \quad \Rightarrow \quad \left\| \frac{W_{SS}}{1 + \Delta W_T} \right\|_\alpha$$

Note: Robust Performance with level  $\alpha$

if the performance is much smaller than 1, i.e.

$$\|WT\|_{\infty} < 1 \text{ and } \left\| \frac{W_{SS}}{1 + \Delta WT} \right\|_{\infty} < \alpha$$

Hence

$$\| |W_{SS}| + \alpha |WT| \|_{\infty} < \alpha$$

The optimum performance is obtained if  $\alpha_{\min}$  can be found

This problem is Solved by duality and Convex Optimizat

By (Amen, Zames) 1993.

No Numerical Software is developed yet.

Summary: The nominal feedback system is assumed to be internally stable. Then the Robust Stability is  $\|WT\|_{\infty} < 1$

where  $W$  is the plant Mult. uncer. profile.

For Nominal performance & R. stability  $\| \max(W_{SS}, |WT|) \|_{\infty} < 1$  \*

The robust performance is achieved By

$$\underbrace{\|W_{\Delta T}\|_{\infty} < 1}_{\text{Robust St.}} + \left\| \frac{W_{SS}}{1 + \Delta WT} \right\|_{\infty} < 1 \neq \alpha \quad (\text{I})$$

$$\text{OR} \quad \|W_S| + |WT| \|_{\infty} < 1 \quad (\text{II})$$

Since this problem proves hard to be solved, do the

Compromise by the following

$$\max(|W_{SS}|, |WT|) \leq |W_{SS}| + |WT| \leq 2 \max(|W_{SS}|, |WT|)$$

If  $\neq (*)$  are not far apart, there is a safety factor of 2 only

$$\|W_{SS}\|_\infty < \frac{1}{2} \Leftrightarrow \|WT\|_\infty < 1$$

Then the robust performance is obtained. But this is a conservative look at the problem. Since  $\|\cdot\|_\infty$  is conservative itself, Not very pleasant approach.

Look another point

$$\| |W_{SS}| + |WT| \|_\infty < 1 \xrightarrow{\text{Compromise}} \left\| \left( W_{SS}^2 + WT^2 \right)^{1/2} \right\|_\infty \leq 1$$

L<sub>2</sub> norm

from simple geometry :

$$\frac{1}{\sqrt{2}} (|W_{SS}| + |WT|) \leq \left( |W_{SS}|^2 + |WT|^2 \right)^{1/2} \leq \sqrt{2} \max(|W_{SS}|, |WT|)$$

which is fairly a better approximation

This is Called Mixed-Sensitivity problem and is Solved

$$\left\| \begin{pmatrix} W_{SS} \\ WT \end{pmatrix} \right\|_\infty \leq 1$$

if this is smaller than  $\sqrt{2}$  the robust performance is guaranteed.

As illustrated in last section we have shown that robust characteristics are related to a min-max problem  $\min \|W^*\|_\infty$ , of  $S$  or  $T$  where in

Both t.f.  $\frac{1}{1+CP}$  exists, an unknown t.f. is in the denominator.

By coprime factorization, we reduce those problems into a problem where the unknown parameter is given in the formulation as an affine parameter.

### 3-4-1 Coprime Factorization, Stable Plant

Assume  $P$  is already stable, we will parametrize  $\forall C$ 's which stabilizes the closed loop system. Let us introduce the class of all

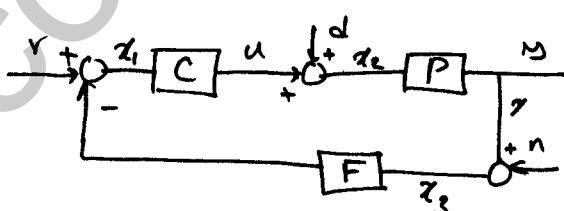
stable, proper, real-rational transfer function as  $RH^\infty$  (rational  $H^\infty$ )

(Due to the stability of these class of function  $H^\infty$  norms exists)

Note if  $F, G \in RH^\infty \Rightarrow F+G \in RH^\infty, FG \in RH^\infty$  Hence

$RH^\infty$  is a commutative ring with identity

For internal stability Consider a general feedback system as:



$$\begin{aligned} u &= \text{inputs} \begin{pmatrix} r \\ d \\ n \end{pmatrix} \\ y &= \text{outputs} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &\quad \begin{aligned} x_1 &= r - Fx_3 \\ x_2 &= d + cx_1 \\ x_3 &= n + Px_2 \end{aligned} \end{aligned}$$

The transfer matrix from  $\tilde{u}$  to  $y$  can be evaluated from

$$\begin{bmatrix} 1 & 0 & F \\ -c & 1 & 0 \\ 0 & -P & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \\ d \\ n \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = T_{yu} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

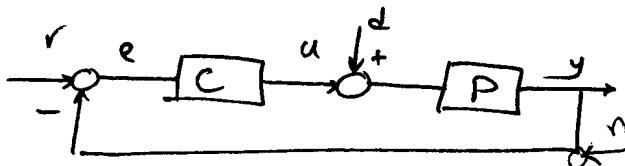
$$T_{yu} = \begin{bmatrix} 1 & 0 & F \\ -c & 1 & 0 \\ 0 & -P & 1 \end{bmatrix}^{-1} = \frac{1}{1+PCF} \begin{bmatrix} 1 & -PF & -F \\ c & 1 & -CF \\ PC & P & 1 \end{bmatrix}$$

For internal stability all 9 elements must be stable.

For unity Feed back

$$T_{Yr} = \frac{1}{1+PC} \begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix}$$

Theorem: Assume for  $P \in RH^\infty$ , The set of all  $C$ 's for which the feedback system is internally stable equals  $\{C : = \frac{Q}{1-PQ} \text{ where } Q \in RH^\infty\}$



Proof: For internal stability The Transfer Matrix  $T_{Yr}$  becomes:

$$C = \frac{Q}{1-PQ} \Rightarrow Q = \frac{C}{1+PC} \Rightarrow$$

$$T_{Yr} = \begin{bmatrix} 1-PQ & -P(1-PQ) & -(1-PQ) \\ Q & 1-PQ & -Q \\ PQ & PC(1-PQ) & 1-PQ \end{bmatrix}$$

where all entries belongs to  $RH^\infty$  □

Note1:  $S = 1 - PQ$  with coprime factorization

$$T = PQ$$

Both are affine function of  $Q$ ; Hence, Optimization of min-max problem  $\|W_s S\|_\infty$  or  $\|WT\|_\infty < 1$  are easier to obtain.

Note2: Factorization is the first part of design, we need performance measure to be added to choose from the large set  $Q \in RH^\infty$ .

For example: Stability + asymptotically tracking to a step input  $r$  (no steady state)<sup>error</sup>

$$L_S = \lim_{S \rightarrow 0} S \cdot E(S) = \lim_{S \rightarrow 0} S \cdot S(s) \cdot \frac{1}{S} = \lim_{S \rightarrow 0} S(s) = 0 \Rightarrow S(0) = 1 - P(0)Q(0) = 0$$

$$\Rightarrow P(0)Q(0) = I. \Rightarrow Q(0) = 1/P(0) \Rightarrow \{C = \frac{Q}{1-PQ}, Q \in RH^\infty, Q(0) = 1/P(0)\}$$

Ex: For  $P(s) = \frac{1}{(s+1)(s+2)}$ , characterize stabilizing controller ( $C(s)$ ) which gives zero steady-state error to unit ramp input.

$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s^2} S(s) \cdot s = 0 \Rightarrow \lim_{s \rightarrow 0} \frac{s C(s)}{s} = 0 \Rightarrow C(s) \text{ has two zeros at } s=0$$

$$Q(s) \in \mathbb{RH}^\infty, \text{ Let's suppose 1}^{\text{st}} \text{ order } Q(s) = \frac{as+b}{s+1}$$

$$S(s) = 1 - PQ = 1 - \frac{as+b}{(s+1)^2(s+2)} = \dots = \frac{s^3 + 4s^2 + (5-a)s + (2-b)}{(s+1)^2(s+2)}$$

Hence  $a=5$  and  $b=2$  For multiple zeros at  $s=0$

$$\Rightarrow Q(s) = \frac{5s+2}{s+1} \Rightarrow C(s) = \frac{Q}{1-PQ} = \frac{(5s+2)(s+1)(s+2)}{s^2(s+4)}$$

→ Note For a 2<sup>nd</sup> order Syst we found 3<sup>rd</sup> order Controller = degree of Generalized P.

### 3.4.2 Coprime Factorization: Unstable Plant

Suppose that P is not Stable and we want to find the set of Stabilizing Plants C. Write P as the ratio of coprime polynomials

$$P = \frac{N}{M}$$

where we can find two other polynomials X, Y satisfying "Bezout Identity"  $NX + MY = 1$

We claim by right choice of N, M, X, Y, the stabilizing Controller will be

$$C = \frac{X}{Y} \left/ \left( \frac{X+MQ}{Y-NQ} \right) \right.^{HII}$$

by proper choice, we mean the controller must get proper and  $Y \neq 0$

3-31

Ex:  $P(s) = \frac{1}{s}$

one choice  $N(s) = 1$ ,  $M(s) = 0$

$\rightarrow 1) Y(s) = 1, Y(s) = 0 \Rightarrow XN + YM = 1$

But  $C(s) = \frac{1}{s}$  undefined  $\rightarrow$

$\rightarrow 2) X(s) = -s+1, Y(s) = 1 \Rightarrow XN + YM = 1$

but  $C(s) = \frac{-s+1}{1}$  is non proper

Non appropriate choice

To remedy the suitable choice:

choose  $N, M, X, Y$  all  $\in RH^\infty$ , being rational, proper, and stable T.F.

Def: for  $P(s) = \frac{N}{M}$ ,  $N, M$  are coprime ( $\text{no common element}$ ) if  $\exists$

two other T.F.  $X, Y \in RH^\infty$  satisfying "Begout Identity".

$$N \cdot X + M \cdot Y = 1$$

Note: for coprimeness to hold,  $N \neq M$  can have no common zeros in  $R(s) \geq 0$

nor at the point  $s = \infty$  (proper), for this point  $s_0$

$$N(s_0)X(s_0) + M(s_0)Y(s_0) = 0 + 0 = 0 \neq 1$$

Ex:  $P(s) = \frac{1}{s-1} \Rightarrow N(s) = \frac{1}{(s+1)^k}, M(s) = \frac{(s-1)}{(s+1)^k} k \geq 1$

The simplest choice is  $N(s) = \frac{1}{s+1}, M(s) = \frac{s-1}{s+1} \in RH^\infty$

Finding  $X, Y$  is not so trivial and need some manipulation

To find  $x, y$  generally we may use Euclid's algorithm. This alg. computes the greatest common divisor of two polynomials  $n(s), m(s)$ . When  $n, m$  are coprime, the algorithm can be used to compute  $x, y$

Example:  $G(s) = \frac{1}{(s-1)(s-2)}$

$$N(s) = \frac{1}{(s+1)^2}, \quad M(s) = \frac{(s-1)(s-2)}{(s+1)^2}$$

$n(\lambda)$  is the numerator of  $N(s)$  ( $s \rightarrow \lambda$ ),  $m(\lambda)$  is the numerator of  $M(s \rightarrow \lambda) \Rightarrow$

degree  $n \geq \text{degree } m$  :

For this to happen choose a change of variable  $s = \frac{1-\lambda}{\lambda} \rightarrow \lambda = \frac{1}{1+s}$

$$G(\lambda) = \frac{\lambda^2}{6\lambda^2 - 5\lambda + 1} \rightarrow n(\lambda) = \lambda^2, \quad m(\lambda) = 6\lambda^2 - 5\lambda + 1$$

Now we are ready to give Euclid's Procedure:

→ Euclid's Algorithm

Input: polynomials  $n(\lambda), m(\lambda) \Rightarrow \text{degree}(n) \geq \text{degree } m$

Step 1: divide  $m$  into  $n$  to get quotient & remainder

$$n = mq_1 + r_1$$

Step 2: Continue for  $r_1$  into  $m$

$$m = r_1 q_2 + r_2$$

Step 3: Continue for  $r_2, r_1$

$$r_1 = r_2 q_3 + r_3$$

Continue until  $r_k$  is a nonzero constant

→ Euclid's algorithm

Now  $x, y$  are obtained by simple manipulation, for example for  $k=3$

$$\begin{cases} r_1 = n - mq_1 \\ r_2 = m - r_1 q_2 = m - nq_1 + mq_1 q_2 \\ r_3 = r_1 - r_2 q_3 = n - mq_1 - mq_3 + nq_1 q_3 - mq_1 q_2 q_3 = \text{constant} \end{cases}$$

$$r_3 = (1+q_2 q_3) n + (-q_3 - q_1(1+q_2 q_3)) m$$

$$1 = \frac{(1+q_2 q_3)}{r_3} n + \frac{-q_3 - q_1(1+q_2 q_3)}{r_3} m$$

$$\text{Hence } x = \frac{1+q_2 q_3}{r_3} \quad \leftarrow y = \frac{-q_3 - q_1(1+q_2 q_3)}{r_3}$$

### Summary of The Procedure "Coprime factorization"

Input  $G$ :

Step 1: if  $G$  is stable, Set  $N=G$ ,  $M=1$ ,  $X=0$ ,  $Y=I$ , and Stop, else

Step 2: Transform  $G(s)$  to  $G(\lambda)$  by  $s = \frac{1-\lambda}{\lambda}$ , write  $G(\lambda)$  as

a coprime of two polynomials  $G(\lambda) = \frac{n(\lambda)}{m(\lambda)}$

~ Step 3: Using Euclid's algorithm, find  $x(\lambda), y(\lambda) \Rightarrow$

$$nx + my = 1$$

Step 4: Transform  $n(\lambda), m(\lambda), x(\lambda), y(\lambda)$  to  $N(s), M(s), X(s), Y(s)$

by  $\lambda = \frac{1}{s+1} \Rightarrow$  The stabilizing controller is  $C = \frac{X(s)}{Y(s)}$ .

Example:

$$G(s) = \frac{1}{(s-1)(s-2)}$$

$$G(\lambda) = \frac{\lambda^2}{6\lambda^2 - 5\lambda + 1}$$

$$n(\lambda) = \lambda^2$$

$$m(\lambda) = 6\lambda^2 - 5\lambda + 1$$

$$q_1(\lambda) = \frac{1}{6}$$

$$r_1(\lambda) = \frac{5}{6}\lambda - \frac{1}{6}$$

$$q_2(\lambda) = \frac{36}{5}\lambda - \frac{114}{25}$$

$$r_2(\lambda) = \frac{6}{25}$$

$$\left\{ \begin{array}{l} r_1 = n - mq_1 \\ r_2 = m - r_1 q_2 = m - nq_2 + mq_1 q_2 \end{array} \right.$$

$$r_2 = -nq_2 + m(1+q_1 q_2) \Rightarrow 1 = n\left(-\frac{q_2}{r_2}\right) + m\frac{(1+q_1 q_2)}{r_2}$$

$$\left\{ \begin{array}{l} x(\lambda) = \frac{-q_2(\lambda)}{r_2(\lambda)} = -30\lambda + 19 \\ y(\lambda) = \frac{1+q_1 q_2(\lambda)}{r_2(\lambda)} = 5\lambda + 1 \end{array} \right.$$

$$\left\{ \begin{array}{l} n(\lambda) = \lambda^2 \\ m(\lambda) = 6\lambda^2 - 5\lambda + 1 \\ x(\lambda) = -30\lambda + 19 \\ y(\lambda) = 5\lambda + 1 \end{array} \right.$$

$$\lambda = \frac{1}{s+1}$$

$$\left\{ \begin{array}{l} N(s) = \frac{1}{(s+1)^2} \\ M(s) = \frac{(s-1)(s+2)}{(s+1)^2} \\ X(s) = \frac{19s-11}{s+1} \\ Y(s) = \frac{s+6}{s+1} \end{array} \right.$$

$$- stabilizing Controller : C(s) = \frac{X(s)}{Y(s)} = \frac{19s-11}{s+6}$$

### 3-4-4/ Controller Parametization

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until now, we found one  $C(s) = \frac{X(s)}{Y(s)}$  by means of coprime factorization which we claimed would stabilize the plant, Now the set of all stabilizing plants will be:

Theorem 2: The set of all  $C(s)$ 's for which the feedback system is stable internally

equals:  $\mathcal{C} : \left\{ \frac{X+MQ}{Y-NQ} : Q \in RH^\infty \right\}$

Note: for stable plants we have theorem 1: by  $N=P, M=1, X=0, Y=I \Rightarrow$

$$\frac{X+MQ}{Y-NQ} = \frac{Q}{1-PQ}$$

The benefit of this factorization, in addition to find all stabilizing plants controllers for a general plant is, that  $S + T$  becomes a fine function of  $Q$

$$S = \frac{PC}{1+PC} \quad \text{and} \quad S = \frac{1}{1+PC}$$

$$S = \frac{1}{1 + \frac{N}{M} \cdot \frac{X+MQ}{Y-NQ}} = \frac{M(Y-NQ)}{My - MNQ + Nx + NMQ} = \frac{M(Y-NQ)}{Nx + My}$$

From Bezout identity

$$S = M(Y-NQ) \quad \text{affine function of } Q \in RH^\infty$$

$$\text{similarly } T = N(X+MQ) \quad //$$

We will use these important property in next chapter for controller design

Until now we present away to classify all stabilizing controller for nominal stability; other than that we require performance indices to come into picture. But while the set of all stabilizing controller is parametric which is an  $\infty$  set, we can choose ones, which perform as we wish;

Look at an example:

Ex: For  $P(s) = \frac{1}{(s-1)(s+2)}$  characterize all stabilizing controller

- ⇒ 1) the steady state error for a unit step error is 0
- 2) The steady state error for a sinusoidal input disturbance d with frequency 10 rad/sec is zero.

From previous example we found coprime factorization of the plant  $\Rightarrow$

$$C(s) = \frac{X + MQ}{Y - NQ} \quad \text{in the set of all stabilizing controller } \forall Q \in \mathbb{RH}^\infty$$

By Final Value Theorem for 1) to hold  $T(0) = 1$ . or  $S(0) = 0$ .

$$\Rightarrow T(0) = N(0) [X(0) + M(0)Q(0)] = 1$$

$$\text{For } N(s) = \frac{1}{(s+1)^2}, M(s) = \frac{(s-1)(s-2)}{(s+1)^2}, X(s) = \frac{19s-11}{s+1}; Y(s) = \frac{s+6}{s+1}$$

$$1 = 1 (-11 + 2 Q(0)) \Rightarrow \boxed{Q(0) = 6} \quad (1)$$

$$\text{For 2) to hold } P(0)S(10j) = 0 \Rightarrow S = M(Y - NQ) : \frac{Y(s)}{N(s)} = P.S$$

$$M(10j) [Y(10j) - N(10j)Q(10j)] = 0$$

$$\Rightarrow Q(10j) = \frac{Y(10j)}{N(10j)} = \frac{(6+10j)(1+10j)}{1} = \boxed{-94 + 70j} \quad (2)$$

every  $Q \in \mathbb{RH}^\infty$  which satisfies (1) & (2) is a solution to this problem

To find admissible  $Q \in \mathbb{RH}^\infty$  we use a polynomial function

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of  $\frac{1}{s+1}$  for example

$$Q(s) = q_1 + q_2 \frac{1}{s+1} + q_3 \frac{1}{(s+1)^2}$$

$$Q(0) = 6 \Rightarrow q_1 + q_2 + q_3 = 6$$

$$Q(10j) = -94 + 70j \dots \begin{cases} -99q_1 + q_2 + q_3 = 99 \times 94 - 20 \times 70 \\ 20q_1 + 10q_2 = -99 \times 70 - 94 \times 20 \end{cases}$$

Solve for  $q_1, q_2, q_3$

$$\Rightarrow q_1 = -79; q_2 = -723; q_3 = 808$$

$$\Rightarrow Q(s) = \frac{-79s^2 - 881s + 6}{(s+1)^2}$$

$$\dots C(s) = \frac{X + MQ}{Y - NQ} = \dots = \frac{-60s^4 - 598s^3 + 2515s^2 - 1774s + 1}{s(s^2 + 100)(s + 9)}$$

Note that the controller is unstable & nonminimum phase itself.

As far as the feedback exists this doesn't cause any problem, but many practicing engineers like to have stable controller, especially for stable plants. we call this as:

Def: A plant is "Strongly Stabilizable" if internal stability will be achieved with a Stable Controller  $C(s)$ .

Theorem 3:  $P$  is strongly stabilizable iff it has an even number of real poles between every pair of real zeros in  $\text{Re } s \geq 0$

Readout from book to establish the procedure.

### 3-4-3/ Coprime Factorization by State-space

use the notation

$$G(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \Rightarrow G(s) = D + C(SI - A)^{-1}B$$

we need to find  $G = \frac{N}{M} \Leftrightarrow X, Y \Rightarrow NX + MY = 1$  "Begout Identity"

From State-space Model of a Syst

$$\begin{cases} \dot{X} = Ax + Bu \\ Y = cx + du \end{cases}$$

choose a real matrix  $F \Rightarrow A + BF$  is stable

Define  $v := u - Fx$  then

$$\dot{x} = (A + BF)x + Bv$$

$$u = Fx + v$$

$$Y = (C + DF)x + DV$$

$u = Fx + v$  external input  
state F.b.

then

$$M(s) := \left[ \begin{array}{c|c} A + BF & B \\ \hline F & 1 \end{array} \right] = \frac{U(s)}{V(s)}$$

$$N(s) := \left[ \begin{array}{c|c} A + BF & B \\ \hline C + DF & D \end{array} \right] = \frac{Y(s)}{V(s)}$$

Therefore,  $U(s) = M(s) \cdot V(s) \quad \& \quad Y(s) = N(s) \cdot V(s)$

$$\Rightarrow Y(s) = N(s) \cdot M^{-1}(s) \cdot U(s) \quad \text{that is } G(s) = \frac{N(s)}{M(s)} \left( N(s) M^{-1}(s) \right)$$

and  $N$  &  $M$  are proper, then they are stable because  $A + BF$  is stable

Hence  $N, M \in \mathbb{RH}^\infty$ . Similarly  $X(s) \& Y(s)$  can be found from

$$X(s) := \left[ \begin{array}{c|c} A + HC & H \\ \hline F & 0 \end{array} \right]$$

$$Y(s) := \left[ \begin{array}{c|c} A + HC & -B - HD \\ \hline F & 1 \end{array} \right]$$

where  $H$  is a real matrix  $\Rightarrow A + HC$  is stable.

3-4-3 | Cont'd

State-space Methods for Coprime factorization (MIMO)

Theorem: Suppose  $P(s)$  is a proper & real rational Transfer matrix and

$$P = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

is a stabilizable & detectable realization. Let  $F$  and  $L$  be such that  $A+BF$  and  $A+LC$  are both stable, and define

$$\left[ \begin{array}{cc} M & -\tilde{u} \\ N & \tilde{v} \end{array} \right] = \left[ \begin{array}{c|cc} A+BF & B & -L \\ F & I & 0 \\ CDF & D & I \end{array} \right]$$

and

$$\left[ \begin{array}{cc} \tilde{v} & u \\ -\tilde{N} & \tilde{M} \end{array} \right] = \left[ \begin{array}{c|cc} A+LC & -(B+LD) & L \\ F & I & 0 \\ C & -D & I \end{array} \right]$$

Then  $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$  are rcf & lcf, respectively and,

further  $\left[ \begin{array}{cc} v & u \\ -\tilde{N} & \tilde{M} \end{array} \right] \left[ \begin{array}{cc} M & -\tilde{u} \\ N & \tilde{v} \end{array} \right] = \left[ \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right] = I$

(Double c.f. Bezout Identity)

Proof: we can easily verify that D.c.f. Bezout Identity holds

Note: if  $P$  is stable, then we can take  $v=\tilde{v}=I$   $u=\tilde{u}=0$

$$N=\tilde{N}=P, \quad M=\tilde{M}=I$$

Note: The coprime factorization of a transfer matrix can be given a feedback control interpretation

Therefore with the stabilizing controller  $K$  we can find

$$G \text{ or } P = NM^{-1} = \tilde{M}^{-1} \tilde{N}$$

$$K = UV^{-1} = \tilde{V}^{-1} \tilde{U}$$

→ No Need

Lemma: Consider the feedback system in fig(2)

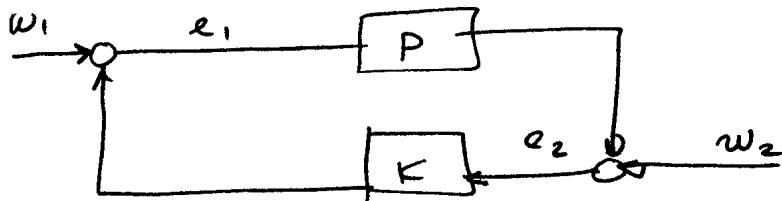


Fig (2)

The following conditions are equivalent:

- 1) The feedback system is internally stable.
- 2)  $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$  is invertible in  $RH_\infty$
- 3)  $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$  is invertible in  $RH_\infty$
- 4)  $\tilde{M}V - \tilde{N}U$  is invertible in  $RH_\infty$
- 5)  $\tilde{M}M - \tilde{U}N$  is invertible in  $RH_\infty$

Proof in page 74 Zhou Book.

Corollary: Let  $P$  be a proper real rational matrix and  $P = NM^{-1} = \tilde{M}^{-1} \tilde{N}$  be the corresponding rcf & lcf over  $RH_\infty$

Then there exists a controller  $K = UV^{-1} = \tilde{V}^{-1} \tilde{U}$  with

$U, V, \tilde{U}, \tilde{V} \in RH_\infty \Rightarrow$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = I$$

## Procedure

1) Get a stabilizable & detectable realization for  $P$

$$P = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \Rightarrow P(s) = D + C(sI - A)^{-1}B$$

2) Find stabilizable controller & observer gains  $F, L$ , respectively  
either using lqr & lqe OR place routines

$$F = -\text{lqr}(A, B, I_{nxn}, J_{mxm})$$

OR

$$F = -\text{place}(A, B, P_f) \quad \therefore P_f \text{ is the desired pole of } A+BF$$

Similarly

$$F = -\text{lqr}(A', C', I_{nxn}, J_{pxp}) \text{ OR}$$

$$= -\text{lqe}(A, C, \alpha, \beta) \text{ OR}$$

$$= -\text{place}(A', C', P_I) \quad \therefore P_i \text{ poles of } A+LC$$

Then find the coprime factors by

$$M(s) = \left[ \begin{array}{c|c} A+BF & B \\ \hline F & I \end{array} \right]$$

$$\tilde{M}(s) = \left[ \begin{array}{c|c} A+LC & L \\ \hline C & I \end{array} \right]$$

$$N(s) = \left[ \begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array} \right]$$

and

$$\tilde{N}(s) = \left[ \begin{array}{c|c} A+LC & B+LD \\ \hline C & D \end{array} \right]$$

$$U(s) = \left[ \begin{array}{c|c} A+LC & L \\ \hline F & 0 \end{array} \right]$$

$$\tilde{U}(s) = \left[ \begin{array}{c|c} A+BF & L \\ \hline F & 0 \end{array} \right]$$

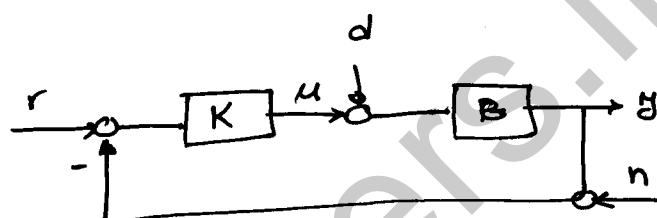
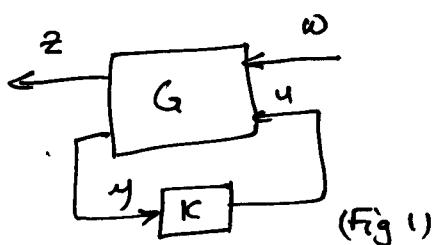
$$V(s) = \left[ \begin{array}{c|c} A+LC & -(B+LD) \\ \hline F & I \end{array} \right]$$

$$\tilde{V}(s) = \left[ \begin{array}{c|c} A+BF & -L \\ \hline C+DF & I \end{array} \right]$$

## Stabilizing Controllers:

In this section we occupy our attention to systematically characterizing controllers, which make the closed loop system, internally stable. State-Space representation is used here to have generalized MIMO systems and based.

Consider the generalized plant Model compared to physical implementation Block diagram



To introduce the well-posedness of the controller Consider the simple example in which  $G(s) = \frac{1-s}{s+2}$ ;  $K=1$  are two causal and proper system, However

$$u = K(r - n - y) ; y = G(d + u)$$

$$u = K(r - n) - KG(d + u) \rightarrow u = \frac{K}{1+KG}(r - n) - \frac{KG}{1+KG}d$$

$$u = \frac{1}{1 + \frac{1-s}{s+2}}(r - n) - \frac{\frac{1-s}{s+2}}{\frac{3}{s+2}}d$$

$$u = \frac{s+2}{3}(r - n) - \frac{(1-s)}{3}d !$$

both transfer functions from  $r, n$  to  $u$  & from  $d$  to  $u$  are non proper and hence the controller  $K$  is not well-posed.

Def: A feedback system is well-posed, if all closed-loop transfer matrices are proper and well-def.

Lemma 1: The feedback system in Fig (2) is well-posed, iff

(2)

$$I - K(\infty) G(\infty)$$

is invertible.

More generally in state-space representation if the System  $G$  is represented with

$$\begin{aligned} \dot{x}(t) &= Ax(t) + [B_1 \quad B_2] \begin{bmatrix} w(t) \\ u(t) \end{bmatrix} \\ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(t) + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} w(t) \\ u(t) \end{bmatrix}; \end{aligned}$$

$$\Rightarrow G = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right]$$

$$\Rightarrow G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

and the controller  $K$  is represented with

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\ u(t) &= C_K x_K(t) + D_K y(t) \end{aligned}$$

$$\Rightarrow K = \left[ \begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$$

Then Lemma 1 is translated to

prop 1. The connection of  $G$  &  $K$  in fig(1) is well-posed : iff

$$I - D_{22} D_K$$

is non singular

proof: write the overall state-space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ \dot{z}(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \end{cases}$$

$$\dot{x}_K(t) = A_K x_K(t) + B_K y(t)$$

and

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_K(t) \end{bmatrix} + \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} w(t)$$

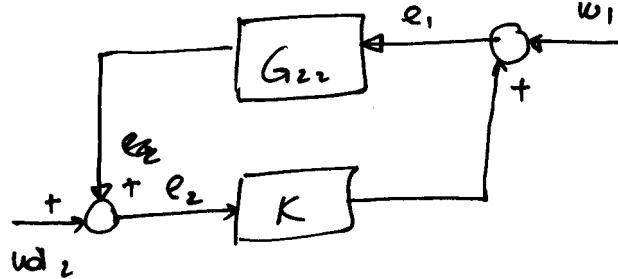
this can be welldefined iff,  $I - D_{22} D_K$  is non singular

(Dull RFT) Note: if either  $D_{21}$  or  $D_K = 0$  (strictly proper)  $\Rightarrow$  well-posed

Def 2. The Feedback System in Fig(1) or fig(2) is internally stable, if it is well-posed and for every initial condition  $x(0)$  of  $h$ , and  $x_c(0)$  of  $K$

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x_c(t) = 0 \quad \text{when } \omega = 0.$$

This can be immediately checked if for the system



$$\begin{cases} e_1 = \omega_1 + K e_2 \\ e_2 = \omega_2 + G_{22} e_1 \end{cases}$$

$$\Rightarrow \begin{cases} e_1 - K e_2 = \omega_1 \\ e_2 - G_{22} e_1 = \omega_2 \end{cases}$$

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

Lemma 2. The system in Figure (3) is internally stable, if the

transformation from  $(\omega_1, \omega_2)$  to  $(e_1, e_2)$  belongs to  $RH_\infty$ .

$$T_{\omega \rightarrow e} = \begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KG_{22})^{-1} & K(I - G_{22}K)^{-1} \\ G_{22}(I - KG_{22})^{-1} & (I - G_{22}K)^{-1} \end{bmatrix} \text{ belongs to } RH_\infty$$

but using Matrix Inversion Lemma this has different other representation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\Delta^{-1}CA^{-1} & -A^{-1}B\Delta^{-1} \\ -\Delta^{-1}CA^{-1} & \Delta^{-1} \end{bmatrix} \text{ where } \Delta = D - CA^{-1}B$$

$$= \begin{bmatrix} \hat{\Delta}^{-1} & -\hat{\Delta}^{-1}BD^{-1} \\ -D^{-1}C\hat{\Delta}^{-1} & D^{-1} + D^{-1}C\hat{\Delta}^{-1}BD^{-1} \end{bmatrix} \text{ where } \hat{\Delta} = A - BD^{-1}C$$

Note for simple case

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -D^{-1}C A^{-1} & D^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$$

hence other more difficult forms for  $T_{ew}$  is as follows

$$T_{ew} = \begin{bmatrix} I & K \\ -G_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} I + K(I - G_{22}K)^{-1}B_{22} & K(I - G_{22}K)^{-1} \\ (I - G_{22}K)^{-1}G_{22} & (I - G_{22}K)^{-1} \end{bmatrix} \quad \text{from } \alpha$$

$$= \begin{bmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ G_{22}(I - KG_{22})^{-1} & I + G_{22}(I - KG_{22})^{-1}K \end{bmatrix} \quad \text{from } \delta$$

in which using the simplest formulations in each case results into

$$T_{ew} = \begin{bmatrix} (I - KG_{22})^{-1} & K(I - G_{22}K)^{-1} \\ G_{22}(I - KG_{22})^{-1} & (I - G_{22}K)^{-1} \end{bmatrix} \quad \text{as before.}$$

**Proposition 2.** Suppose that  $(A, B_2, G)$  is stabilizable & detectable, Then Interconnected System in Fig(3) is internally stable, iff, the transfer matrix of  $\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \tilde{\omega}$  to  $\xi = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$  is in  $\mathbb{R}H^\infty$ .

Note that the state-space representation can be found by

$$1) A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_C \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_C \end{bmatrix} \begin{bmatrix} I & -D_C \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_C \\ C_2 & 0 \end{bmatrix}$$

$$2) T_{ew}^{(S)} = \bar{D}^{-1} \begin{bmatrix} 0 & C_C \\ C_2 & 0 \end{bmatrix} (I_S - A_{cl})^{-1} \begin{bmatrix} B_2 & 0 \\ 0 & B_C \end{bmatrix} \bar{D}^{-1} + \bar{D}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$$

in which  $\bar{D} = \begin{bmatrix} I & -D_C \\ -D_{22} & I \end{bmatrix}$

**Proposition 3.** The System in Fig(1) is internally stable iff,  $I - D_{22}D_C$  is invertible &  $A_{cl}$  as given above is Hurwitz.

Proposition 4. Nec. & Suf. condition for the existence of an internally ⑤ stabilizing controller for Fig1 is that  $(A_1, B_2, C_2)$  is stabilizable & detectable. In that case, one such controller is given by

$$K(s) = \begin{bmatrix} A + B_2 F + L C_2 + L D_{22} F & -L \\ F & 0 \end{bmatrix}$$

where  $F$  &  $L$  are matrices such that  $A + B_2 F$  &  $A + L C_2$  are Hurwitz.

( $F$  is an state feed back Controller gain &  $L$  is a state observer gain )

Note: that the above controller is an output-Controller for the System or a state-feedback / state observer controller ( $u = K \hat{x}$ ) in which a stabilizing controller is added to a stable observer. the closed-loop eigenvalues of the system is exactly that of  $(A + B_2 F)$

&  $(A + L C_2)$  by separation theorem.

Note 2: This controller is not LQR/LQG and hence not optimal.

→ Full information state-feedback stabilization via LMI's

Consider full state-feedback of a system as:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = I \cdot x \end{cases} \quad (C = I \text{ & } D = 0 \text{ for full state information available})$$

$$u(t) = F \cdot y(t)$$

and

for convenience we assume there is no dynamics in the controller and  $D_K = F$ . for such case we have

$$A_C = A + B_2 F$$

$F$  exists if  $(A, B_2)$  is stabilizable

Consider Lyapunov inequality for stability analysis

(6)

if  $A_{CL}X + XA_{CL}^* < 0$

has a solution  $X > 0 \Rightarrow$  syst is internally stable.

Substitute  $(A_{CL} = A + B_2 F)$

$$(A + B_2 F)X + X(A + B_2 F)^* < 0$$

$$AX + B_2(FX) + XA^* + (XF^*)B_2^* < 0$$

The above inequality is not an LMI in  $F$  &  $X$ , since their products appears  
Let's introduce  $Z = FX$  by this change of variable this results  
into an LMI

$$AX + B_2 Z + XA^* + Z^*B_2^* < 0$$

Theorem 1) If a full-state feedback  $D_K = F$ , stabilizes Fig(1)

If,  $\exists$  matrices  $X > 0$  &  $Z$  such that  $F = Z^{-1}$

and  $[A \quad B_2] \begin{bmatrix} X \\ Z \end{bmatrix} + [X \quad Z^*] \begin{bmatrix} A^* \\ B_2^* \end{bmatrix} < 0$  is satisfied

which is an LMI Condition for stability

→ Stabilization problem : An LMI approach

Now generalize previous approach for any (fixed-order) controller  $K : (A_K, B_K, C_K, D_K)$ , and impose the closed-loop matrix  $A_{CL}$  be Hurwitz by means of Lyapunov inequality

$$\lambda_{CL} > 0, A_{CL} \lambda_{CL} + \lambda_{CL} A_{CL}^* < 0$$

Can we compute  $\lambda_{CL}, A_K, B_K, C_K, D_K$  from the above equation tractably?

Theorem 2: Consider the system of Fig(1) with  $D_{22}=0$ , There exists a controller of order  $n_K$ , which internally stabilizes the system if and only if,  $\exists n \times n$  matrices  $X > 0 \in Y > 0 \Rightarrow$

$$N_0^*(A^*X + XA) N_0 \leq 0 \quad (1)$$

$$N_C^*(AY + YA^*) N_C \leq 0 \quad (2)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \text{ and} \quad (3)$$

$$\text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + n_K \quad (4)$$

where  $N_0$  &  $N_C$  are full column rank matrices such that

$$\text{image of } N_0 \rightarrow \text{Im } N_0 = \text{Ker } C_2 \quad (\text{Kernel } C_2)$$

$$\text{Im } N_C = \text{Ker } B_2^*$$

Note if  $A = U \Sigma V^*$  is singular value decomposition of

then  $\text{Im } A = \text{Im } [u_1, \dots, u_r]$  and

$$\text{Ker } A = \text{Im } [v_{r+1}, \dots, v_n]$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}$$

Note: (1) & (2) are equivalent to detectability and stabilizability of  $(A_2, B_2, C_2)$ , respectively. (2) is equivalent to the result of full-state LTI condition for Controller.

Note 2: Since (1), (2) are homogeneous in the unknowns, the solutions can always be scaled up to satisfy (3). This means that if  $\exists$  constant in the order  $n_k$ , the existence of a stabilizer  $\Leftrightarrow$  the detectability & stabilizability of  $(A_2, B_2, C_2)$ , consistent with previous analysis.

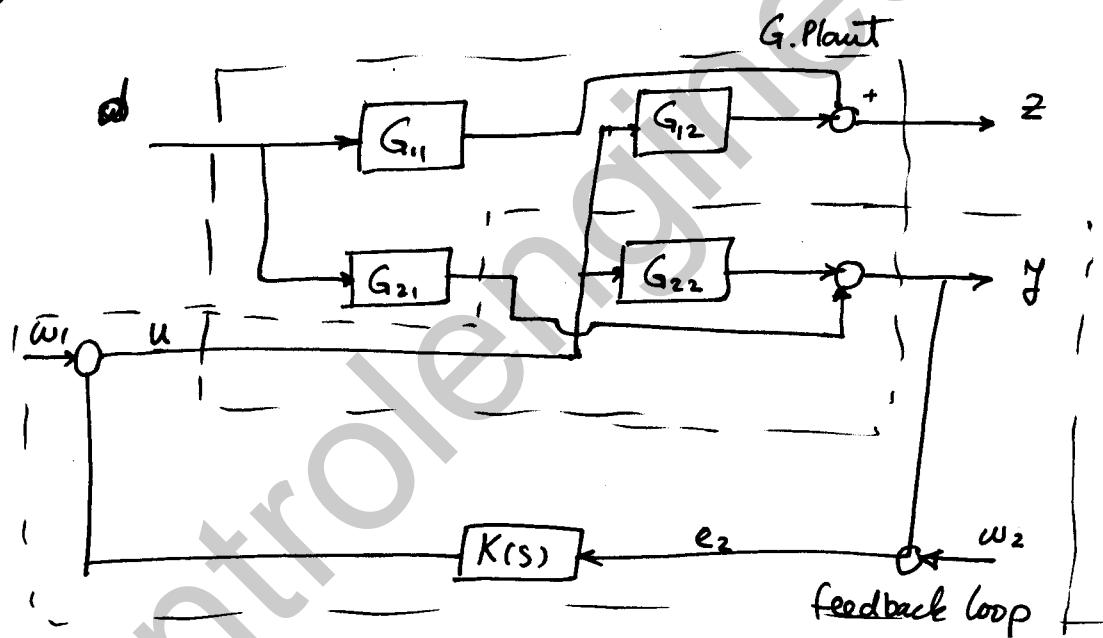
N3: rank condition (4) is not felt when  $n_k > n \Rightarrow$  we never need controller with order  $> n$  to stabilize.  
this condition is not convex and makes the computation of  $X, Y$  more difficult  
But the above analysis gives good insight into the role of controller order  
 $\Rightarrow \exists$  always a  $n_k = n$  degree controller to stabilize if  $(A_2, B_2, C_2)$  is detectable and stabilizable.

## Parametrization of Stabilizing Controllers

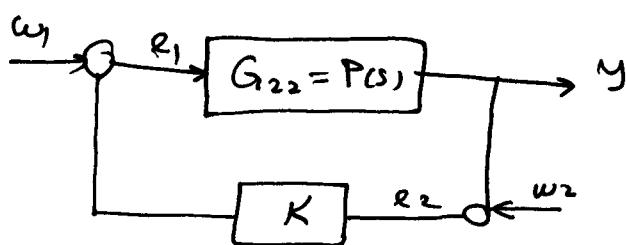
The parametrization of controller will be strongly based on transfer function methods, and is basically giving a parametrization of  $K$  with respect to  $Q \in \mathbb{R}^{H_\infty}$  which makes all the elements of  $T_{\infty} \in \mathbb{R}^{H_\infty}$ . But generally in MIMO Case and with state-space formulations

### Coprime factorization:

The main issue to parameterize the controller is the well-posedness of the closed-loop system, for simplicity in notation use  $G_{22}(s) = P(s)$  and recall that, for the overall system :



for internal Stability (for  $G_{22} = P(s)$ ) we must have  
(ignore  $d$  &  $e_2$  and consider only



$$\begin{cases} e_1 = \omega_1 + K e_2 \\ e_2 = \omega_2 + P e_1 \end{cases} \rightarrow \begin{cases} e_1 - K e_2 = \omega_1 \\ -P e_1 + e_2 = \omega_2 \end{cases} \rightarrow \begin{bmatrix} I & -K \\ -P & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$T_{\omega\omega} = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} \in \mathbb{R}H_\infty \text{ for internal stability}$$

And as we derived earlier

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - KP)^{-1} \\ (I - PK)^{-1} P \\ = P(I - KP)^{-1} \end{bmatrix} \left[ \begin{array}{l} (I - KP)^{-1} K \in \\ K(I - PK)^{-1} \\ (I - PK)^{-1} \end{array} \right] \in \mathbb{R}H_\infty$$

all elements

to parameterize the stabilizing controller, Consider the following

left + right Coprime factorization for  $P(s)_{m \times n}$  possibly unstable real-rational trans matrix

$$P(s) = \underset{m \times n}{N(s)} \underset{m \times n}{M(s)}^{-1} = \underset{n \times n}{\tilde{M}(s)}^{-1} \underset{m \times m}{\tilde{N}(s)}$$

where  $N(s)$ ,  $M(s)$ ,  $\tilde{N}(s)$  &  $\tilde{M}(s)$  are all in  $\mathbb{R}H_\infty$

Corresponding to the above Coprime factors of  $P(s)$   $\Rightarrow$  a Bezout Identity Solution for MIMO Case:  $\exists X_r, Y_r, X_L, Y_L \in \mathbb{R}H_\infty \Rightarrow$

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = X_r M + Y_r N = I$$

$$\begin{bmatrix} \tilde{M} & \tilde{N}(s) \end{bmatrix} \begin{bmatrix} X_L \\ Y_L \end{bmatrix} = \tilde{M} X_L + \tilde{N} Y_L = I$$

or integrate in its Doubly Coprime Bezout Identity:

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M(s) & -Y_L \\ N(s) & X_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

For Right Coprime factorization

$$P(s) = N(s) \tilde{M}^{-1}(s) \quad \text{right-coprime factorization}$$

$$U(s)N(s) + V(s)M(s) = I \quad \text{Bogout Identity}$$

claim :  $K(s) = -V(s) \cdot U(s)$  via stabilizing controller

$$\text{(or generally } K(s) = - (V(s) + Q(s)\tilde{N}(s))^{-1}(U(s) - Q(s)\tilde{M}(s)) \text{)}$$

$Q(s) \in \mathbb{R}_{\infty}$

For left coprime factorization

$$P(s) = \tilde{M}^{-1}(s) \tilde{N}(s)$$

$$N(s)\tilde{U}(s) + M(s)\tilde{V}(s) = I$$

claim :  $K(s) = -(\tilde{U}(s) - M(s)Q(s))(\tilde{V}(s) + N(s)Q(s))^{-1}$

$Q(s) \in \mathbb{R}_{\infty}$  transforms

stabilizes the closed loop syst.

Proof (1,1) : Right CF +  $K(s) = -V(s) \cdot U(s)$

The closed loop Transfer function consists of four elements:

$$\begin{aligned} 1) \quad (I - KP)^{-1} &= (I + V(s)U(s)N(s)\tilde{M}^{-1}(s))^{-1} \\ &= [V(s) \{ V(s) + U(s)N(s)\tilde{M}^{-1}(s) \}]^{-1} \\ &= [V(s) \{ V(s)M(s) + U(s)N(s) \} \tilde{M}^{-1}(s)]^{-1} \\ &= [V(s) \{ I \} \tilde{M}^{-1}(s)]^{-1} \\ &= [V(s) \tilde{M}^{-1}(s)]^{-1} \\ &= M(s)V(s) \in \mathbb{R}_{\infty} \end{aligned}$$

(12)

Similarly

$$(1,1) \quad (I - KR)^{-1}K = M(s)V(s) \left[ -V(s)U(s) \right] \\ = -M(s).U(s) \in RH_\infty$$

$$(2,1) \quad (I - PK)^{-1}P = P(I - KP)^{-1} = N(s)M^{-1}(s)(M(s)V(s)) \\ = N(s)V(s) \in RH_\infty$$

$$(2,2) \quad (I - PK)^{-1} = I + P(I - KP)^{-1}K = I \leftarrow N M^{-1}(s) M(s)V(s)V(s)U(s) \\ = I - N(s)U(s) \in RH_\infty$$

Now for general controller:

$$K(s) = -(V + Q\tilde{N})^{-1}(U - Q\tilde{M})$$

$$(I - KP)^{-1} = \left\{ I + (V + Q\tilde{N})^{-1}(U - Q\tilde{M})(NM^{-1}) \right\}^{-1}$$

$$= \left\{ (V + Q\tilde{N})^{-1} \left\{ (V + Q\tilde{N}) + (U - Q\tilde{M})NM^{-1} \right\} \right\}^{-1}$$

$$= \left\{ (V + Q\tilde{N})^{-1} \left[ (V + Q\tilde{N})M + (U - Q\tilde{M})N \right] \tilde{M}^{-1} \right\}^{-1}$$

$$= \left\{ (V + Q\tilde{N})^{-1} \left[ \underbrace{VM + UN}_{I \rightarrow \text{from Bez. id (1,1)}} + Q \underbrace{(\tilde{N}M - \tilde{M}N)}_{0} \right] \tilde{M}^{-1} \right\}^{-1}$$

$$= M(V + Q\tilde{N}) \in RH_\infty$$

Furthermore this is an affine form w.r.t. Q

Similarly

$$H_{12} = (I - KP)^{-1}K = -M(U - Q\tilde{M}) \in RH_2$$

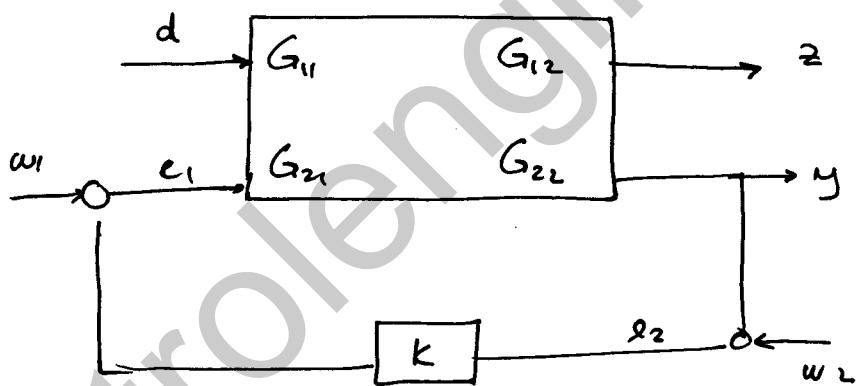
$$H_{21} = (I - PK)^{-1}P = P(I - KP)^{-1} = N(V + Q\tilde{N}) \in RH_2$$

$$H_{22} = I + P(I - KP)^{-1}K = I - N(U - Q\tilde{M}) \in PH_2$$

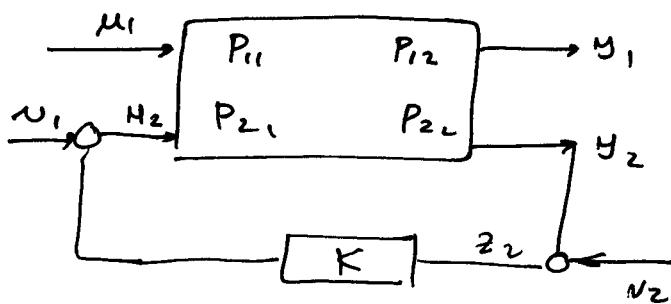
$$T_{ew} = \begin{bmatrix} P(V + Q\tilde{N}) & -M(U - Q\tilde{M}) \\ N(V + Q\tilde{N}) & I - N(U - Q\tilde{M}) \end{bmatrix} \in RH_2$$

and affine fraction of  $Q$ .

Now consider the full System



To make the notation easier to proceed



First take  $u_1 = 0$  but take  $y_1$  into account

$$\begin{aligned}
 Y_{11} &= P_{11} u_1 + P_{12} u_2 = P_{12} u_2 \\
 &= P_{12} [I \ 0] \begin{bmatrix} u_2 \\ z_2 \end{bmatrix} \in \begin{bmatrix} u_2 \\ z_2 \end{bmatrix} = T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
 &= P_{12} [I \ 0] \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
 &= P_{12} [T_{11} \quad T_{12}] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
 &= [P_{12} M(v+Q\tilde{N}) \quad -P_{12} M(u-Q\tilde{M})] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
 \end{aligned}$$

for complete stability

$$[P_{12} M(v+Q\tilde{N}) \quad -P_{12} M(u-Q\tilde{M})] \in RH_\infty$$

Assume that this is the case then

$$P_{12} M(v+Q\tilde{N}) + P_{12} M(u-Q\tilde{M}) \in RH_\infty$$

$$P_{12} M(v+Q\tilde{N})M + P_{12} M(u-Q\tilde{M})N \in RH_\infty$$

$$P_{12} M \{ (v+Q\tilde{N})M + (u-Q\tilde{M})N \} \in RH_\infty$$

Bezout ID.

$$\Rightarrow \boxed{P_{12} M \in RH_\infty} \quad \text{the converse is also true.}$$

Similarly if we set  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$  to get the other Transfer functions

$$Y_1 = P_{11} u_1 + P_{21} u_2$$

$$\left. \begin{aligned}
 Y_2 = Z_2 &= P_{21} u_1 + P_{22} u_2 \\
 u_2 &= K_{22}
 \end{aligned} \right\} \quad \begin{aligned}
 Z_2 &= P_{21} u_1 + P_{22} K_{22} \\
 Z_2 &= (I - P_{22} K)^{-1} P_{21} u_1
 \end{aligned}$$

$$\Rightarrow u_2 = K(I - P_{22}K)^{-1}P_{21}u_1$$

$$y_1 = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]u_1$$

$$\Rightarrow \text{Must have } \begin{bmatrix} P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \\ K(I - P_{22}K)^{-1}P_{21} \\ (I - P_{22}K)^{-1}P_{21} \end{bmatrix} \in RH_\infty \quad (*)$$

$$\text{But } P_{22} = \tilde{M}^{-1}\tilde{N}$$

$$\& K = -(\tilde{\alpha} - MQ)(\tilde{V} + NQ)^{-1}$$

$$(I) \Rightarrow (I - P_{22}K)^{-1} = ((\tilde{V} + NQ)\tilde{M})$$

$$(II) \Rightarrow K(I - P_{22}K)^{-1} = -(\tilde{\alpha} - MQ)\tilde{M}$$

$$\Rightarrow (*) = \begin{bmatrix} P_{11} + P_{12}(MQ - \tilde{\alpha})\tilde{M}P_{21} \\ (MQ - \tilde{\alpha})\tilde{M}P_{21} \\ (\tilde{V} + NQ)\tilde{M}P_{21} \end{bmatrix} \in RH_\infty$$

Similar to previous derivation

$$(MQ - \tilde{\alpha})\tilde{M}P_{21} + (\tilde{V} + NQ)\tilde{M}P_{21} \in RH_\infty$$

$$= \dots = \boxed{\tilde{M}P_{21} \in RH_\infty}$$

$$\text{and } P_{11} + P_{12}(MQ - \tilde{\alpha})\tilde{M}P_{21} =$$

$$= (P_{11} - P_{12}\tilde{\alpha}\tilde{M}P_{21}) + P_{12}MQ\tilde{M}P_{21}$$

$$\Rightarrow \boxed{P_{11} - P_{12}\tilde{\alpha}\tilde{M}P_{21} \in RH_\infty}$$

Thm: For a General LFT Configuration,  $K$  stabilizes  $P$  iff

a)  $P$  satisfies admissibility conditions ① to ③

$$\textcircled{1} \quad T_{12} \triangleq P_{12} M \in \mathbb{R}^{H_\infty}$$

$$\textcircled{2} \quad T_{21} \triangleq P_1 P_{21} \in \mathbb{R}^{H_\infty}$$

$$\textcircled{3} \quad T_{11} \triangleq P_{11} - P_{12} \tilde{M} P_{21} \in \mathbb{R}^{H_\infty}$$

b)  $K$  stabilizes  $P_{22}$

Note 1: The Coprime factorization development is carried out only for  $P_{22}$  and not the other components of  $P$  namely  $P_{11}, P_{12}$ , and  $P_{21}$ . But admissibility conditions ① to ③ is checked for these three components. It's nice that if admissible conditions holds for one derived  $M$  (Coprime factorization) it holds for others, too.

Note 2: The  $H_\infty$  performance optimization for the LFT Configuration

min  $\|T_{11y}\|_\infty$  for all stabilizing controllers, we can easily show  $Q(s) \in \mathbb{R}^{H_\infty}$

$$\text{that } \|T_{11y}\| = \|(P_{11} - P_{12} \tilde{M} P_{21}) + P_{12} M Q \tilde{M} P_{21}\|_\infty$$

$$\text{in general form } = \|T_{11}(s) + T_{12}(s) Q T_{21}(s)\|_\infty$$

which has an affine form w.r.t.  $Q$  for given  $P_{11}, P_{12}, P_{21}$  &

Coprime factorization of  $P_{22}$

### 3-5/ Design Constraints

3-40

Before giving the methods for robust stability and Mixed Sensitivity problem, let us determine the basic limitations causes by, stability concerns, robustness, actuator & sensor limitations

#### 3-5-1/ Algebraic Constraints

① Identity  $S + T = 1$  must hold for all frequencies

$\Rightarrow |S(j\omega)| + |T(j\omega)|$  cannot get both small at a frequency (less than  $\frac{1}{2}$ )

② A necessary condition for robust performance is that

$$\min \{|W_S(j\omega)|, |W(j\omega)|\} < 1 \quad \forall \omega$$

otherwise the problem of robust performance is not conceivable.

③ Interpolation Condition:

To have internal stability, if unstable poles of the plants  $P_i$  and if unstable zeros of the plant  $Z_i$ , interpolation condition holds

$$S(P_i) = 0 \Rightarrow T(P_i) = 1$$

$$S(z_i) = 1 \Rightarrow T(z_i) = 0$$

For multiple unstable poles & zeros we must have: (as stated before)

$P_o$  is unstable pole with multiplicity  $m$

$z_0 \quad \sim \quad \text{zero} \quad " \quad " \quad "$

$$\left\{ \begin{array}{l} S(P_o) = \frac{d^i S}{ds^i}(P_o) = 0 \quad i=1, \dots, m-1 \\ T(P_o) = 1 + \frac{d^i T}{ds^i}(P_o) = 0 \end{array} \right. \quad P_o \text{ unstable} \quad \left\{ \begin{array}{l} T(z_0) = \frac{d^i T}{dz^i}(z_0) = 0 \quad i=1, \dots, m-1 \\ S(z_0) = 1 - \frac{d^i S}{dz^i}(z_0) = 0 \end{array} \right. \quad z_0 \text{ unstable}$$

### 3-5-2] Analytic Constraints

3-4)

To introduce these constraints, remind three Theorem:

A) Maximum Modulus Theorem:

Suppose  $\Omega$  is a region (nonempty, close, simply connected set) in the complex plane and  $F$  is a function that is analytic in  $\Omega$ , and  $F$  is not constant. Then  $|F|$  will attain its maximum on the boundaries of  $\Omega$ .

B) Cauchy's Theorem:

Suppose that  $\Omega$  is a bounded open set with connected Complement and  $D$  is a nonintersecting closed contour in  $\Omega$ . If  $F$  is analytic in  $\Omega$  then

$$\oint_D F(s) ds = 0$$

C) Cauchy's Integral Formula:

Suppose that  $F$  is analytic on a non-self-intersecting closed contour  $D$  and in its interior  $\Omega$ . Let  $s_0$  be a point in  $\Omega$ . Then

$$F(s_0) = \frac{1}{2\pi j} \oint_D \frac{F(s)}{s - s_0} ds$$

A direct implicat of C.I.F is Poisson integral formula.

Poisson integral Formula: Let  $F$  be analytic and of bounded magnitude in  $\operatorname{Re}(s) \geq 0$  and let  $s_0 = \delta + j\omega_0$  be a point in the complex plane with  $\delta > 0$ . Then:

$$F(s_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} F(j\omega) \frac{\delta}{\delta^2 + (\omega - \omega_0)^2} d\omega$$

From these Theorems, we can conclude the following:

① Bounds on  $W_S$  &  $W$

Suppose that the loop transfer function  $L(s) = C(s) \cdot P(s)$  has an unstable zero  $z_0 \in RHP$ : Then  $|S(z_0)| = 1 - T(z_0) = 1$  int. cond.

$$\boxed{\|W_S S\|_\infty \geq |W_S(z_0)|}$$

direct conclusion of H.M.Th.

$$|W_S(z_0)| = |W_S(z_0) S(z_0)| \leq \sup_{Re(s) \geq 0} |W_S(s) S(s)| = \|W_S S\|_\infty$$

For  $\|W_S S\|_\infty < 1$  be achievable  $\Rightarrow |W_S(z)| < 1$

Similarly

$$\boxed{\|WT\|_\infty \geq |W(P_0)|}$$

$P_0$  is unstable pole of  $L$

For  $\|WT\|_\infty < 1$  robust stability to hold  $\Rightarrow |W(P_0)| < 1$

② inner-outer factorization

Def: a transfer function is inner or all poles of its magnitude =  $1 + \omega$

Def: a transfer " is outer or minimum phase if it has no zeros in  $Re(s) > 0$

every general plant can be factorized into inner-outer factorization.

Lemma: If fact  $G \in RH^\infty$ , it can be represented by inner-outer factorizat

$$G(s) = G_i(s) \cdot G_o(s) \text{ or } G_{ap}(s) \cdot G_{mp}(s)$$

These factors are unique up to sign (Blaschke product).

$$\text{Example: } G_i(s) = \frac{4(s-2)}{s^2+s+1} = \frac{-s+2}{s+2} \cdot \frac{-4(s+2)}{s^2+s+1}$$

$$G_o(s) = \frac{s^2-s+2}{(s+1)(s+2)^3} = \frac{s^2-s+2}{s^2+s+2} \cdot \frac{s^2+s+2}{(s+1)(s+2)^3}$$

Note: Special care must be done for poles on the imaginary axis.

Suppose  $L(s)$  has no poles on the  $j\omega$ -axis, from now on.

Let us factorize  $S(s) + T(s)$

$$S(s) = S_i(s) \cdot S_o(s) \quad \text{and} \quad T(s) = T_i(s) \cdot T_o(s)$$

Suppose  $P(s)$  has a zero  $z_0 \in \text{Re}(s) > 0$  & a pole  $p_0 \in \text{Re}(s) > 0$  and no other poles & zeros. Then

$$S_i(s) = \frac{s - p_0}{s + p_0} \quad \text{and} \quad T_i(s) = \frac{s - z_0}{s + z_0}$$

From interpolation conditions  $S(z_0) = T(p_0) = 1$ , hence,

$$\begin{cases} S_o(z_0) = S_i^{-1}(z_0) = \frac{z_0 + p_0}{z_0 - p_0} \\ T_o(p_0) = T_i^{-1}(p_0) = \frac{p_0 + z_0}{p_0 - z_0} \end{cases}$$

Then

$$\|W_S\|_\infty = \|W_S S_o\|_\infty \geq |W_s(z_0) S_o(z_0)| = \left| W_s(z_0) \frac{z_0 + p_0}{z_0 - p_0} \right|$$

$$\Rightarrow \boxed{\begin{aligned} \|W_S S\|_\infty &\geq \left| W_s(z_0) \frac{z_0 + p_0}{z_0 - p_0} \right| \\ \|W_T\|_\infty &\geq \left| W(p_0) \frac{p_0 + z_0}{p_0 - z_0} \right| \end{aligned}}$$

we would like to decrease the above infinity norms for robust stability and performance; However, they both have lower limits if  $p_0 \neq z_0$  exists (system is unstable & non-minphase). The restriction is more stringent if the right half plane pole & zeros are close  $(p_0 - z_0) \ll 1$

### ③ Waterbed Effect

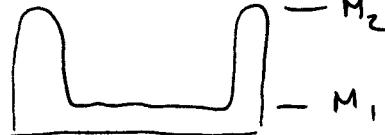
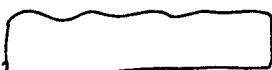
To establish this adverse effect, we can prove the following lemma

from Poisson integral formula:

Lemma: For every point  $s_0 = b_0 + j\omega_0$  with  $b_0 > 0$  (unstable node)

$$\log |S_o(s_0)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \log |S(j\omega)| \cdot \frac{b_0}{b_0^2 + (\omega - \omega_0)^2} d\omega$$

The water bed effect, as it is understood from the name is that as in a water bed



You may not reduce the value of  $S(j\omega)$  in some frequency range  $[\omega_1, \omega_2]$  unless you increase the value in some other frequency range. The water bed effect exist for the plants possessing unstable zeros:

Consider  $M_1$  denotes the maximum magnitude of  $S$  in the frequency band  $[\omega_1, \omega_2]$ ,

$$M_1 = \max_{\omega_1 \leq \omega \leq \omega_2} |S(j\omega)|$$

and let  $M_2$  denote  $M_2 = \|S\|_\infty$  for all frequencies. for getting good performance in the bandwidth  $[\omega_1, \omega_2] \Rightarrow M_1 \ll 1$ .

The question is that is it possible to make  $M_1$  as small as we want while keeping  $M_2$  small as well? For minimum phase system we have

Theorem 1: Suppose that  $P$  has a zero at  $z_0$  with  $\operatorname{Re}(z_0) > 0$ . Then

$\exists c_1, c_2 \in \mathbb{R}^+$ , depending only on  $\omega_1, \omega_2$  and  $z_0 \Rightarrow$

$$c_1 \log M_1 + c_2 \log M_2 \geq \log |S_i(z_0)| \geq 0$$

Example:  $P(s) = \frac{s-1}{(s+1)(s-p)}$  from interpolation cond.

$$S(p) = 0 \Rightarrow S_i(s) = \frac{s-p}{s+p} \cdot G_i(s) \quad G_i(s) \text{ is all pass too}$$

From M.M.Th Max  $G_i(s)$  is at boundary = 1  $\Rightarrow |G_i(s)| \leq 1$

$$\Rightarrow S_i(1) \leq \frac{1-p}{1+p} \Rightarrow c_1 \log M_1 + c_2 \log M_2 \geq \log \left| \frac{1+p}{1-p} \right|$$

again if  $z_i \neq p_i$  are close RHS gets too large.

If  $\not\in$  unstable zero we can show that  $M_c < \epsilon, M_c < \delta$

can be found for any positive  $\epsilon + \delta \Rightarrow$  No adverse effect.

on the other hand  $\exists$  another Theorem for  $L$  with relative degree  $> 2$

Theorem: Area Formula

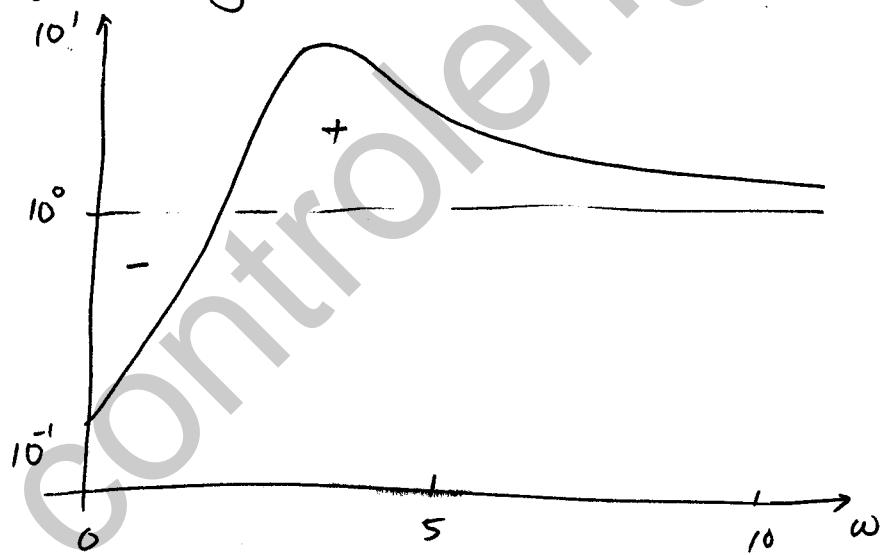
Let  $\{p_i\}$  denote the set of poles of  $L$  in  $\text{Re}(s) > 0$ , and assume the relative degree  $\geq 2$ . Then,

$$\int_0^\infty \log |S(j\omega)| d\omega = \pi (\log e) \left( \sum \text{Re } p_i \right)$$

Example: for  $L(s) = \frac{10}{(s-1)(s+2)}$

The feedback system is stable, relative degree = 2 hence plot

$|S(j\omega)|$  in log scale, versus  $\omega$



The Theorem says that the net area is positive & equals to

$$\pi \log_e \left( \sum \text{Re } p_i \right) - \pi \log_e$$

hence for general plant as above which is unstable  $S$  cannot get small at a frequency range, unless it rises up at some freq.

## 4/ Robust Stability and Nominal Performance Solution

4-1

In this chapter we would like to design a controller in which the similar problems of  $\|W_S S\|_\infty < 1$  or  $\|W T\|_\infty < 1$  is found.

From coprime factorization :

$$W_S S = W_S M(Y - NQ) = W_S M Y - W_S M N Q = T_1 - T_2 Q$$

$$W T = W N(X + M Q) = W N X + W N M Q = T_3 + T_4 Q$$

both have similar form. This problem is solved earlier by some other application named model-matching problem:

### 4-1II/ The Model-Matching problem

Let  $T_1(s), T_2(s) \in \mathbb{R}H^\infty$ , Find a stable transfer function  $Q(s)$  that with anumption of  $T_2$  has no zeros on the imaginary axis find the minimum

$$\gamma_{opt} := \min_{\text{stable } Q} \|T_1 - T_2 Q\|_\infty$$

(Note  $Q$  already stabilizes the system), a  $Q$  achieving the minimum is called "the optimal Solution".

a trivial case is when  $T_1/T_2$  is stable  $\Rightarrow Q = T_1/T_2$  and  $\gamma_{opt} = 0$  in the

Suppose nontrivial case where  $T_2$  has unstable zero at  $s_0$  (Special Case)

If  $Q$  is stable and  $T_2 Q$  has finite  $\infty$ -norm ( $T_2 Q \in \mathbb{R}H^\infty$ ) then

By M.M.Th.

$$\|T_1 - T_2 Q\|_\infty \geq |T_1(s_0)| \text{ hence } \gamma_{opt} \geq |T_1(s_0)|$$

On the other hand the function  $Q = \frac{T_1 - T_1(s_0)}{T_2}$  is stable and gives the

model matching error  $= |T_1(s_0)| = \gamma_{opt}$  is an optimal  $Q$ , which is the unique optimal solution.

Example:  $\gamma_{\text{opt}} = \|T_1 - T_2 Q\|$ ,  $T_1 = \frac{4}{s+3}$ ,  $T_2 = \frac{s-2}{(s+1)^3}$  [4-2]

$$\gamma_{\text{opt}} = |T_1(2)| = \frac{4}{5} \quad \& \quad Q_{\text{opt}} = \frac{T_1 - T_1(2)}{T_2} = \frac{\frac{4}{s+3} - \frac{4}{5}}{\frac{s-2}{(s+1)^3}}$$

$$Q_{\text{opt}} = -\frac{4(s+1)^3}{5(s+3)}$$

To solve the problem in general case, we need Nevanlinna-Pick Problem

### 4-1-2 | The Nevanlinna-Pick Problem

Similar to  $\mathbb{RH}^\infty$  class, define the space  $\mathbb{HT}^\infty$  for stable, proper, complex-rational functions in which  $\infty$ -norm would be similarly defined, as maximum magnitude on the imaginary axis.

Let  $\{a_1, \dots, a_n\}$  be a set of points in the open RHP (unstable poles or zeros) and  $\{b_1, \dots, b_n\}$  a set of points in  $\mathbb{C}$ , for simplicity assume  $a_i$ 's are distinct. NP interpolation problem is to find  $G$  in  $\mathbb{HT}^\infty$  where

$$\left. \begin{array}{l} \|G\|_\infty \leq 1 \\ G \text{ is to interpolate the value } b_i \text{ at point } a_i \end{array} \right\} G(a_i) = b_i \quad i=1, \dots, n$$

write the array  $a_1 \ a_2 \ \dots \ a_n$  for further use  
 $b_1 \ b_2 \ \dots \ b_n$

NP Problem is not always solvable, a necessary condition for solvability is  $|b_i| \leq 1$  for  $i=1, \dots, n$ .

Def:  $H$  is Hermitian matrix if  $H^* = H$ , (<sup>\*</sup> complex conjugate)

if  $H$  is real, it is Hermitian if it is symmetric

Def:  $H$  is pos.def. if  $\forall \begin{cases} x \in \mathbb{C} \\ x \neq 0 \end{cases} \quad x^* H x > 0 \rightarrow (H > 0 \text{ if all eigenvalues } > 0)$

pos.semi-def "  $x^* H x \geq 0 \quad (H \geq 0 \text{ " } \geq 0)$

Def : Pick matrix  $Q$  is defined from  $a_1, \dots, a_n$   
 $b_1, \dots, b_n$

$$Q_{ij} = \frac{1 - \bar{b}_i b_j}{a_i + \bar{a}_j} \quad \text{which is Hermitian}$$

Example : data array  $\begin{matrix} b+j \\ 0.1-0.1j \\ 0.1+0.1j \end{matrix}$

$$Q = \begin{bmatrix} 0.0817 & 0.0814 - 0.0119j \\ 0.0814 + 0.0119j & 0.0817 \end{bmatrix}$$

whose eigenvalues are  $-0.0005$  and  $0.1639$ :  $Q$  is indefinite  $\Rightarrow$

NP problem not Solvable

Theorem 1: The NP problem is solvable iff  $Q \geq 0$  (Pick Th.)

proof in the book.

#### 4-1-3| Nevanlinna Algorithm

We are to begin the procedure to find NP solution when it is solvable

The solution is developed inductively: First, the case  $n=1$  is solved, then

the case of  $n$  points is reduced to the case of  $n-1$  points.

Denote  $D$  the open unit disk,  $|z| < 1$ , and  $\overline{D}$  the closed unit disk  
 $|z| \leq 1$ . A Möbius function has the form

$$M_b(z) = \frac{z-b}{1-\bar{z}b} \quad \text{where } |b| < 1$$

and has the properties:

- 1)  $M_b$  has a zero at  $z=b$  and a pole at  $z=\frac{1}{b}$  hence it is analytic in  $D$
- 2)  $|M_b| = 1$  on the unit circle
- 3)  $M_b$  maps  $D$  onto  $D$
- 4)



4) The inverse map is

4-4

$$M_b^{-1}(z) = \frac{z+b}{1+z\bar{b}}$$

Note  $M_b^{-1} = M_{-b}$ . So the inverse map is a Möbius function too.

We will also need the all-pars function  $A_a(s) = \frac{s-a}{s+\bar{a}}$   $\operatorname{Re}(a) > 0$

Now let's solve NP problem for one point  $\begin{matrix} a_1 \\ b_1 \end{matrix}$  with the aid of M.F.

data array

$$\begin{matrix} a_1 \\ b_1 \end{matrix}$$

Two Cases:

Case 1:  $|b_1| = 1$  A Solution is  $G(s) = b_1$ , By the M.M.Th it is unique

Case 2:  $|b_1| < 1$ , there are an infinite number of Solutions:

Lemma: The set of all Solutions is

$$\left\{ G : G(s) = M_{-b_1} [G_1(s) A_{a_1}(s)], G_1 \in \mathcal{H}^\infty, \|G_1\|_\infty \leq 1 \right\}$$

if  $G_1$  is an all-pars function, so is  $G$ .

Proof: Let  $G_1 \in \mathcal{H}^\infty, \|G_1\|_\infty \leq 1$ . define

$$G(s) = M_{-b_1} [G_1(s) A_{a_1}(s)]$$

$G(s)$  is generated in two steps  $s \rightarrow G_1(s) A_{a_1}(s) + z \mapsto M_{-b_1}(z)$

the first is analytic in the closed right half-plane and maps it into the closed disk  $\overline{D}$ ; the second is analytic in  $\overline{D}$  and maps it back into  $\overline{D}$   
it follows that  $G \in \mathcal{H}^\infty$  and  $\|G\|_\infty \leq 1$ .

also  $G$  interpolates  $b_1$  at  $a_1$ ,  $G(a_1) = M_{-b_1}(G_1(a_1) A_{a_1}(a_1)) = M_{-b_1}(0) = b_1$

Hence  $G$  solves the NP problem. Moreover, if  $G_1$  is an all-pm function, then so is  $G_1 A_{a_1}$ , hence so is  $G$ , because  $M_{-b_1}$  maps unit circle onto itself. [4-5]

Conversely, suppose that  $G$  solves the NP problem,

Define  $G_1$  so that  $G(s) = M_{-b_1} [G_1(s) A_{a_1}(s)]$

$$\text{that is } G_1(s) = \frac{M_{-b_1}[G(s)]}{A_{a_1}(s)}$$

The function  $M_{-b_1}[G(s)]$  belongs to  $\mathcal{CH}^\infty$ , has  $\infty$ -norm  $\leq 1$  (maps  $\bar{\mathbb{D}}$  into  $\bar{\mathbb{D}}$ ), and has a zero at  $s=a_1 \Rightarrow$   
 $G_1 \in \mathcal{CH}^\infty$  and  $\|G_1\|_\infty \leq 1$

Example : Solve NP problem for  $\begin{matrix} 2 \\ 0.6 \end{matrix}$  data array

from the Lemma

$$M_{-b} = \frac{s+b}{1+zb}$$

$$G(s) = M_{-b} \left[ G_1(s) \cdot \frac{s-2}{s+2} \right]$$

$$= \frac{G_1(s) \frac{s-2}{s+2} + 0.6}{1 + 0.6 G_1(s) \frac{s-2}{s+2}}$$

where  $G_1(s)$  can be any all pm functi

$$\text{for } G_1(s) = 1 \rightarrow G(s) = \frac{1.6s - 0.8}{1.6s + 0.8} \text{ all pm}$$

$$G_1(s) = \frac{s-1}{s+1} \Rightarrow G(s) = \frac{s^2 - 0.75s + 2}{s^2 + 0.75s + 2} \text{ again all pm}$$

Now return to NP problem with  $n$  data points, the problem is named Solvable; See how it is reduced to  $n-1$  point 4-6

Case 1  $|b_1| = 1$ .

Since the problem is solvable, by the M.M.Th. it must be that  $G(s) = b_1$  is the unique solution (and hence that  $b_1 = \dots = b_n$ ).

Case 2:  $|b_1| < 1$ .

Pose a new problem, Labeled NP' problem, with  $n-1$  data points

$$\begin{matrix} a_2 & \dots & a_n \\ b'_2 & \dots & b'_n \end{matrix}$$

$$\text{where } b'_i = \frac{M_{b_1}(b_i)}{A_{a_1}(a_i)}$$

Lemma: The set of all solutions to the NP problem is given by the same previous formula  $G(s) = M_{-b_1}[G_i(s) A_{a_1}(s)]$

where  $G_i$  ranges over all solutions to the NP' problem if  $G_i$  is all  $a_i$  other is  $G$ .

Example: Solve NP for the data array

$$\begin{matrix} 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{matrix}$$

The Pick matrix

$$Q_{ij} = \frac{1 - b_i b_j}{a_i + \bar{a}_j} \quad Q = \begin{bmatrix} 0.375 & 0.2778 & 0.2188 \\ 0.2778 & 0.2222 & 0.1833 \\ 0.2188 & 0.1833 & 0.1563 \end{bmatrix}$$

The smallest eigenvalue is 0.0004,  $Q > 0 \Rightarrow$  NP Solvable.

A Solution can be found by reducing into two points.

$$\begin{matrix} a_2 & a_3 \\ b'_2 & b'_3 \end{matrix} \rightarrow \begin{bmatrix} 2 & 3 \\ -0.6 & -0.5714 \end{bmatrix}$$

$$b'_i = \frac{M_{b_1}(b_i)}{A_{a_1}(a_i)}$$

reduce to NP" problem with

$$\begin{array}{rcl} a_3 & = & 3 \\ b_3'' & = & 0.02174 \end{array} \quad b_3'' = \frac{M_{b_2'}(b_3')}{Aa_2(a_3)}$$

Now Solve the problem with reverse order

$$G_2(s) = M_{-b_3''} [G_3(s) Aa_3(s)]$$

where  $G_3(s)$  is arbitrary in  $\mathbb{C}H^\infty$  of Norm  $\leq 1$ , Let's take  $G_3(s) = 1$ .

the simplest all-pole function Then:

$$G_2(s) = \frac{1.2174s - 2.3478}{1.2174s + 2.3478}$$

The induced Solution to NP' problem is

$$G_1(s) = M_{-b_2'} [G_2(s) Aa_2(s)] = \frac{0.487s^2 - 7.6522s + 1.8785}{" + " + " + "}$$

Finally the Solution to NP prob

$$G(s) = M_{-b_1} [G_1(s) Aa_1(s)] = \frac{0.730s^3 - 4.0696s^2 + 14.2957s - 0.9391}{" + " + " + " + "}$$

The order of  $G(s)$  is 3 as the data points.

In general always  $\exists$  an all-pole solution of degree  $\leq n$

$\rightarrow$  Read the case of complex-conjugate unstable nodes  $a_i \neq a_i^*$

FROM Book

Note: if in the data array we have complex-conjugate Nodes  
 then if  $(a_i, b_i)$  appears  $\Rightarrow (\bar{a}_i, \bar{b}_i)$  will also appear  
 If we want to give a  $G(s) \in RH_\infty$  instead of  $CH_\infty$  we can  
 use the following interpretation

$$G(s) = G_R(s) + j G_I(s)$$

$G(s) \in CH_\infty$  but  $G_R(s) + G_I(s) \in RH_\infty$

if  $G(s)$  is absolute to N.P.  $\Rightarrow G_R(s)$  is also absolute  
 hence use  $G_R(s) = \frac{1}{2} (G(s) + \underline{G}(s))$  as the solution  
 where  $\underline{G}(s)$  is found by conjugating all coefficients of  $G(s)$

Example: For the data

$s+2j$	$s-2j$
$0.1 - 0.1j$	$0.1 + 0.1j$

The N.P. problem is solvable ( $\underline{\lambda}(Q) = 0.0051$ )

The N.P. Solution exists using  $G_1(s) = 1$

$$G(s) = \dots = \frac{(0.5268 + 0.1213j)s^2 - (9+j)s + (47.1073 + 1.1410j)}{(0.5268 - 0.1213j)s^2 + (9-j)s + (47.1073 - 1.1410j)}$$

Now obtain

$$\underline{G}(s) = \frac{1}{2} (G + \underline{G})$$

It results into

$$G_R(s) = \frac{0.2628s^4 - 30.6418s^2 + 2217.7775}{0.2923s^4 + 9.7255s^3 + 1314.1186s^2 + 850.4137s + 2220.702}$$

Example 2: Consider the array:

$$(NP) \quad a_i : \begin{matrix} 1-j & & 1+j \\ & \frac{1}{3.4496}(2-j) & \frac{1}{3.4496}(2+j) \\ & 0.58 - j0.29 & 0.58 + j0.29 \end{matrix}$$

The NP problem is solvable

$$Q = \begin{bmatrix} 0.1028 + j0.2710 & 0.2498 \\ 0.2498 & 0.1028 - j0.2710 \end{bmatrix}$$

$$\lambda(Q) = 0.2053, 0.0004 \Rightarrow Q \geq 0$$

$$(NP)' \Rightarrow b'_2 = M_{b_1}(b_2) / A_{a_1}(a_2)$$

$$= \frac{0.58 + j0.29 - (0.58 - j0.29)}{1 - (0.58 + j0.29) * (0.58 - j0.29)} \times \frac{4j + 1 + j}{1 + j - 1 + j}$$

$$b'_2 = 0.3550 + 0.9350j$$

$$G_1(s) = \frac{\frac{s - (1+j)}{s + (1-j)} + (0.3550 + 0.9350j)}{1 + (0.355 - 0.935j) \left( \frac{s - (1+j)}{s + (1-j)} \right)}$$

$$G_1(s) = \frac{(0.355 + 0.935j) \times (s - (2.89 \times 10^{-5} + 0.31j))}{s + (2.89 \times 10^{-5} - 0.31j)}$$

$$\hookrightarrow \begin{bmatrix} 1-j \\ 0.58 - j0.29 \end{bmatrix} \Rightarrow G_2(s) = \frac{G_1 \times \frac{s - (1-j)}{s + (1+j)} + (0.57 - 0.29j)}{1 + G_1 \times \frac{s - (1-j)}{s + (1+j)} \times (0.58 + 0.29j)}$$

$$G_2(s) = \frac{(s - (0.45 - 8.23 \times 10^{-5}j))(s - (1.12 \times 10^{-4} + 0.31j))}{(s + (0.45 + 8.23 \times 10^{-5}j))(s + (1.12 \times 10^{-4} - 0.31j))}$$

$$G_2(s) = \frac{s^2 - (0.45 + 0.31j)s + (7.9 \times 10^{-5} + 0.14j)}{s^2 + (0.45 - 0.31j)s + (7.9 \times 10^{-5} - 0.14j)}$$

$$G_R(s) = \frac{1}{2}(G_1 + G_2)$$

$$G_R(s) = \frac{s^4 - 0.1057s^2 - 0.0197s}{s^4 + 0.899s^3 + 0.2984s^2 + 0.08653s + 0.01943}$$

check interpolations

$$G_R(s=1+j) = \dots = 0.58 + 0.29j$$

$$G_R(s=1-j) = \dots = 0.58 - 0.29j \quad \checkmark$$

## Some Notes on the Solutions:

1) if you have an 1 entry array, you don't need to use NP  
use M.M.Th. directly

$$a_1 \Rightarrow G(a_1) = b_1 \Rightarrow \|G\|_\infty \leq b_1$$

$b_1$  From M.M.Th.  $x_{opt} = b_1 \rightarrow G(s)$  must be flat =  $b_1$

$$G(s) = T_1 - T_2 Q(s) = b_1 \Rightarrow Q(s) = \frac{T_1 b_1 - b_1}{T_2(s)}$$

3) if you have two complex-conjugate nodes in the array

$$\begin{array}{ll} a_R + j a_I & a e^{-j a I} \\ b_R + j b_I & b e^{-j b I} \end{array}$$

Although there are two interpolator points which should be satisfied

But Note that  $\|b_R + j b_I\| = \|b_R - j b_I\|$ , hence the M.M.Th.

can be directly used to find the Solution

2) if you have two points in the array but  $b_1 = b_2$  OK

$\|b_1\| = \|b_2\|$  then From M.M.Th. we can use the flat

function  $G(s) = b_1 = b_2$  and interpolate two points in it.

optimal Solution  $T_1 - T_2 Q(s) = b_1$  as before

$$\Rightarrow Q(s) = \frac{T_1(s) - b_1}{T_2(s)}$$

$$\begin{aligned} T_1(s=a_1) &= b_1 \\ T_1(s=a_2) &= b_1 \end{aligned}$$

## 4-2-11 Solution of the Model-Matching Problem

NP solution can be applied on model-matching problem with the following formulation:

For the minimum model-matching error,  $\gamma_{opt}$ , equals the minimum  $\gamma \geq$

$$\|T_1 - T_2 Q\|_\infty \leq \gamma$$

For some stable  $Q$ , fix  $\gamma > 0$  and consider  $Q \rightarrow G$  defined by

$$G = \frac{1}{\gamma} (T_1 - T_2 Q)$$

if  $Q$  is stable  $\Rightarrow G$  is stable too, but the inverse needs interpolation condition to hold for  $Q \in \text{RH}^\infty$ : (Assume No repeated unstable zeros for  $T_2$ )

Let  $\{z_i \neq i=1, \dots, n\}$  be unstable zeros of  $T_2$ , interpolation conditions are

$$G(z_i) = \frac{1}{\gamma} T_1(z_i) \quad i=1, \dots, n$$

Therefore,  $\gamma_{opt}$  equals the minimum  $\gamma \geq \exists$  a function  $G$  in  $\text{RH}^\infty$ :

$$\int \|G\|_\infty < 1$$

$$\left\{ G(z_i) = \frac{1}{\gamma} T_1(z_i) \quad i=1, \dots, n \right.$$

precisely NP Problem with the array

$$\begin{array}{ccccccccc} a_1 & \dots & a_n & = & z_1 & \dots & z_n \\ \frac{b_1}{\gamma} & \dots & \frac{b_n}{\gamma} & & \frac{T_1(z_1)}{\gamma} & \dots & \frac{T_1(z_n)}{\gamma} \end{array}$$

The associated Pick matrix equals to

$$A - \gamma^2 B \quad \begin{cases} A_{ij} = \frac{1}{a_i + \bar{a}_j} \\ B_{ij} = \frac{\bar{b}_i b_j}{a_i + \bar{b}_j} \end{cases}$$

The problem is solvable if  $A - \bar{\gamma}^2 B \geq 0$ .

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But Both  $A$  &  $B$  are Hermitian.  $A$  is pos. def because all  $\alpha_i$ 's are distinct. Such a matrix has a positive definite square root (i.e. a matrix  $A^{1/2}$  where  $A^{1/2} \cdot A^{1/2} = A$ ). If the inverse of  $A^{1/2}$  is  $A^{-1/2}$ , from the next lemma  $\gamma_{opt}$  can be calculated:

Lemma:  $\gamma_{opt} = \bar{\gamma} (A^{-1/2} B A^{-1/2})$  ( $\bar{\gamma} = \sqrt{\lambda_{\max}}$ )

### Summary of the Solution

Due to the problem by coprime factorization reduce the problem into

$$\|T_1 - T_2 Q\| < \delta$$

get  $T_1 \rightarrow T_2$

Step 1: Determine  $\{z_i : i=1, \dots, n\}$  The unstable zeros of  $T_2$   $\text{Res} > 0$

Step 2: Define  $b_i := T_1(z_i)$ ,  $i=1, \dots, n$

$$\text{and } A_{ij} = \frac{1}{z_i - \bar{z}_j} \quad B := \frac{b_i b_j}{z_i + \bar{z}_j}$$

Step 3: Compute  $\gamma_{opt} = \bar{\gamma} (A^{-1/2} B A^{-1/2})$

Step 4: Solve the NP problem  $\gamma_{opt}^{-1} T(z_1) \dots \gamma_{opt}^{-1} T(z_n)$

Find Solution  $G$  wh  $\|G\|_\infty < 1$

Step 5:  $Q := \frac{T_1 - \gamma_{opt} G}{T_2}$

Calculate  $C$  by inverse coprime factorization

$G$  is an inner solution  $\Rightarrow T_1 - T_2 Q = \gamma_{opt} \cdot G$

a constant times an inner function.

$$\underline{\text{Ex1:}} \quad T_1(s) = \frac{s+1}{10s+1} \quad T_2(s) = \frac{(s-1)(s-5)}{(s+2)^2}$$

$$\text{Step 1: } z_1 = 1 \quad z_2 = 5$$

$$\text{Step 2: } b_1 = T_1(z_1) = \frac{2}{11} \quad b_2 = T_1(z_2) = \frac{6}{51}$$

$$A = \begin{bmatrix} 0.5 & \frac{1}{6} \\ \frac{1}{6} & 0.1 \end{bmatrix} \quad B = \begin{bmatrix} 0.0165 & 0.0036 \\ 0.0036 & 0.0014 \end{bmatrix}$$

$$\text{Step 3: } \gamma_{\text{opt}} = \bar{\gamma} (A^{-1/2} B A^{-1/2})$$

$$\text{Step 4: } z_1 = 1 \quad z_2 = 5$$

$$\gamma_{\text{opt}}^{-1} b_1 = 0.8997 \quad \gamma_{\text{opt}}^{-1} b_2 = 0.5821$$

$$G(s) = \frac{-1.0035s + 18.9965}{1.0035s + 18.9965}$$

$$\text{Step 5: } Q(s) = \frac{0.3021s^2 + 1.2084s + 1.2084}{s^2 + 19.0308s + 1.8931}$$

$$\underline{\text{Ex2:}} \quad \text{for the plant} \quad P(s) = \frac{1}{(s-1)(s+2)} \quad \text{with uncertainty mult.}$$

weighting function  $W(s) = \frac{(s+1)^2}{25}$ ; Solve for the optimal Robust stabilizing Controller.

Solution: for Robust Stability  $\|WT\|_\infty < 1$

optimal Solvent is  $\gamma_{\text{opt}} = \min \|WT\|_\infty$

use coprime factorization to reduce into affine problem

(4-11)

from previous example:

$$P(s) = \frac{N(s)}{M(s)} \Rightarrow \left\{ \begin{array}{l} N(s) = \frac{1}{(s+1)^2} \\ M(s) = \frac{(s-1)(s-2)}{(s+1)^2} \end{array} \right., \quad \left\{ \begin{array}{l} X(s) = \frac{19s-11}{s+1} \\ Y(s) = \frac{s+6}{s+1} \end{array} \right. \quad W(s) = \frac{(s+1)^2}{25}$$

$$\|W\Gamma\|_\infty = \|WN(X + MQ)\|_\infty < 1$$

$$\text{Find } \gamma_{\text{opt}} = \min \|WNX - (-WNMQ)\|_\infty \\ = \min \|T_1 - T_2 Q\|_\infty$$

$$\text{where } T_1 = WNX = \frac{(s+1)^2}{25} \cdot \frac{1}{(s+1)^2} \cdot \frac{19s-11}{(s+1)} = \frac{19s-11}{25(s+1)}$$

$$T_2 = -WNM = \frac{(s+1)^2}{25} \cdot \frac{1}{(s+1)^2} \cdot \frac{(s-1)(s-2)}{(s+1)^2} = \frac{(s-1)(s-2)}{25(s+1)^2}$$

Solve NP problem:

$$a_1 = z_1 = 1 \quad a_2 = z_2 = 2$$

$$S1) \quad b_1 = T_1(z_1) = \frac{8}{50} = 0.16 \quad S2) \quad b_2 = T_1(z_2) = \frac{27}{25} = 0.36$$

$$A = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.25 \end{bmatrix}; \quad B = \begin{bmatrix} 0.0128 & 0.0192 \\ 0.0192 & 0.0324 \end{bmatrix}$$

$$S3) \quad A^{1/2} = \begin{bmatrix} 0.6223 & 0.3357 \\ 0.3357 & 0.3705 \end{bmatrix} \rightarrow S = A^{-1/2} B A^{-1/2} = \begin{bmatrix} 0.0455 & -0.1275 \\ -0.1275 & 0.4297 \end{bmatrix}$$

$$\sqrt{\lambda_{\max}(S)} = \sqrt{0.4681} = 0.6842 = \gamma_{\text{opt}}$$

$$S4) \quad \text{NP :} \quad \begin{matrix} 1 & 2 \\ 0.2338 & 0.5262 \end{matrix}$$

$$\text{Pick Matrix : } Q = \begin{bmatrix} 0.4727 & 0.2923 \\ 0.2923 & 0.1808 \end{bmatrix} \quad \text{eig}(Q) = \begin{pmatrix} 0.6534 & 0 \\ 0 & 0 \end{pmatrix} \quad Q \geq 0$$

$\Rightarrow$  NP problem Solvable.

NP'

$$b_2' = \frac{M_{b_1}(b_2)}{A_{a_1}(a_2)} = \frac{\frac{b_2 - b_1}{1 - b_2 b_1}}{\frac{a_2 - a_1}{a_2 + a_1}} = \dots = 1$$

NP' =

2

1 → makes it simple  $G_1(s) = 1, G_2(s) = 1$ 

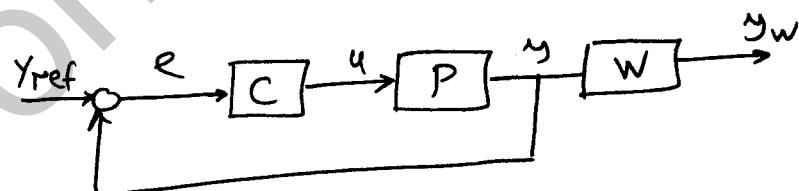
$$\Rightarrow Q = \frac{T_1 - \gamma_{opt} \cdot G_1}{T_2} = \frac{\frac{19s-11}{25(s+1)} - 0.6842}{-\frac{(s-1)(s-2)}{25(s+1)^2}} = \frac{28.105 - 1.8750s}{s^2 - 3s + 2}$$

$$\Rightarrow C(s) = \frac{X + MQ}{Y - NQ} \quad \text{where } X, Y, M, N, Q \text{ are given}$$

$$(C(s)) = \frac{as^4 + \dots}{s^4 + \dots} \quad 4^{\text{th}} \text{ order} \equiv \text{the degree of the G. Plant}$$

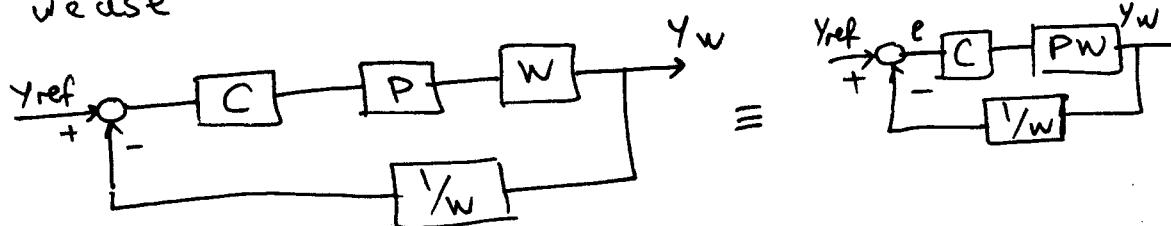
$$(C(s)) =$$

Solution By Numerical interpolation (N-synthesis)

To solve  $\|WT\|_\infty < 1$  generate an input-output system with input-outputT.F equal to  $WT = \frac{WCP}{1+CP}$  :

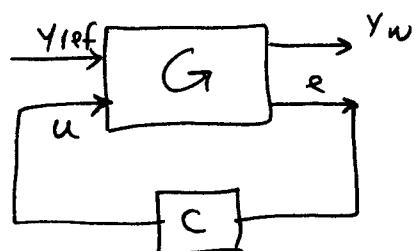
$$Ty_w y_{ref} = \frac{W \cdot CP}{1+CP} = WT, \text{ Note } W(s) = \frac{(s+1)^2}{25} \text{ is not proper to}$$

realize it we use



write to  $\mu$ -Synthesis format

14-13



$$PW = \frac{(s+1)^2}{25(s-1)(s-2)}$$

$$H_w = \frac{25}{(s+1)^2}$$

$$\text{num}_{pw} = [1 \ 2 \ 1];$$

$$\text{den}_{pw} = 25 \times [1 \ -3 \ 2];$$

$$PW = \text{nd2sys}(\text{num}_{pw}, \text{den}_{pw});$$

$$\text{num}_{w-1} = 25;$$

$$\text{den}_{w-1} = [1 \ 2 \ 1];$$

$$W-1 = \text{nd2sys}(\text{num}_{w-1}, \text{den}_{w-1});$$

The list of program is enclosed:

$\delta_{opt}$  is found by : Iterat :  $\delta_{opt} = 0.684255$   
 $\delta_{opt} = \bar{\delta} = 0.684241$

$$C(s) = \frac{5.52 \times 10^5 (s - 6.2095 \times 10^{-1})}{(s + 5.6215)(s + 3.2306 \times 10^4)} = \frac{5.52 \times 10^5 (s + 0.621)}{s^2 + 3.2311 \times 10^4 s + 1.8161 \times 10^5}$$

```
% Copyright Dr. Hamid D. Taghirad 2005

clc
echo on
% This is the matlab file to construct robust stability solution
% to the course note example. New LTI system structures, and
% robust control tools are used in this example.
% Matlab ver 7 and Robust Control toolbox ver 3 is required
%
%           1
% Plant = -----
%                   (s - 1)(s - 2)
%
%           (s + 1)^2
% Wt   = -----
%                   25
pause % strike any key to continue

clc
%
% The Nominal plant and weights are build in zero-pole format
%
s=zpk('s'); % zero-pole format
plant=1/((s-1)*(s-2)); % plant
wt=(s+1)^2/25; % weighting function
wt_1=25/((s+1)^2); % inverse of the weight
aug_pl=plant*wt; % Augmented plant and weight
%
%
% Build multiplicative uncertainty block and uncertain system
%
%
delta = ultidyn('delta',[1 1]); % 1 by 1 full delta block
wt_prop = wt/(1e-3*s+1)^2; % weighting function propered
Pertplant = plant*(1+wt_prop*delta); % Perturbed plant
Psample = usample(Pertplant,10); % 10 random samples of the plant
% Note in each run the samples vary

pause % strike any key to continue
clc

% Do a frequency_response for the nominal system,
% 10 samples of uncertain system and the uncertainty weight
%
%
% Wait a few seconds for calculations

echo off
omega = logspace(-1,2,200); % frequency vector
bodemag(plant,'b',Psample,'g:',omega); % frequency response of
% nominal and perturbed plants
grid;
title('Frequency Response of the Nominal Plant vs. 10 uncertain plant samples');
xlabel('Frequency (rad/s)');
ylabel('Magnitude');
echo on
```

```
%  
% The frequency responses are plotted in Figures Window  
%  
pause % strike any key to continue  
clc  
  
% Now that the plant and the weight have been defined,  
% construct the interconnection structure, Plant_IC;  
  
systemnames = 'aug_pl wt_1';  
inputvar = '[yref; u]';  
outputvar = '[ aug_pl; yref - wt_1]';  
input_to_aug_pl = '[ u ]';  
input_to_wt_1 = '[ aug_pl]';  
sysoutname = 'plant_ic';  
cleanupsysic = 'yes';  
sysic  
  
pause % strike any key to continue  
clc  
%  
% We can change the properties of the generated plant using  
% get and set commands or as following.  
% See page 1-25 Control Toolbox user guide for the details.  
%  
% plant_ic.InputName={'yd';'u'}; % Set the input names  
% plant_ic.OutputName={'y';'e'}; % Set the output names  
%  
% We can make sure that our augmented system has its minimal  
% realization to avoid uncontrallability for some hidden modes  
% Check zero-poles patterns and use minreal  
%  
plant_ic(2,2)  
plant_ic = minreal(plant_ic);  
plant_ic(2,2)  
  
pause % strike any key to continue  
clc  
  
% The next step is to design an H-infinity control law for PLANT  
% The hinfsyn computes a stabilizing H optimal LTI/SS controller  
% K for a partitioned LTI plant P.  
%  
% [K,CL,GAM,INFO]=hinfsyn(P,NMEAS,NCON,KEY1,VALUE1,KEY2,VALUE2,...)  
%  
% The controller, K, stabilizes the P and has the same number  
% of states as P. The SYSTEM P is partitioned where inputs to  
% B1 are the disturbances, inputs to B2 are the control inputs,  
% output of C1 are the errors to be kept small, and outputs of  
% C2 are the output measurements provided to the controller.  
% B2 has column size (NCON) and C2 has row size (NMEAS).  
% The optional KEY and VALUE inputs determine tolerance,
```

```
% solution method and so forth--see Figure 1 for details.  
%  
% The closed-loop system is returned in CL and the achieved H-inf  
% cost in GAM. INFO is a STRUCT array that returns additional  
% information about the design.  
%  
% 'GMAX' real initial upper bound on GAM (default=Inf)  
% 'GMIN' real initial lower bound on GAM (default=0)  
% 'TOLGAM' real relative error tolerance for GAM (default=.01)  
% 'S0'real(default=Inf) frequency S0 at which entropy is evaluated,  
% only applies to METHOD 'maxe'  
% 'METHOD' `ric'(default) standard 2-Riccati solution, OR  
% 'lmi' LMI solution Or  
% 'maxe' maximum entropy solution  
% 'DISPLAY''off' or 'on'(default) no command window display, or  
% command window displays synthesis progress information  
%  
%  
% In this example, the system interconnection structure is  
% PLANT_IC, with 1 measurements, 1 controls,  
  
pause % strike any key to continue  
clc  
[k1,g1,gamma1,info1]=hinfssyn(plant_ic,1,1,'Display','on');  
  
% An H-infinity control law has been designed which achieves an  
% infinity norm of 0.6855 for the interconnection structure.  
%  
%  
% First, we will examine aspects of the controller  
% that was just designed, starting with the controller POLES  
% and zeros.  
pause % strike any key to continue  
clc  
k1=zpk(k1)  
  
% Next, a Bode plot of the frequency response of k1  
  
echo off  
clf  
bodemag(k1)  
title('Bode plot of controller, k1')  
xlabel('strike any key to continue')  
  
format short e  
echo on  
pause % strike any key to continue  
clc  
  
% Onto the closed loop, first checking that it is stable.  
  
echo off  
g1=zpk(g1);  
g1=minreal(g1)
```

```
echo on
% Now bode plot for a system

bodemag(g1);

pause % strike any key to continue

% Next, find the peak norm of the frequency response, using
% PKVNORM. This corresponds to the maximum singular of the
% closed loop system.
echo off

[g1peak,freqpeak]=norm(g1,inf);

fprintf( '\n          final gamma : %g',gamma1)
fprintf( '\n      Max singular value: %g \n \n',g1peak)
echo on

%
% Now check the robustness of the controller in the closed loop
% First find the perturbed closed loop for 10 samples

echo off
CLpert = feedback( Pertplant*k1 , 1);
CLsample = usample(CLpert,10);
CLnom = CLpert.Nominal;

impulse(CLnom,'r',CLsample,:g')
title('Closed loop impulse response of nominal and 10 perturbed system')

echo on
%
% Now See the impulse response of the nominal closed system
% and 10 perturbed ones.
% As it is seen in the figure window all responses are stable.
pause % strike any key to continue
clc

%% Robust Stability Analysis
% Does the closed-loop system remain stable for all values of |k|, |delta|
% in the ranges specified above? We use |robuststab| to answer this basic
% robustness question:

[stabmarg,destabunc,report,info] = robuststab(CLpert);

stabmarg

%%
% The variable |stabmarg| gives upper and lower bounds on the *robust
% stability margin*, a measure of how much uncertainty on |k|, |delta| the
% feedback loop can tolerate before becoming unstable. For example, a
% margin of 0.8 indicates that as little as 80% of the specified
% uncertainty level can lead to instability. Here the margin is 1.46,
% which means that the closed loop will remain stable for up to 146% of the
% specified uncertainty.
```

```
pause % strike any key to continue
clc
%The |report| function summarizes these results:

report
pause % strike any key to continue
clc

%%
% The variable |destabunc| contains the smallest combination of (|k|,|delta|)
% perturbations that causes instability.

destabunc

pause % strike any key to continue
clc

%%
% We can substitute these values into |ClosedLoop|, and verify that these
% values cause the closed-loop system to be unstable:
pole(usubs(CLpert,destabunc))

%%
% Note that the natural frequency of the unstable closed-loop pole is
% given by |stabmarg.DestabilizingFrequency|:

stabmarg.DestabilizingFrequency

%
% This concludes the example
%
```

### 4-3 | Design for Performance

As described before performance criterion  $\|W_S S\|_\infty < 1$  was introduced.  
 to find a proper controller  $C \Rightarrow$  the system is internally stable and the  
 above norm is smaller than 1. A general procedure is given for min-phase  
 and non-min phase systems summarizing the results of model-matching  
 to finding proper controller

#### 4-3.1 min-phase System.

Assume  $P$  has no zero in  $\text{Re}(s) \geq 0$ . The weighting function  $W_S$  is  
 assumed to be stable and strictly proper. We will show that with those  
 conditions it is always possible to design a proper  $C \Rightarrow \|W_S S\|_\infty < 1$   
 Let  $K$  be a positive integer and  $\tau$  a positive real number: Consider

$$J(s) = \frac{1}{(\tau s + 1)^K}$$

This is a lowpass filter with corner frequency  $\omega = 1/\tau$  and rolls off with  
 -  $20K$  db/decade. So for low frequency  $J(j\omega) \approx 1$ . It has a very useful  
 property:

Lemma: If  $G$  is stable and strictly proper, then

$$\lim_{\tau \rightarrow 0} \|G(1-J)\|_\infty = 0.$$

Now, Suppose  $P$  is also stable;  $C = \frac{Q}{1-PQ} \quad Q \in \mathbb{RH}^\infty$  characterizes  
 all stabilizing controllers for  $P$ . hence,

$$W_S S = W_S (1 - PQ)$$

To make  $\|W_S S\|_\infty < 1$  we are prompted to set  $PQ = I$  or  $Q = P^{-1}$ .

This is indeed stable ( $P$  is stable) but probably not proper, hence not in  $\mathbb{RH}^\infty$

So let's try  $Q = P^{-1}J$  with  $k$  large enough to make  $P^{-1}J$

propn: (i.e.  $k$  equals the relative degree of  $P$ )

$$\text{Then } W_S S = W_S (I - J)$$

whose  $\infty$ -norm is  $< 1$  for sufficiently small  $\epsilon$  by Lemma.

### Summary of Procedure

①  $P + P^{-1}$  stable

input  $P, W_S$

Step 1: Set  $k = \text{relative degree of } P$

Step 2: choose  $\epsilon$  so small that

$$\|W_S(I - J)\|_\infty < 1$$

where

$$J(s) = \frac{1}{(\epsilon s + 1)^k}$$

Step 3: Set  $Q = P^{-1}J$

Step 4: Set  $C = Q / (I - PQ)$

②  $P^{-1}$  stable

input,  $P, W_S$

Step 1: Do coprime factorization of  $P$

Find four factors in  $RH^\infty \rightarrow$

$$P = N/M, NX + MY = I.$$

Step 2: Set  $k = \text{rel. degree of } P$

Step 3: choose  $\epsilon$  so small that

$$\|W_S NY(I - J)\|_\infty < 1$$

Step 4: Set  $Q = YN^{-1}J$

Step 5: Set  $C = (X + MQ) / (Y - NQ)$

Example:

$$P(s) = \frac{1}{(s-2)^2}, W_S(s) = \frac{100}{s+1}$$

(in 1 rad/sec b.w. only 1% st.st.error)

$$\textcircled{1} \quad N(s) = \frac{1}{(s+1)^2}; M(s) = \frac{(s-2)^2}{(s+1)^2}$$

$$X(s) = 27 \frac{s-1}{s+1} \quad Y(s) = \frac{s+7}{s+1}$$

$$\textcircled{2} \quad k=2$$

$$\textcircled{3} \quad \text{choose } \epsilon \Rightarrow \|W_S NY(I - J)\|_\infty < 1$$

$$W_S NY(I - J) =$$

$$\frac{100 (s-2)^2 (s+7)}{(s+1)^4} \left[ 1 - \frac{1}{(\epsilon s + 1)^2} \right]$$

$\epsilon$	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$
$\infty$ -norm	199	19.97	1.997	0.1997

choose  $\epsilon = 10^{-4}$

$$\textcircled{4} \quad Q(s) = \frac{(s+1)(s+7)}{(10^{-4}s + 1)^2}$$

$$\textcircled{5} \quad C(s) = 10^4 \frac{(s+1)^3}{s(s+7)(10^{-4}s + 2)}$$

## 4-3-2) Non-min phase Systems ( $P^{\text{unstable}}$ )

14-16

For non-min phase case we must use interpolate condition: Assume that

- 1)  $P$  has no poles or zeros on the imaginary axis, only distinct poles and zeros in  $\text{Re}(s) > 0$ , and at least one unstable zero  $\text{Re}(s) > 0$
- 2)  $W_S(s)$  is stable and strictly proper.

To motivate, let us review the interpolate conditions. To find an internally stabilizing  $C \geq \|W_S(s)\|_{\infty} < 1$ , for internal stability

$$\begin{aligned} S(z) &= 1 & z = \text{unstable poles of } P \\ S(p) &= 0 & p = \text{" pole of } P \end{aligned}$$

Let's call  $G(s) = W_S(s) \cdot S(s)$  hence

$$\begin{aligned} G(z) &= W_S(z) & + \|G\|_{\infty} < 1 & \text{this is the NP problem.} \\ G(p) &= 0 \end{aligned}$$

Hence we need to solve the N.P. problem, but  $Q$  may be improper  $\Rightarrow$

$Q$  also not proper, what we do, first find  $Q_{\text{im}} \rightarrow Q = Q_{\text{im}} \times J$  to make it proper

Procedure:

input:  $P \in \mathbb{N}_S$

(1) Do a coprime factorization

$$P = N/M \quad NX + MY = I$$

(2) Find a stable  $Q_{\text{im}} \Rightarrow$  (NP soln)

$$\|W_S M(Y - NQ_{\text{im}})\|_{\infty} < 1.$$

+ interpolation Condition holds

using NP solution:

(3) Set  $L$  large enough  $\Rightarrow$

$Q_{\text{im}} \cdot J$  is proper &  $\epsilon$  small enough  $\Rightarrow$

$$\|W_S M(Y - NQ_{\text{im}} J)\|_{\infty} < 1$$

(4) Set  $Q = Q_{\text{im}} \cdot J$

(5) Set  $C = (X + MQ)/(Y - NQ)$ .

Note: If  $\lambda_{\text{opt}} < 1$ .

#### 4-4 A Comprehensive Example:

A non-min phase model of a flexible-beam 7-joint robot to be

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)}$$

The poles : 0, -0.0007, -0.3565 ± 5.27j

• zeros : -4.9881, 5.5308 Non-min-phase plant.

Note 1: 3 pole s=0 at j-w axis, we perturb it to a small No.  $\underline{\underline{10^{-6}}}$

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{5s^4 + 3.5682s^3 + 139.5021s^2 + 0.0929s + \underline{\underline{10^{-6}}}}$$

Note 2: weighting Selection:

Suppose we want a settling time  $\approx 8$  sec & overshoot  $\leq 10\%$  for a step reference input.  $\Rightarrow$  The desired closed loop syst must comply to a 2<sup>nd</sup> order system

$$T_{id}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\begin{aligned} t_s = 8 \text{ sec} \Rightarrow & \quad \gamma_0 \leq 1.0 \\ \frac{4.6}{\zeta\omega_n} = 8 & \quad e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.1 \quad \left. \begin{array}{l} \zeta = 0.5912 \\ \omega_n = 0.9585 \end{array} \right\} \end{aligned}$$

$$\text{choose } \zeta = 0.6 \text{ & } \omega_n = 1 \Rightarrow T_{id} = \frac{1}{s^2 + 1.2s + 1}$$

$$S_{id}(s) = 1 - T_{id}(s) = \frac{s(s+1.2)}{s^2 + 1.2s + 1}$$

$$f_N \|W_S S\| < 1 \rightarrow W_S = \frac{1}{S_{id}} = \frac{s^2 + 1.2s + 1}{s(s+1.2)} \quad \text{this is not strictly proper}$$

$$\Rightarrow W_S(s) = \frac{s^2 + 1.2s + 1}{(s+0.001)(s+1.2)(0.001s+1)}$$

Now run the procedure

14-18

① Since  $P \in \text{RRH}^\infty \Rightarrow N = P, M = 1, X = 0, Y = 1$

②  $\|W_S M(Y - NQ_{im})\|_\infty = \|W_S(1 - PQ_{im})\|_\infty \quad T_1 = W_S, T_2 = W_S P$

Phas a zero at  $s = 5.5308 \Rightarrow$

$$\min \|W_1(1 - PQ_{im})\|_\infty = |W_1(5.5308)| = 1.0210$$

hence No solution for this problem (The Conservativeness of  $Sol^n$ )

Relax the Performance criteria with

$$W_{Sn} = \frac{0.9}{1.0210} W_{S\text{old}} \Rightarrow \delta_{\text{opt}} = 0.9$$

Hence the Soln to NP problem is

$$Q_{im} = \frac{W_S - 0.9}{W_S P} \quad (\text{Max. No. Thkener})$$

$$Q_{im} = \frac{s(0.0008s^5 + 0.0221s^4 + 0.1768s^3 + 0.7007s^2 + 3.891s + 0.0026)}{s^3 + 6.1081s^2 + 6.8297s + 4.9801}$$

③ Set  $J(s) = \frac{1}{(Rs+1)^3}$  Compute  $\|W_S(1 - PQ_{im}J)\|_\infty$

$$\begin{array}{c|ccc} s & 0.1 & 0.05 & 0.04 \\ \hline \infty-\text{norm} & 1.12 & 1.01 & 0.988 \end{array}$$

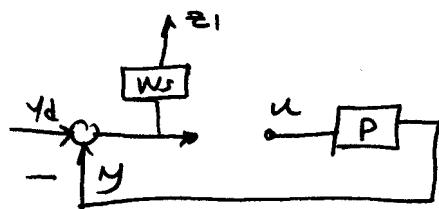
choose  $s = 0.04$

④  $Q = \frac{Q_{im}}{(0.04s+1)^3}$

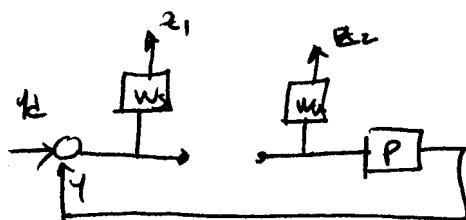
⑤ calculate  $C = Q / (1 - PQ)$

Note:  $Q$  has high order: we may reduce the order by getting stable & proper  
(Not C to not disturb int. st.)

# Solution by $\mu$ -synthesis



This will not satisfy the A2 Assumption  
 $A_{12} \neq \text{full rank}$

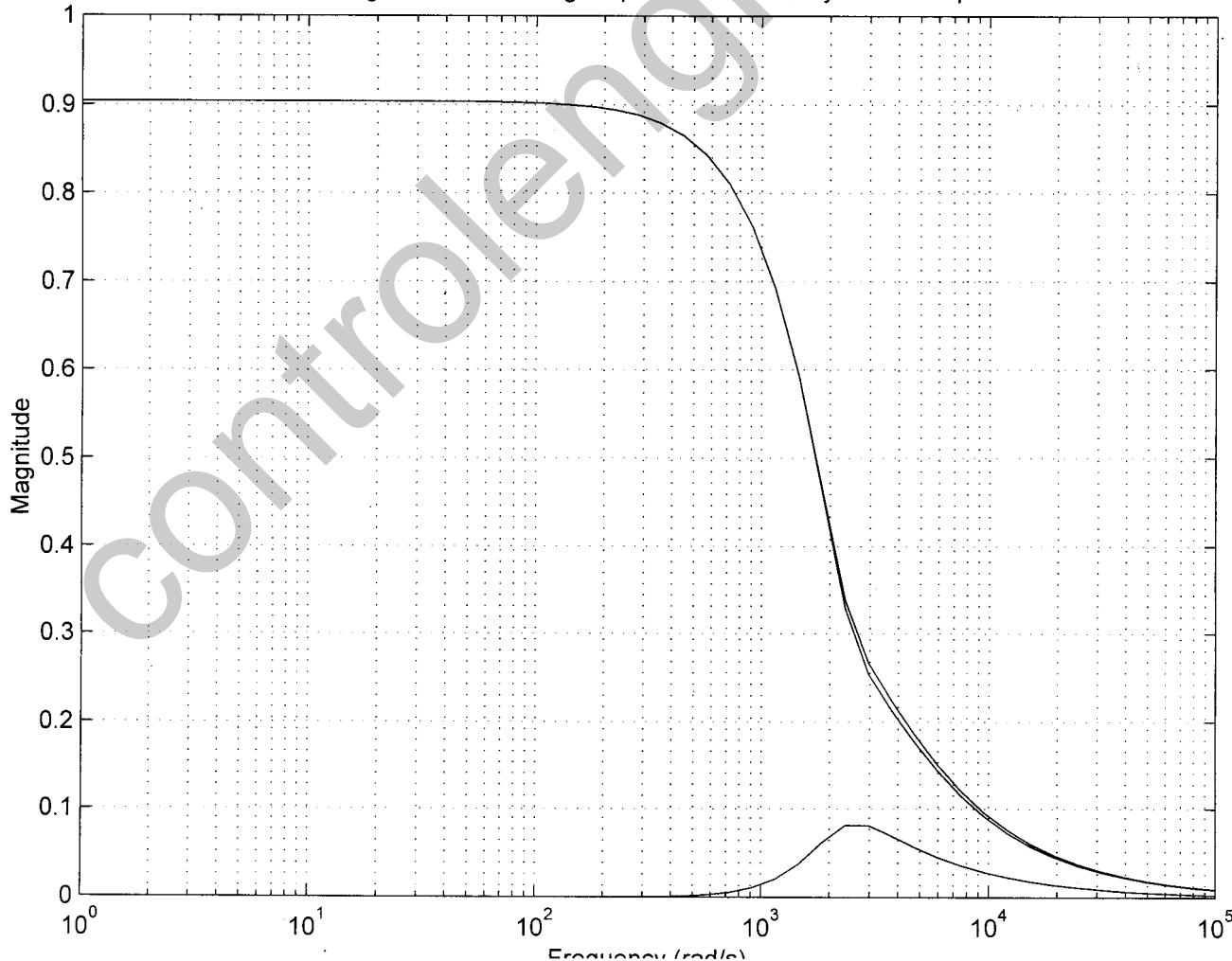


add an auxiliary output  $z_2 = W_u u$   
 $W_u \ll 1$  here  $W_u = 10^{-2}$

$$\gamma_{opt} = 0.904$$

$$K(s) = \frac{2.7538 \times 10^9 \times (s + 124.06)(s + 7.0294)(s + 0.356 + j5.26)}{(s + 1e-3)(s + 1.2)(s + 4.9083)(s + 1e3)(s + 2.9593 \times 10^3)(s + 9.04e2 + j2.26e3)}$$

Singular value and sigma plot of the H-infinity closed loop



```
% Copyright Dr. Hamid D. Taghirad 2005

clc
echo on
% This is the matlab file to construct H_inf solution of
% Nominal Performance solution for the flexible beam
% to the course note example
%
% -6.475s^2+4.0302s+175.77
% Plant = -----
%           5s^4+3.5682s^3+139.5021s+0.0929s+1e-6
%
%           s^2+1.2s+1
% Ws = -----
%           (s+1e-3)(s+1.2)(1e-3s+1)
pause % strike any key to continue

clc
s=tf('s'); % transfer function format

% plant transfer function
plant=(-6.475*s^2+4.0302*s+175.770)/...
(5*s^4+3.5682*s^3+139.5021*s^2+0.0929*s+1e-6);

% weighting function

ws= 0.9 / 1.021 * (s^2 + 1.2 * s + 1)/...
((s+1e-3)*(s + 1.2)*(1e-3*s+1));

% Actuator effort weighting function
wu=tf(1e-3,1); % To relax hinf solution assumption b1

clc

% Now that the plant and the weight have been defined,
% construct the interconnection structure, Plant_IC;

echo off
systemnames = 'plant ws wu';
inputvar = '[yd; u]';
outputvar = '[ws; wu; yd-plant]';
input_to_plant = '[u]';
input_to_ws = '[yd-plant]';
input_to_wu = '[u]';
sysoutname = 'plant_ic';
cleanupsysic = 'yes';
sysic
echo on

pause % strike any key to continue
clc
%
% We can change the properties of the generated plant using
% get and set commands or as following.
% See page 1-25 Control Toolbox user guide for the details.
%
```

```
plant_ic.InputName={'yd' 'u'}; % Set the input names
plant_ic.OutputName={'Nom_p' 'Act_u' 'e'}; % Set the output names

pause % strike any key to continue
clc
%%%
% Alternatively, we can use the |icsignal| and |iconnect|
% functions to build the closed-loop model:
%
P = iconnect;
yd = icsignal(1);
u = icsignal(1);
e = icsignal(1);
z = icsignal(2);
P.Input = [yd; u];
P.Output = [z; e];
P.Equation{1} = equate(z(1),ws*(yd-plant*u));
P.Equation{2} = equate(z(2),wu*u);
P.Equation{3} = equate(e,yd-plant*u);
P_ic = P.System;
P_ic.InputName = {'yd' 'u'};
P_ic.OutputName = {'nom_p' 'Act_u' 'e'};

pause % strike any key to continue
clc

% The next step is to design an H-infinity control law for PLANT
%
% In this example, the system interconnection structure is
% plant_ic, or P_ic with 1 measurements, 1 controls,
%
pause % strike any key to continue
clc
[k1,g1,gamma1,info1]=hinfssyn(plant_ic,1,1,'Display','on','Tolgam',1e-3);

%
% An H-infinity control law has been designed which achieves an
% infinity norm of 0.9066 for the interconnection structure.
%
% First, we will examine aspects of the controller
% that was just designed, starting with the controller POLES
% and zeros.
%
pause % strike any key to continue
clc
k1=minreal(k1);
k1=zpk(k1)

%
% Next, a Bode plot of the frequency response of k1
%
echo off
clf
omega=logspace(-1,6,200);
bodemag(k1,omega)
title('Bode plot of controller, k1')
xlabel('strike any key to continue')
```

```
format short e
echo on
pause % strike any key to continue
clc

% Now bode plot for the closed loop system

echo off
bodemag(g1(1,1),g1(2,1),':g',omega);
title('Closed loop Frequency responses; Nominal Performane, Actuator effort')
grid

%
% This concludes the example
%
```

## 4-9] Design for Robust Stability

[4-21]

As mentioned earlier Robust Stability  $\|WT\|_\infty < 1$  is very similar to the nominal performance  $\|WSS\|_\infty < 1$ , while using coprime factorization. Similar concern  $\Rightarrow$  properties of Q, & introducing J may apply here.

Also we assume:

P: has neither poles nor zeros on the imaginary axis

N: has no zeros on the imaginary axis

Example:  $P(s) = \frac{s-1}{(s+1)(s-P)}$   $0 < P \neq 1$

with uncertainty profile  $W(s) = \frac{s+0.1}{s+1}$

Solution:  $N(s) = \frac{s-1}{(s+1)^2}$   $M(s) = \frac{s-P}{s+1}$

$$X(s) = \frac{(P+1)^2}{P-1} \quad Y(s) = \frac{s - (P+3)/(P-1)}{s+1}$$

Since  $N(s)$  is Non-min phase, use inner-outer factorization

$$N(s) = N_i(s) \cdot N_o(s)$$

$$N_i(s) = \frac{s-1}{s+1} \quad N_o(s) = \frac{1}{s+1}$$

$$\|WN(X + MQ)\|_\infty = \|WN_o(X + MQ)\|_\infty$$

Then

$$T_1 = WN_o X \quad ; \quad T_2 = -WN_o M$$

with Model-Matching  $T_1 = WN_o X$  ;  $T_2 = -WN_o M$

$$T_1(s) = \frac{(P+1)^2(s+0.1)}{(P-1)(s+1)^2}$$

$$T_2(s) = -\frac{(s+0.1)(s-P)}{(s+1)^3}$$

Since  $\exists$  one zero at RHP :  $s = P$  for  $T_2$

$$\delta_{opt} = |T_1(P)| = \left| \frac{P+0.1}{P-1} \right|$$

as  $P \rightarrow 1$   $\delta_{opt} \nearrow$ , less uncertainty can be tolerated where  $P \rightarrow 3$

for  $p=0.5$   $\gamma_{opt} = 1.2 \rightarrow$  Nonrobust st. solut.

for  $p=0.2$   $\gamma_{opt} = \frac{0.3}{0.8} = 0.375$

Since we have only one interpolation constraint; the solution to the Model matching problem satisfies

$$T_1 - T_L Q_{im} = T_1(P)$$

$$Q_{im} = \left( \frac{1.2^2(s+0.1)}{-0.8(s+1)^2} - \frac{1.2^2 \cdot 0.3}{-0.8 \cdot 1/2^2} \right) / -\frac{(s+0.1)(s-0.2)}{(s+1)^3}$$

$$Q_{im} = \frac{(s+1)(1.44(s+0.1) - 0.3(s+1)^2)}{0.8(s+0.1)(s-0.2)}$$

Step 3 : Set  $\gamma_{opt} < \gamma_{opt} = 0.35$  and  $J(s) = \frac{1}{Cs+1}$

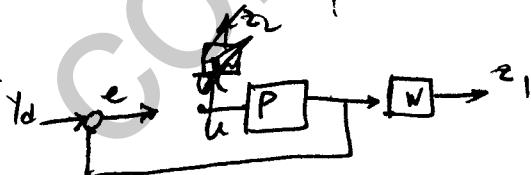
find  $\tau \Rightarrow \|W_2 N(X + MQ_{im} J)\|_\infty < 0.35$

$\tau = 10^{-3}$  will do that

$$\text{Step 4} \Rightarrow Q(s) = \frac{Q_m}{10^{-3}s + 1}$$

$$\text{Step 5} \Rightarrow C(s) = \frac{X + NQ}{Y - NQ} = \dots$$

→ Solution by pole synthesis:  $\text{with } 10^{-6} \text{ added for A21 ansatz}$



$$\gamma_{opt} = 0.375$$

$$C(s) = \frac{-2.6974 \times 10^2 \times (s + 1.0172)(s - 0.98591)}{(s + 6.72 \times 10^6)(s + 1.4724 \times 10^6)(s + 1.001)}$$

$$= \mu s^2 -$$

## 4-5] Design for Robust Stability

[4-21]

As mentioned earlier Robust Stability  $\|WT\|_\infty < 1$  is very similar to the nominal performance  $\|WSS\|_\infty < 1$ , while using coprime factorization. Similar concern  $\Rightarrow$  properties of  $Q$ , & introducing  $J$  may apply here.

Also we assume:

P: has neither poles nor zeros on the imaginary axis

N: has no zeros on the imaginary axis

Example:  $P(s) = \frac{s-1}{(s+1)(s-P)} \quad 0 < P \neq 1$

with uncertainty profile  $W(s) = \frac{s+0.1}{s+1}$

Solution:  $N(s) = \frac{s-1}{(s+1)^2} \quad M(s) = \frac{s-P}{s+1}$   
 $X(s) = \frac{(P+1)^2}{P-1} \quad Y(s) = \frac{s - (P+3)/(P-1)}{s+1}$

Since  $N(s)$  is Non min phase, use inner-outer factorization

$$N(s) = N_i(s) \cdot N_o(s)$$

$$N_i(s) = \frac{s-1}{s+1} \quad N_o(s) = \frac{1}{s+1}$$

$$\|WN(X+MQ)\|_\infty = \|WN_o(X+MQ)\|_\infty$$

Then  $\|WN(X+MQ)\|_\infty = \|WN_o(X+MQ)\|_\infty$

with Model-Matching  $T_1 = WN_o X ; T_2 = -WN_o M$

$$T_1(s) = \frac{(P+1)^2(s+0.1)}{(P-1)(s+1)^2}$$

$$T_2(s) = -\frac{(s+0.1)(s-P)}{(s+1)^3}$$

Since  $\exists$  one zero at RHP :  $s = P$  for  $T_2$

$$\delta_{opt} = |T_1(P)| = \left| \frac{P+0.1}{P-1} \right|$$

as  $P \rightarrow 1 \delta_{opt} \nearrow$ , less uncertainty can be tolerated where  $P \rightarrow 3$

```
% Copyright Dr. Hamid D. Taghirad 2005

clc
clear all
format short e
echo on
% This is the matlab file to construct robust stability solution
% to the course note example. New LTI system structures, and
% robust control tools are used in this example.
% Matlab ver 7 and Robust Control toolbox ver 3 is required
%
%          ( s - 1 )
% Plant =  -----
%                  (s + 1)(s - 0.2)
%
%          ( s + 0.1 )
% Wt   =  -----
%                  (s + 1)

pause % strike any key to continue

clc
%
% The Nominal plant and weights are build in zero-pole format
%
echo off
s=tf('s'); % zero-pole format
plant=(s-1)/((s+1)*(s-0.2)); % plant
wt=(s+0.1)/(s+1+eps); % weighting function
% Actuator effort weighting function
wu=tf(1e-7,1); % To relax hinf solution assumption b12
%
%
% Build multiplicative uncertainty block and uncertain system
%
%
delta = ultidyn('delta',[1 1]); % 1 by 1 full delta block
Pertplant = plant*(1+wt*delta); % Perturbed plant
Psample = usample(Pertplant,10); % 10 random samples of the plant
% Note in each run the samples vary
echo on
% Do a frequency_response for the nominal system,
% 10 samples of uncertain system and the uncertainty weight
%
%
% Wait a few seconds for calculations

echo off
omega = logspace(-3,2,200); % frequency vector
bodemag(plant,'b',Psample,'g:',omega); % frequency response of
% nominal and perturbed plants
grid;
title('Frequency Response of the Nominal Plant vs. 10 uncertain plant samples');
xlabel('Frequency (rad/s)');
ylabel('Magnitude');
echo on
```

```
%  
% The frequency responses are plotted in Figures Window  
%  
pause % strike any key to continue  
clc  
  
% Now that the plant and the weight have been defined,  
% construct the interconnection structure, Plant_IC;  
% We use the |icsignal| and |iconnect|  
% functions to build the closed-loop model  
%  
echo off  
  
P = iconnect;  
yd = icsignal(1);  
u = icsignal(1);  
e = icsignal(1);  
z = icsignal(1);  
P.Input = [yd; u];  
P.Output = [z; e];  
P.Equation{1} = equate(z,wt*plant*u);  
P.Equation{2} = equate(e,yd-(plant*u));  
plant_ic = P.System;  
plant_ic.InputName = {'yd' 'u'};  
plant_ic.OutputName = {'Robust_s' 'e'};  
echo on  
%  
%  
% We can make sure that our augmented system has its minimal  
% realization to avoid uncontrallability for some hidden modes  
% Check zero-poles patterns and use minreal  
%  
plant_ic = minreal(plant_ic);  
  
pause % strike any key to continue  
clc  
  
% The next step is to design an H-infinity control law for PLANT  
% In this example, the system interconnection structure is  
% plant_ic, with 1 measurements, 1 controls,  
  
[k1,g1,gamma1,info1]=hinfsyn(plant_ic,1,1,'TOLGAM',1e-4,'Display','on');  
  
%  
% First, we will examine aspects of the controller  
% that was just designed, starting with the controller POLES  
% and zeros.  
pause % strike any key to continue  
clc  
k1=minreal(k1);  
k1=zpk(k1)  
  
% Next, a Bode plot of the frequency response of k1
```

```
echo off
clf
bodemag(k1)
title('Bode plot of controller, k1')
xlabel('strike any key to continue')

format short e
echo on
pause % strike any key to continue
clc

% Onto the closed loop, first checking that it is stable.

echo off
g1=minreal(g1);
g1=zpk(g1)

echo on
% Now bode plot for a system

omega=logspace(0,5,200);
bodemag(g1,omega);

pause % strike any key to continue

% Next, find the peak norm of the frequency response, using
% PKVNORM. This corresponds to the maximum singular of the
% closed loop system.
echo off

[g1peak,freqpeak]=norm(g1,inf);

fprintf( '\n final gamma : %g',gamma1)
fprintf( '\n Max singular value: %g \n \n',g1peak)
echo on

pause % strike any key to continue
clc
%
% Now check the robustness of the controller in the closed loop
% First find the perturbed closed loop for 10 samples

echo off
CLpert = feedback( Pertplant*k1 , 1);
CLSsample = usample(CLpert,10);
CLnom = CLpert.Nominal;
t=0:1e-5:3e-3;
impulse(CLnom,'r',CLSsample,:g',t)
title('Closed loop impulse response of nominal and 10 perturbed system')

echo on
%
% Now See the impulse response of the nominal closed system
% and 10 perturbed ones.
```

```
% As it is seen in the figure window all responses are stable.  
pause % strike any key to continue  
clc  
%  
%% Robust Stability Analysis  
% Does the closed-loop system remain stable for all values of |k|, |delta|  
% in the ranges specified above? We use |robuststab| to answer this basic  
% robustness question:  
  
[stabmarg,destabunc,report,info] = robuststab(CLpert);  
report  
  
%  
% This concludes the example  
%
```

## 5 Mixed Sensitivity Problem

15-1

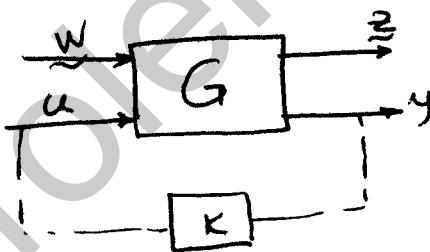
As we explained before, robust performance problem:

$$\|WSS\| + \|WT\|_{\infty} < 1$$

is modified into  $\|WSS\|_{\infty} < 1$  to have simpler solution. The analytic solution to this problem is a bit involved and may be not suitable for hand calculations, we use a matrix ST. Space solution approach, based on [DGK F, 89], which will be developed in this chapter:

### 5-1 Definition of the Design Problem

Refer to our perspective problem illustrated before and by following figure

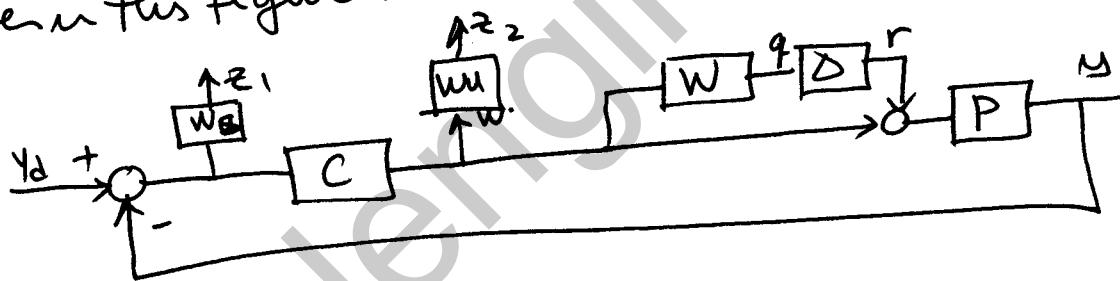


where  $w$  groups exogenous inputs (not in our hand), such as disturbance, setpoints or test inputs. The vector  $z$  represents performance variables, that must in a norm-based sense kept small. More precisely, the design objective is to keep the norm of transmission matrix  $T_{wz}$  small. This is in fact a generalization of mixed sensitivity problem, in which  $T_{wz}$  has one input  $y_d$  and two outputs representing  $WSS$  &  $WT$  in a suitable form.

The vector  $\underline{y}$  contains the control inputs, and  $\underline{y}$  are the measurements used for the feedback purposes. The problem definition is, hence, to identify the vectors  $\underline{w}$  and  $\underline{z}$ . Because, Specification for both stability, robustness & performance, are given in the form of the weighted norms of certain transmissions, we must locate input-output pairs that give the required transmissions:

### Comprehensive Example:

The most common example which illustrates problem definition is given in this figure:



The transmission of interests are the weighted Sensitivity  $W_S S$ , the weighted complementary Sensitivity  $W_T$  which is equal to the transfer function from  $r$  to  $q$  (with a block removed) and measure to illustrate the size of control effort, we may show that with Full weighted control effort T.F..

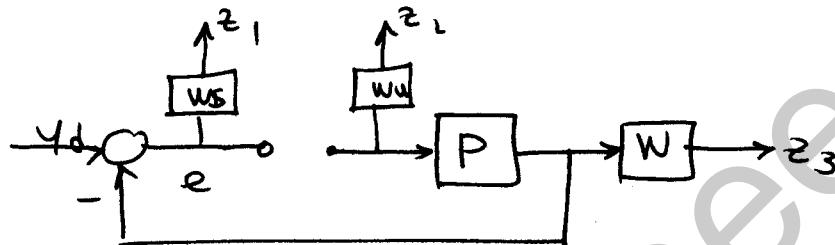
We have to define an input vector  $\underline{w}$  and the output  $\underline{z}$  which yields to this 3 desired transfer functions

a)  $W_S S$  is the transfer function from  $Y_d$  to  $z_1$   $T_{Y_d z_1} = \frac{W_S S}{1 + CP} = W_S S$

$$\textcircled{2} \quad T_{y_d} z_2 = W_u U = W_u \cdot F.S = W_u P^T \cdot T$$

gives the control effort measure

- \textcircled{3} To get WT, we may remove ( $\Delta$ ) block and have the transfer function  $T_{rq}$ . However, with this combination we have two inputs  $y_d$  and  $r$  and 3 outputs  $z_1, z_2, q$  and hence 6 transfer matrices, we may rearrange to have only 3 needed transfer functions.

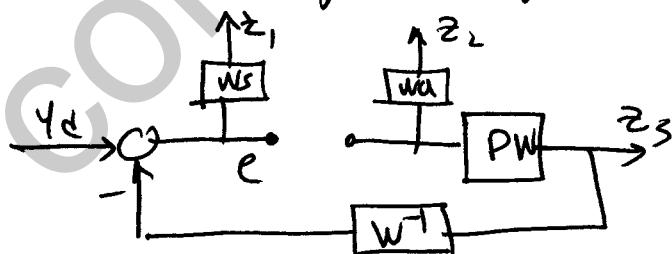


in which  $\tilde{W} = W_u$  and  $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$

and  $T_{y_d} z = \begin{pmatrix} W_u S \\ W_u U \\ W T \end{pmatrix}$  without introducing any extra unwanted T.F.

usually  $W$  is nonproper we may use  $W^\dagger$  for properness if

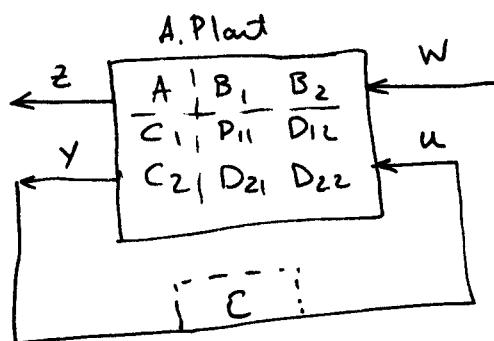
$P$  has relative degree  $>$  degree of  $W$



This is what has been used in the previous reference papers given.

## 5-2 The Augmented State-Space Model

The state-space representation of the system can be written in the form of



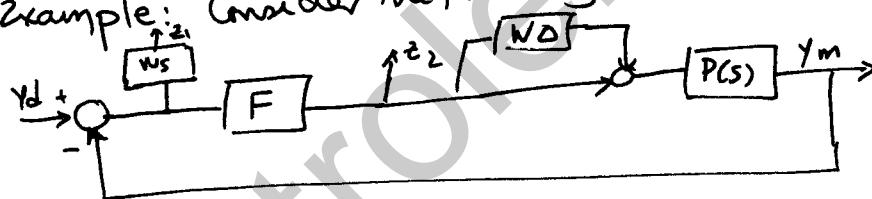
arrows in this direction make the system representation

which represents:

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases}$$

This representation combines the system model and the models of the various weight functions, as shown by the following example.

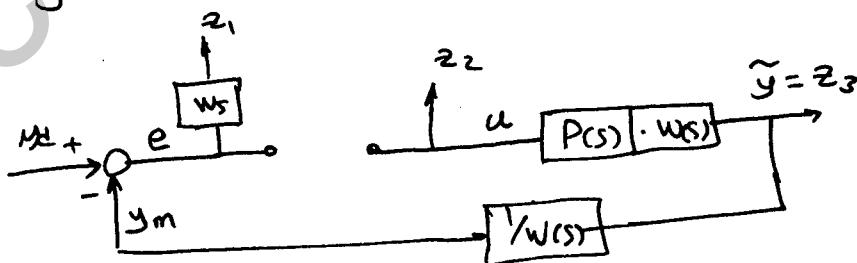
Example: Consider the following System :



objective //  $\frac{W_S S}{W_T U}$  //

$$\text{where } P(s) = \frac{1}{s^2 + 0.2s + 1}, \quad W_S(s) = \frac{s+1}{s(s+10)}, \quad W(s) = 0.002(s+10)^2$$

removing the "NO" block to reduce the complexity



$$\frac{z_3}{u} = \frac{2(s^2 + 20s + 100)}{1000(s^2 + 0.2s + 1)} = 2 \times 10^{-3} \left[ 1 + \frac{19.8s + 99}{s^2 + 0.2s + 1} \right]$$

or use tf2ss  $\Rightarrow$   $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = x_1$

$$y = x_1$$

$$e = y_d - x_1$$

$$z_3 = 2 \times 10^{-3} [99x_1 + 19.8x_2] + 2 \times 10^{-3} u$$

"c"                          "d"

$$\text{Now: } \frac{z_1}{e} = w_g(s) = \frac{s+1}{s(s+10)} \rightarrow \exists \text{ a pole on jw-axis}$$

$$\text{perturb the pole slightly} \quad \frac{s+1}{(s+0.01)(s+10)} = \frac{s+1}{s^2 + 10.01s + 0.1}$$

$$\Rightarrow \frac{z_1}{e} = \frac{s+1}{s^2 + 10.01s + 0.1}$$

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.1 & -10.01 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (y_d - x_1)$$

"use Controllable Canonical form"

$$z_1 = x_3 + x_4$$

Augment all the components

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -0.1 & -10.01 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} y_d + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [-1 \ 0 \ 0 \ 0] \tilde{x} + y_d$$

$$\tilde{z} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0.198 & 0.0396 & 0 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 1 \\ 2 \times 10^{-3} \end{bmatrix} u$$

$\mu$ -synthesis solution:

We may use sysic command in  $\mu$ -synthesis for this

Note: State-space representation are not unique use minreal to get minimal realization when T.F. are given as above.

## 5-3] $H^2$ Solution

5-6

You are familiar with LQR solutions. When no notion of robustness is required we may minimize  $\|T_{wz}(s)\|_2$ . We bring the solution here to get acquainted with  $H^2$  solution, and to compare. This solution is less conservative, But has no claim on robustness.

### $H^2$ Solution Assumptions:

A1:  $(A_1, B_1)$  and  $(A_2, B_2)$  are stabilizable.

(Note not controllable, weaker cond. only unstable modes controllable)

A2:  $(C_1, A)$  and  $(C_2, A)$  are detectable.

A3:  $D_{12}^T D_{22}$  and  $D_{21} D_{21}^T$  nonsingular

all components of  $u$  affect  $z$  (controllable); all components of  $w$  affect  $y$

A4:  $D_{11} = 0$

$w$  is not affecting  $z$  directly.

We write  $T_{wz}$  by columns, as

$$T_{wz} = [(T_{wz})_1 \ (T_{wz})_2 \ \dots \ (T_{wz})_m].$$

To calculate  $H^2$ -norm, from

$$\text{tr } T_{wz}^T(-s) T_{wz}(s) = \sum_{i=1}^m (T_{wz})_i^T(-s) (T_{wz})_i(s).$$

The square of the  $H^2$ -norm is obtained by integrating the RHS of the above eq. over the  $j$ -axis : use Parseval

$$\|T_{wz}\|_2^2 = \sum_{i=1}^m \int_0^\infty \| (T_{wz})_i(t) \|^2 dt$$

where  $(Twz)_i(t) = \mathcal{L}^{-1}[(Twz)_i(s)]$  is the response  $z(t)$  to a unit impulse  $w(t)$ . Now this is equal to initial state response  $b_i$ , the  $i$ th column of the matrix  $B_1$ . Let  $z(x_0, t)$  be the response to an initial state  $x_0$ . We may then write  $\|Twz\|_2^2$  as

$$\begin{aligned}\|Twz\|_2^2 &= \sum_{i=1}^m \int_0^\infty \|z(b_i, t)\|^2 dt \\ &= \sum_{i=1}^m \int_0^\infty \|C_1 x(b_i, t) + D_{12} u(b_i, t)\|^2 dt\end{aligned}$$

The problem, then, is to minimize the RHS of the above equation

with:

$$\dot{x} = Ax + B_2 u$$

$$y = C_2 x + D_{21} w + D_{22} u$$

Let us examine the "fullstate feedback"  $\tilde{y} = \tilde{x}$  and expand the term corresponding to  $i=1$ :

$$\begin{aligned}J_1 &= \int_0^\infty [x^T(b_1, t) C_1 C_1^T x(b_1, t) + x^T(b_1, t) C_1^T D_{12} u(b_1, t) \\ &\quad + u^T(b_1, t) D_{12}^T C_1 x(b_1, t) + u^T(b_1, t) D_{12}^T D_{12} u(b_1, t)] dt\end{aligned}$$

This form an LQ problem, even with the inclusion of cross terms

The solution of that problem is of the form  $\tilde{u} = -k \tilde{x}$

and independent of the initial state. The same control law minimizes all  $J_i$ 's, and hence minimizes the sum of them  $J = \sum_{i=1}^m J_i$

To remove the cross terms define a new input

$$v = u + (D_{12}^T D_{12})^{-1} D_{12}^T C_1 x.$$

it is easy to show that :

$$\ddot{x} = [A - B_2(D_{12}^T D_{12})^{-1} D_{12}^T C_1]x + B_2 w$$

$$\mathcal{J}_1 = \int_0^\infty [x^T C_1^T [I - D_{12}(A_2^T D_{12})^{-1} D_{12}^T] C_1 x + v^T (D_{12}^T A_2) v] dt$$

This presents a standard LQ problem. Its solution is :

$$0 = -Kx \Rightarrow u = -[K + (P_{12}^T D_{12})^{-1} A_2^T C_1]x = -Kx$$

There is no need to do this transform since, all numerical packages solve their corresponding Riccati Eq. with the cross terms.

If the states are not all available, then state estimation is also required. Since the solution is a bit involved, we summarize the

Solution as follows:

For the system

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + 0 + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{cases}$$

1. The Control gain:

Solve the LQ Control for

$$\ddot{x} = Ax + B_2 u$$

$$\mathcal{J} = \int_0^\infty [x^T u^T] \begin{bmatrix} C_1^T C_1 & C_1^T A_2 \\ D_{12}^T C_1 & D_{12}^T D_{12} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

and obtain the gain matrix K

2. Solve the LQ Estimation (K.F.) for

$$\ddot{x} = Ax + w$$

$$y = C_2 x + 0$$

with noise intensity matrix  $E\{[w][w^T v^T]\} = \begin{bmatrix} B_1 B_1^T & B_1 D_{21}^T \\ D_{21} B_1^T & D_{21} D_{21}^T \end{bmatrix}$  and find matrix G

### 3. The controller

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B_2u + G(y - D_{22}u - C_2\hat{x}) \\ = (A - B_2K - GC_2 + GD_{22}K)\hat{x} + Gy \\ u = -K\hat{x} \end{cases}$$

### 4. The optimal norm

$$\min \|Twz\|_2^2 = \|T_c\|_2^2 + \|T_f\|_2^2$$

where  $T_c$  is the transmission of

$$\begin{aligned}\dot{x} &= (A - B_2K)x + B_1w \\ z &= (C_1 - D_{12}K)x\end{aligned}$$

and  $T_f$  is the transmission of

$$\begin{aligned}\dot{\tilde{x}} &= (A - GC_2)\tilde{x} + (B_1 - GD_{21})w \\ v &= K\tilde{x}\end{aligned}$$

$\mu$ -synthesis Solution:

By generating augmented plant, the  $H^2$  solution is calculated

By h2syn

## 5-4 $H^\infty$ Mixed Sensitivity Solution

15-10

The sub-optimal Solution of M.S.P. is to find a stabilizing controller which makes  $\|T_{wz}\|_\infty < \gamma$

The optimal Solution can be find analytically, but is not of that importance since we may iterate by decreasing  $\gamma$  until a solution fails to exist.

The detail of Solution is give in [DGKF], we don't go through details. Start with Hamiltonian matrix

$$H = \begin{bmatrix} A & -R \\ -Q & A^T \end{bmatrix}$$

with  $R$  &  $Q$  are symmetric but not necessarily pos. def. All submatrices in  $H$  are of  $n \times n$  dim. The matrix  $H$  is symplectic and has the property that its eigenvalues are located symmetrically wr.t. the  $j\omega$ -axis as well as the real axis.

Assume  $H$  has no eigenvalues on the  $j\omega$ -axis: "the stability Condition"

We form the  $2n \times n$  matrix  $X$  with columns from the eigenvectors and generalized eigenvectors of  $H$  that corresponds to LHP eigenvalues for complex conjugate pairs, we use the real and imaginary parts.

The resulting matrix is partitioned into  $n \times n$  matrices  $X_1$  and  $X_2$  as follows:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

If  $X_1$  is nonsingular,  $H$  is said to satisfy the complementary condition.

If  $H$  has no  $j\omega$ -axis eigenvalues and  $X_1^{-1}$  exists, then  $P = X_2 X_1^{-1}$  satisfies the Riccati equation,  $A^T P + PA - PRP + Q = 0$

and in addition,  $A - RP$  is a stable matrix.

[5-11]

The converse is not true:  $P$  may satisfy the Riccati equation and  $A - RP$  may be stable without the complementarity and stability conditions being satisfied.

If  $R > 0$  and  $Q \geq 0$ , a squared root method can be used to express  $R$  as  $BB^T$  and the Riccati equation is the familiar one.

If, in addition,  $(A, B)$  is stabilizable and  $(Q^{1/2}, A)$  is detectable then the complementarity and stability conditions can be shown to hold.

[DGKF] gives a solution under the following assumptions:

$$\left\{ \begin{array}{l} D_{11} = D_{22} = 0 \\ D_{12}^T D_{12} = D_{21} D_{21}^T = I \\ B_1 D_{21}^T = D_{12}^T C_1 = 0 \end{array} \right.$$

(These assumption can be relaxed in numerical packages)

The  $H_\infty$  solution revolves around the two Hamiltonian matrices

$$H_C = \begin{bmatrix} A & \gamma^2 B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}; \quad H_f = \begin{bmatrix} A^T & \gamma^2 C_1^T C_1 - C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}$$

The technical assumptions are:

A1:  $(A, B_2)$  is stabilizable and  $(C_2, A)$  is detectable.

A2:  $D_{12}^T D_{12}$  and  $D_{21} D_{21}^T$  are nonsingular

A3 [DGKF]:

$(A, B_1)$  is stabilizable and  $(C_1, A)$  is detectable

OR: A3 [GD]:  $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{21} \end{bmatrix}$

and  $\begin{bmatrix} A^T - j\omega I & C_2^T \\ B_1^T & D_{21}^T \end{bmatrix}$

have full column rank for all  $\omega$

(5-12)

Both A3 assumptions disallow uncontrollable modes of  $(A, B_1)$  and unobservable modes of  $(G, A)$  on the imag axis. However, (A3b) would allow open RHP uncontrollable or unobservable modes, whereas (A3a) would not. On the other hand, (3Ab) states that  $(A, B_2, C_1, D_{11})$  and  $(A^T, C_2^T, B_1^T, D_{11}^T)$  shall have no transmission zeros on the jw-axis, whereas (3Aa) has nothing to say about zeros.

The Theorem calculates the solution

Theorem: Mixed Sensitivity Solution

There exists an admissible controller  $\Rightarrow \|Tw\|_\infty < \gamma$ , iff,

- i)  $H_c$  and  $H_f$  both satisfy the complementarity and stability conditions.
- ii) The Riccati-equation solutions  $P_c$  and  $P_f$  associated with  $H_c$  and  $H_f$  are positive Semidefinite
- iii) The spectral radius (i.e. The maximum eigenvalues)

$$\rho(P_c P_f) < \gamma^{-2}$$

when these condition holds, one such controller is

$$\tilde{x} = A_c \tilde{x} + G_c y$$

$$u = -K_c \hat{x}$$

where:

$$A_c = A + (\gamma^2 B B_1^T - B_2 B_2^T) P_c - (I - \gamma^2 P_f P_c)^{-1} P_f C_2^T C_2$$

$$G_c = (I - \gamma^2 P_f P_c)^{-1} P_f C_2^T$$

$$K_c = B_2^T P_c$$

Notes:

- ① As  $\gamma \rightarrow 0$ , the solution tends to the  $H^2$  solution. Therefore, a solution always exists for small enough  $\gamma$  under the technical condition of the  $H^2$  solution section.
- ② The difference between this and standard Riccati-equation solution is that the terms of  $H_c$  and  $H_f$  may not be neg. def., in order to comp. & stability properties are satisfied.
- ③ The design objectives are stated in terms of  $\infty$ -norm of submatrices of  $T_{WZ}$ . The design process is iterative to tend to  $\gamma_{opt}$ . If the  $\infty$ -norm of a particular submatrix exceed the desired performance limit its weight is increased in the following iteration, conversely a weight may be decreased if the  $\infty$ -norm is well below its target bound.
- ④ In calculating the Setup problem, it is important to reduce the size of elements of  $T_{WZ}$  to what is really needed for performance. If 3 norm of interest are desired to be minimized simultaneously  $T_{WZ}$  must be  $3 \times 1$ , if we don't remove '0' from Block diagram size of  $T_{WZ}$  is increased and the solution may be more conservative and only may not exists.

# State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems

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**Abstract**—Simple state-space formulas are derived for all controllers solving a standard  $\mathcal{H}_\infty$  problem: for a given number  $\gamma > 0$ , find all controllers such that the  $\mathcal{H}_\infty$  norm of the closed-loop transfer function is (strictly) less than  $\gamma$ . A controller exists if and only if the unique stabilizing solutions to two algebraic Riccati equations are positive definite and the spectral radius of their product is less than  $\gamma^2$ . Under these conditions, a parametrization of all controllers solving the problem is given as a linear fractional transformation (LFT) on a contractive, stable free parameter. The state dimension of the coefficient matrix for the LFT, constructed using these same two Riccati solutions, equals that of the plant, and has a separation structure reminiscent of classical LQG (i.e.,  $\mathcal{H}_2$ ) theory. This paper is also intended to be of tutorial value, so a standard  $\mathcal{H}_2$  solution is developed in parallel.

## I. INTRODUCTION

### A. Overview

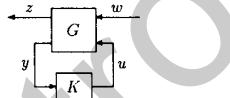
TWO popular performance measures in optimal control theory are  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norms, defined in the frequency-domain for a stable transfer matrix  $G(s)$  as

$$\|G\|_2 := \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[G(j\omega)^* G(j\omega)] d\omega \right)^{1/2}$$

$$\|G\|_\infty := \sup_{\omega} \sigma_{\max}[G(j\omega)] \quad (\sigma_{\max} := \text{maximum singular value}).$$

The former arises when the exogenous signals either are fixed or have a fixed power spectrum; the latter arises from (weighted) balls of exogenous signals.  $\mathcal{H}_2$ -optimal control theory was heavily studied in the 1960's as the linear quadratic Gaussian (LQG) optimal control problem;  $\mathcal{H}_\infty$ -optimal control theory is continuing to be developed.

The basic block diagram used in this paper is



where  $G$  is the generalized plant and  $K$  is the controller. Only finite-dimensional linear time-invariant (LTI) systems and controllers will be considered in this paper. The generalized plant  $G$  contains what is usually called the plant in a control problem plus all weighting functions. The signal  $w$  contains all external inputs,

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including disturbances, sensor noise, and commands; the output  $z$  is an error signal;  $y$  is the measured variables; and  $u$  is the control input. The diagram is also referred to as a linear fractional transformation (LFT) on  $K$ , and  $G$  is called the coefficient matrix for the LFT. The resulting closed-loop transfer function from  $w$  to  $z$  is denoted by  $T_{zw}$ .

The main  $\mathcal{H}_\infty$  output feedback results of this paper as described in the Abstract are presented in Section III. The proofs of these results in Section V exploit the “separation” structure of the controller, which is reminiscent of the classical  $\mathcal{H}_2$  controller. Of course, there are significant differences that reflect the fact that the  $\mathcal{H}_\infty$  criterion corresponds to designing for the worst exogenous signal. These are also discussed in Section V. Special attention will be given to the central controller, obtained by setting to zero the free parameter in the LFT formula for the controller.

If full state feedback is available, then the central controller is simply a gain matrix  $F_\infty$ , obtained through solving a single Riccati equation. Also, the optimal estimator is an observer whose gain is obtained as a solution to a Riccati equation. These special cases are described in Section IV and the proofs are given in Sections VII and VIII. In the general output feedback case the central controller can be interpreted as an optimal estimator for  $F_\infty x$ . Furthermore, the two Riccati equations involved in this solution can be associated with state feedback and estimation problems.

The algebraic Riccati equations in the  $\mathcal{H}_\infty$  solution are those that arise in the theory of linear quadratic differential games. The game theoretic analogy is intuitively appealing for in the  $\mathcal{H}_\infty$  control problem the exogenous input and the control input can be viewed as strategies employed by opposing players in a game: the exogenous input is chosen to maximize the norm of the output and the control input is chosen to minimize it. The Riccati equations have indefinite quadratic terms, however, so solutions cannot be guaranteed as simply as in the  $\mathcal{H}_2$  problem. Indeed, as mentioned in the Abstract, the existence of solutions to the Riccati equations is part of the necessary and sufficient conditions for existence of  $\mathcal{H}_\infty$  (sub)optimal controllers. The preliminary machinery needed to establish these conditions in terms of Riccati equations is developed in Section II.

To facilitate exposition, the problem chosen for treatment in this paper is the simplest special case which captures all the essential features of the general problem. Although the assumptions used in this special case involve some sacrifice of generality, the formulas are simple and easy to interpret. Also, these assumptions are commonly used in treatments of the  $\mathcal{H}_2$  problem. The general formulas are presented in [14]. Our entire approach to the  $\mathcal{H}_\infty$  problem has parallels in the conventional  $\mathcal{H}_2$  theory. In fact, it is interesting to note that as  $\gamma$  tends to  $\infty$ , the central controller actually approaches the  $\mathcal{H}_2$  controller. Because the  $\mathcal{H}_2$  theory is well-understood throughout the control community, these two problems are developed side-by-side. It is hoped that this will enhance the paper's tutorial value.

This paper is organized in a top-down manner and is intended to be accessible to a variety of readers. While this organization may be a bit awkward for the experts who plan to study all the details, it is hoped that it will enhance the tutorial value of the paper. The main results are stated in Sections I–IV. The reader who is only interested in seeing the new theorems and formulas could stop

## Design Example:

15-14

For the previous system

$$P(s) = \frac{1}{s^2 + 0.2s + 1}; W_S = \frac{s+1}{(s+0.01)(s+10)}; W_u(u) = 1; W(s) = \frac{2(s+10)^2}{1000}$$

The design objectives are

- i)  $|T_{ye} W_S(j\omega)| \leq 2 \text{ rad/sec}$  bandwidth  $\approx 1 \text{ rad/sec}$
- ii)  $|T_{yu}(j\omega)| \leq 1 \text{ rad/sec}$  bound on control effort Mag.
- iii)  $|T_{yy}(j\omega)W(j\omega)| \leq 1 \text{ rad/sec}$  Robust Stability

Solution: From previous example augmented system is evaluated

But since  $|W_S(s)S(s)| < 2$  use  $W_{Sn} = 2 W_S = \frac{2(s+1)}{(s+0.01)(s+10)}$  to make the

objective either  $\left\| \begin{bmatrix} W_S \\ W_u U \\ W_T \end{bmatrix} \right\|_2 < 1$  or  $\left\| \begin{bmatrix} W_S \\ W_u U \\ W_T \end{bmatrix} \right\|_\infty < 1$

Hence:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -0.1 & -10.01 \end{bmatrix} \quad \text{correct for } W_{Sn}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0.198 & 0.0376 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2e^{-3} \\ 1 & 0 \end{bmatrix}$$

and use trinfsyn  $\Rightarrow \gamma_{opt} = 0.9993$

$$K(s) = \frac{0.02598(s+9.983)(s+0.1 \pm j0.9499)}{(s+10)(s+0.01)(s+0.2699 \pm j1.0102)}$$

h2syn  $\Rightarrow \text{norm2} = 0.33046$

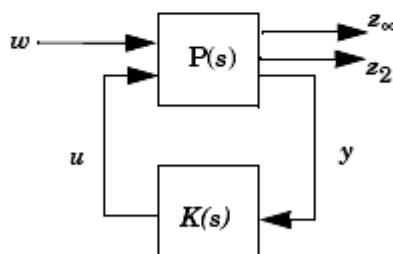
$$K(s) = \frac{0.040803(s+9.971)(s+0.1 \pm j0.9499)}{(s+10)(s+0.01)(s+0.1655 \pm j1.0035)}$$

## Multi-Objective $H_\infty$ Synthesis

In many real-world applications, standard  $H_\infty$  synthesis cannot adequately capture all design specifications. For instance, noise attenuation or regulation against random disturbances are more naturally expressed in LQG terms. Similarly, pure  $H_\infty$  synthesis only enforces closed-loop stability and does not allow for direct placement of the closed-loop poles in more specific regions of the left-half plane. Since the pole location is related to the time response and transient behavior of the feedback system, it is often desirable to impose additional damping and clustering constraints on the closed-loop dynamics. This makes multi-objective synthesis highly desirable in practice, and LMI theory offers powerful tools to attack such problems.

Mixed  $H_2/H_\infty$  synthesis with regional pole placement is one example of multi-objective design addressed by the LMI Control Toolbox. The control problem is sketched in Figure 5-5. The output channel  $z_\infty$  is associated with the  $H_\infty$  performance while the channel  $z_2$  is associated with the LQG aspects ( $H_2$  performance).

Denoting by  $T_\infty(s)$  and  $T_2(s)$  the closed-loop transfer functions from  $w$  to  $z_\infty$  and  $z_2$ , respectively, we consider the following multi-objective synthesis problem:



**Figure 5-5: Multi-objective  $H_\infty$  synthesis**

Design an output-feedback controller  $u = K(s)y$  that

- Maintains the  $H_\infty$  norm of  $T_\infty(s)$  (RMS gain) below some prescribed value  $\gamma_0 > 0$
- Maintains the  $H_2$  norm of  $T_2(s)$  (LQG cost) below some prescribed value  $v_0 > 0$
- Minimizes a trade-off criterion of the form

$$\alpha \|T_\infty\|_\infty^2 + \beta \|T_2\|_2^2$$

with  $\alpha \geq 0$  and  $\beta \geq 0$

- Places the closed-loop poles in some prescribed LMI region  $D$

Recall that LMI regions are general convex subregions of the open left-half plane (see “Pole Placement in LMI Regions” on page 4-5 for details).

## LMI Formulation

Let

$$\begin{cases} \dot{x} = Ax + B_1w + B_2u \\ z_\infty = C_\infty x + D_{\infty 1}w + D_{\infty 2}u \\ z_2 = C_2x + D_{21}w + D_{22}u \\ y = C_y x + D_{y1}w \end{cases}$$

and

$$\begin{cases} \dot{\zeta} = A_K \zeta + B_K y \\ \dot{u} = C_K \zeta + D_K y \end{cases}$$

be state-space realizations of the plant  $P(s)$  and controller  $K(s)$ , respectively, and let

$$\begin{cases} \dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl} w \\ z_\infty &= C_{cl1}x_{cl} + D_{cl1} w \\ z_2 &= C_{cl2}x_{cl} + D_{cl2} w \end{cases}$$

be the corresponding closed-loop state-space equations.

Our three design objectives can be expressed as follows:

- **$H_\infty$  performance:** the closed-loop RMS gain from  $w$  to  $z_\infty$  does not exceed  $\gamma$  if and only if there exists a symmetric matrix  $X_\infty$  such that

$$\begin{pmatrix} A_{cl}\chi_\infty + \chi_\infty A_{cl}^T & B_{cl} & X_\infty C_{cl1}^T \\ B_{cl}^T & -I & D_{cl1}^T \\ C_{cl1}\chi_\infty & D_{cl1} & -\gamma^2 I \end{pmatrix} < 0$$

$$\chi_\infty > 0$$

- **$H_2$  performance:** the  $H_2$  norm of the closed-loop transfer function from  $w$  to  $z_2$  does not exceed  $v$  if and only if  $D_{cl2} = 0$  and there exist two symmetric matrices  $\chi_2$  and  $Q$  such that

$$\begin{pmatrix} A_{cl}\chi_2 + \chi_2 A_{cl}^T & B_{cl} \\ B_{cl}^T & -I \end{pmatrix} < 0$$

$$\begin{pmatrix} Q & C_{cl2}\chi_2 \\ \chi_2 C_{cl2}^T & \chi_2 \end{pmatrix} > 0$$

$$\text{Trace}(Q) < v^2$$

- **Pole placement:** the closed-loop poles lie in the LMI region

$$D = \{z \in \mathbb{C} : L + Mz + M^T \bar{z} < 0\}$$

with  $L = L^T = \{\lambda_{ij}\}_{1 \leq i,j \leq m}$  and  $M = [\mu_{ij}]_{1 \leq i,j \leq m}$  if and only if there exists a symmetric matrix  $\chi_{pol}$  satisfying

$$[\lambda_{ij}\chi_{pol} + \mu_{ij}A_{cl}\chi_{pol} + \mu_{ji}X_{pol}A_{cl}^T]_{1 \leq i,j \leq m} < 0$$

$$\chi_{pol} > 0$$

For tractability in the LMI framework, we must seek a single Lyapunov matrix

$$\chi := \chi_\infty = \chi_2 = \chi_{pol}$$

that enforces all three sets of constraints. Factorizing  $\chi$  as

$$\chi = \chi_1 \chi_2^{-1}, \quad \chi_1 := \begin{pmatrix} R & I \\ M^T & 0 \end{pmatrix}, \quad \chi_2 := \begin{pmatrix} 0 & S \\ I & N^T \end{pmatrix}$$

and introducing the change of controller variables [3]:

$$\begin{cases} B_K &:= NB_K + SB_2D_K \\ C_K &:= C_KM^T + D_KC_yR \\ A_K &:= NA_KM^T + NB_KC_yR + SB_2C_KM^T + S(A + B_2D_KC_y)R, \end{cases}$$

the inequality constraints on  $\chi$  are readily turned into LMI constraints in the variables  $R, S, Q, A_K, B_K, C_K$ , and  $D_K$  [8, 1]. This leads to the following suboptimal LMI formulation of our multi-objective synthesis problem:

Minimize  $\alpha \gamma^2 + \beta \text{Trace}(Q)$  over  $R, S, Q, A_K, B_K, C_K, D_K$ , and  $\gamma^2$  satisfying:

$$\begin{aligned} & \left( \begin{array}{cccc} AR + RA^T + B_2 C_K + C_K^T B_2^T & A_K^T + A + B_2 D_K C_y & B_1 + B_2 D_K D_{y1} & H \\ H & A^T S + SA + B_K C_y + C_y^T B_K^T & SB_1 + B_K D_{y1} & H \\ H & H & -I & H \\ C_\infty R + D_\infty 2 C_K & C_\infty + D_\infty 2 D_K C_y & D_\infty 1 + D_\infty 2 D_K D_{y1} & -\gamma^2 I \end{array} \right) < 0 \\ & \left( \begin{array}{ccc} Q & C_2 R + D_{22} C_K & C_2 + D_{22} D_K C_y \\ H & R & I \\ H & I & S \end{array} \right) > 0 \\ & \left[ \lambda_{ij} \begin{pmatrix} R & I \\ I & S \end{pmatrix} + \mu_{ij} \begin{pmatrix} AR + B_2 C_K & A + B_2 D_K C_y \\ A_K & SA + B_K C_y \end{pmatrix} + \right. \\ & \quad \left. \mu_{ji} \begin{pmatrix} RA^T + C_K^T B_2^T & A_K^T \\ (A + B_2 D_K C_y)^T A^T S + C_y^T B_K^T \end{pmatrix} \right]_{1 \leq i, j \leq m} < 0 \\ & \text{Trace } Q < v_0^2 \\ & \gamma^2 < \gamma_0^2 \\ & D_{21} + D_{22} D_K D_{y1} = 0 \end{aligned}$$

Given optimal solutions  $\gamma^*$ ,  $Q^*$  of this LMI problem, the closed-loop  $H_\infty$  and  $H_2$  performances are bounded by

$$\|T_\infty\|_\infty \leq \gamma^*, \quad \|T_2\|_2 \leq \sqrt{\text{Trace}(Q^*)}.$$

## The Function hinfmix

The function `hinfmix` implements the LMI approach to mixed  $H_2/H_\infty$  synthesis with regional pole placement described above. Its syntax is

```
[gopt,h2opt,K,R,S] = hinfmix(P,r,obj,region)
```

where

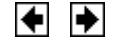
- $P$  is the SYSTEM matrix of the LTI plant  $P(s)$ . Note that  $z_2$  or  $z_\infty$  can be empty, or even both when performing pure pole placement
- $r$  is a three-entry vector listing the lengths of  $z_2$ ,  $y$ , and  $u$
- $obj = [v_0, v_1, \alpha, \beta]$  is a four-entry vector specifying the  $H_2/H_\infty$  constraints and criterion
- $region$  specifies the LMI region for pole placement, the default being the open left-half plane. Use `lmireg` to interactively generate the matrix `region`.

The outputs `gopt` and `h2opt` are the guaranteed  $H_\infty$  and  $H_2$  performances,  $K$  is the controller SYSTEM matrix, and  $R, S$  are optimal values of the variables  $R, S$  (see “LMI Formulation” on page 5-16).

You can perform the following mixed and unmixed designs by setting `obj` appropriately:

obj	Corresponding Design
[0 0 0 0]	pole placement only
[0 0 1 0]	$H_\infty$ -optimal design
[0 0 0 1]	$H_2$ -optimal design
[g 0 0 1]	minimize $\ T_2\ _2$ subject to $\ T_\infty\ _\infty < g$
[0 h 1 0]	minimize $\ T_\infty\ _\infty$ subject to $\ T_2\ _2 < h$
[0 0 a b]	minimize $a\ T_\infty\ _0^2 + b\ T_2\ _2^2$
[g h a b]	most general problem

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**Robust Control Toolbox**

# h2hinfsyn

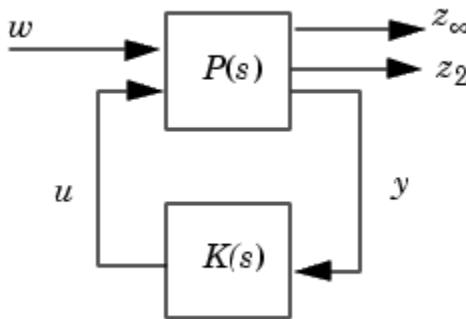
Mixed  $H_2/H^\infty$  synthesis with pole placement constraints

## Syntax

```
[gopt,h2opt,K,R,S] = hinfmix(P,r,obj,region,dkbnd,tol)
```

## Description

h2hinfyn performs multi-objective output-feedback synthesis. The control problem is sketched in this figure.



**Figure 11-4: Mixed  $H_2/H^\infty$  synthesis**

If  $T^\infty(s)$  and  $T_2(s)$  denote the closed-loop transfer functions from  $w$  to  $z^\infty$  and  $z_2$ , respectively, hinfmix computes a suboptimal solution of the following synthesis problem:

Design an LTI controller  $K(s)$  that minimizes the mixed  $H_2/H^\infty$  criterion

$$\alpha \|T^\infty\|_\infty^2 + \beta \|T_2\|_2^2$$

subject to

- $\|T^\infty\|^\bullet < \gamma_0$
- $\|T_2\|_2 < v_0$
- The closed-loop poles lie in some prescribed LMI region D.

Recall that  $\|\cdot\|^\bullet$  and  $\|\cdot\|_2$  denote the  $H^\infty$  norm (RMS gain) and  $H_2$  norm of transfer functions.

P is any SS, TF, or ZPK LTI representation of the plant  $P(s)$ , and r is a three-entry vector listing the lengths of  $z_2$ ,  $y$ , and  $u$ . Note that  $z^\infty$  and/or  $z_2$  can be empty. The four-entry vector obj = [ $\gamma_0$ ,  $v_0$ ,  $\alpha$ ,  $\beta$ ] specifies the  $H_2/H^\infty$  constraints and trade-off criterion, and the remaining input arguments are optional:

- region specifies the LMI region for pole placement (the default region = [] is the open left-half plane). Use lmireg to interactively build the LMI region description region
- dkbnd is a user-specified bound on the norm of the controller feedthrough matrix DK. The default value is 100. To make the controller  $K(s)$  strictly proper, set dkbnd = 0.
- tol is the required relative accuracy on the optimal value of the trade-off criterion (the default is 10-2).

The function h2hinfsyn returns guaranteed  $H^\infty$  and  $H_2$  performances gopt and h2opt as well as the SYSTEM matrix K of the LMI-optimal controller. You can also access the optimal values of the LMI variables R, S via the extra output arguments R and S.

A variety of mixed and unmixed problems can be solved with `hinfmix`. In particular, you can use `hinfmix` to perform pure pole placement by setting `obj = [0 0 0 0]`. Note that both  $z^\infty$  and  $z2$  can be empty in such case.

## Reference

Chilali, M., and P. Gahinet, " $H^\infty$  Design with Pole Placement Constraints: An LMI Approach," to appear in *IEEE Trans. Aut. Contr.*, 1995.

Scherer, C., "Mixed H2 H $\bullet$  Control," to appear in *Trends in Control: A European Perspective*, volume of the special contributions to the ECC 1995.

## See Also

`lmireg` Specify LMI regions for pole placement purposes  
`msfsyn` Multi-model/multi-objective state-feedback synthesis

 gridureal

 h2syn 

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```
% Copyright Dr. Hamid D. Taghirad 2005

clear all
clc
echo on
%
% This is the matlab file to construct H_2 and H_inf and h2/hinf
% solution of course example page 5-14, with second order
% nominal model. An iterative approach is used to refine the hinf
% controller design.
%
% Plant = -----
%           s^2 +0.2s + 1
%
%           10(s + 1)
% ws   = -----
%           (s+0.01)(s + 10)
%
%           2*(s+10)^2
% wt   = -----
%           1000
%
% wu   = 1
%

pause % strike any key to continue

%
% Construct the interconnected plant using sysic
%
s=zpk('s'); % zero-pole format

% plant transfer function
plant=1/(s^2+0.2*s+1);

% uncertainty weighting function
wt= (s+10)^2/500;
wt_1= 1/wt;

% augmented plant
Pw = plant*wt;

% performance weighting function
ws= 2*(s+1)/((s+0.01)*(s+10));

%
% Actuator effort weighting function
wu=tf(1,1);

%
% Now that the plant and the weight have been defined,
% construct the interconnection structure, P_IC;

echo off
systemnames = 'Pw wt_1 wu ws';
inputvar = '[yd; u]';
outputvar = '[Pw; ws ; wu; yd - wt_1]';
```

```

input_to_Pw = '[u]';
input_to_wt_1 ='[Pw]';
input_to_wu = '[u]';
input_to_ws = '[yd - wt_1]';
sysoutname = 'P_ic';
cleanupsysic = 'yes';
sysic
P_ic.InputName={'yd' 'u'}; % Set the input names
P_ic.OutputName={'Rob_st' 'Nom_p' 'Act_e' 'e'}; % Set the output names

P_ic=minreal(P_ic);

%clear plant Pw wt_1 wu ws
echo on
pause % strike any key to continue
clc
[k1,g1,gamma1,info1]=hinfsyn(P_ic,1,1,'Display','on','To',1e-4);

pause % strike any key to continue
clc
%
% Plot and analyze the sigma plot and objective function bode
%
echo off
omega=logspace(-2,3,200);
[mag_g1,phase_g1]=bode(g1,omega);
sig_g1=sigma(g1,omega);
semilogx(omega,sig_g1,omega,mag_g1(1,:),'--r',...
omega,mag_g1(2,:),'-g',omega,mag_g1(3,:),'-.m')

echo on
%
% As it is seen in the plot the actuator effort dominates the objective
% function. A better design would be to add frequency content to the
% actuator weighting function. Here we reduce the actuator effort at
% low frequency with a factor of 0.9 and permit higher amplitude at
% high frequencies, to obtain a better performance.
%
wu=0.9*(0.1*s+1)/(s+1);

pause % strike any key to continue
clc

% Regenerate the system interconnection P_ic

echo off
systemnames = 'Pw wt_1 wu ws';
inputvar = '[yd; u]';
outputvar = '[Pw; ws ; wu; yd - wt_1]';
input_to_Pw = '[u]';
input_to_wt_1 ='[Pw]';
input_to_wu = '[u]';
input_to_ws = '[yd - wt_1]';
sysoutname = 'P_ic';
cleanupsysic = 'yes';
sysic

```

```

P_ic.InputName={'yd' 'u'};           % Set the input names
P_ic.OutputName={'Rob_st' 'Nom_p' 'Act_e' 'e'};       % Set the output names

P_ic=minreal(P_ic);

echo on
pause % strike any key design the controller
clc
[k2,g2,gamma2,info2]=hinfsyn(P_ic,1,1,'Display','on','To',1e-4);

pause % strike any key to continue
clc
%
% Plot and analyze the sigma plot and objective function bode
%
echo off
[mag_g2,phase_g2]=bode(g2,omega);
sig_g2=sigma(g2,omega);
semilogx(omega,sig_g2,omega,mag_g2(1,:),'--r', ...
omega,mag_g2(2,:),':g',omega,mag_g2(3,:),'-.m')

echo on
%
% As it is seen in the plot by this change, the infinity norm of the
% performance index comes down to one.
%
% Look at the robust stability curve the dashed red line, it is
% way below one, so we expect good stability margin.
%
% Look at the nominal performance curve dotted-green. its peak is also
% beyond 0.6. Hence we can better performance by increasing the
% bandwidth.
%
ws=2*(s+5)/((s+0.05)*(s+10));

echo off
systemnames = 'Pw wt_1 wu ws';
inputvar     = '[yd; u]';
outputvar   = '[Pw; ws ; wu; yd - wt_1]';
input_to_Pw = '[u]';
input_to_wt_1='[Pw]';
input_to_wu = '[u]';
input_to_ws = '[yd - wt_1]';
sysoutname = 'P_ic';
cleanupsysic = 'yes';
sysic
P_ic.InputName={'yd' 'u'};           % Set the input names
P_ic.OutputName={'Rob_st' 'Nom_p' 'Act_e' 'e'};       % Set the output names

P_ic=minreal(P_ic);

echo on
pause % strike any key design the controller
clc
[k3,g3,gamma3,info3]=hinfsyn(P_ic,1,1,'Display','on','To',1e-4);

```

```

pause % strike any key to continue
clc
%
% Plot and analyze the sigma plot and objective function bode
%
echo off
[mag_g3,phase_g3]=bode(g3,omega);
sig_g3=sigma(g3,omega);
semilogx(omega,sig_g3,omega,mag_g3(1,:),'--r', ...
omega,mag_g3(2,:),':g',omega,mag_g3(3,:),'-.m')

echo on

%
% Now Solve H2 solution using h2syn
%

[k4,g4,gamma4,info4]=h2syn(P_ic,1,1);
disp('The 2-norm of objective function is');
disp(gamma2);

pause % strike any key to continue
clc
%
% Plot and analyze the sigma plot and objective function bode
%
echo off
[mag_g3,phase_g3]=bode(g3,omega);
sig_g3=sigma(g3,omega);
semilogx(omega,sig_g3,omega,mag_g3(1,:),'--r', ...
omega,mag_g3(2,:),':g',omega,mag_g3(3,:),'-.m')

pause % strike a key to continue
clc
echo on

%
% Design H2/Hinf controller: Nominal Performance (using Robust Toolbox)
%
% [K,CL,NORMZ,INFO] = H2HINFSYN(P,NMEAS,NCON,NZ2,WZ,key1,value1,...)
% Mixed H2/H-infinity synthesis with regional pole-placement
% constraints.      NEW : ROBUST CONTROL TOOLBOX
%
% Given an LTI plant P with partitioned state-space form:
% dx/dt = A * x + B1 * w + B2 * u
% zinf = Ci * x + Di1 * w + Di2 * u
% z2 = C2 * x + D21 * w + D22 * u
% y = Cy * x + Dy1 * w + Dy2 * u
% H2HINFSYN employs LMI techniques to compute an output-feedback control
% law u = K(s)*y that
% * keeps the HINFNORM gain G from w to zinf below the value OBJ(1)
% * keeps the H2NORM H from w to z2 below the value OBJ(2)
% * minimizes a trade-off criterion of the form
%     WEIGHT(1) * G^2 + WEIGHT(2) * H^2
% * places the closed-loop poles in the LMI region specified
% by REGION.

```

```

%
% Input:
% P          LTI plant
% NMEAS      Number of measurements (length of y)
% NCON       Number of control channels (length of u)
% NZ2        length of z2
% WZ         1x2 vector of weights on zinf and z2 (i.e., weights for
%             H-infinity and H2 performance, respectively)
%
% Option keywords and values:
% REGION    Mx(2M) matrix [L,M] specifying the pole
%             placement region as
%             { z : L + z * M + conj(z) * M' < 0 }
% Use the interactive function LMIREG to generate
% REGION. The default REGION=[] enforces just
% closed-loop stability
% H2MAX     upper bound on the H2 norm w->z2 (default = Inf)
% HINFMAX   upper bound on the HINFNORM gain w->zinf (default = Inf)
% DKMAX     bound on the norm of the
%             feedthrough gain DK of K(s)
%             (100=default, 0 yields a strictly proper controller)
% TOL        desired relative accuracy on the objective
%             value (default=1e-2)
% DISPLAY   display synthesis information to screen, (default = 'off')
%
% Output:
% K          optimal output-feedback controller
% CL         lft(P,K) (closed-loop system)
% NORMZ     1x2 vector of closed-loop norms (HINFNORM gain w->zinf
%             and H2 norm w->z2)
%
% INFO      struct array:
%             INFO.R      solution R of LMI solvability condition
%             INFO.S      solution S of LMI solvability condition
nmeas=1;
ncon =1;
nz2 =1;
h2max=1.2;
alpha=1;
beta =0;
wz=[alpha beta]; % alpha = 1; beta = 0 cost:= alpha* inf_norm + beta* 2-norm

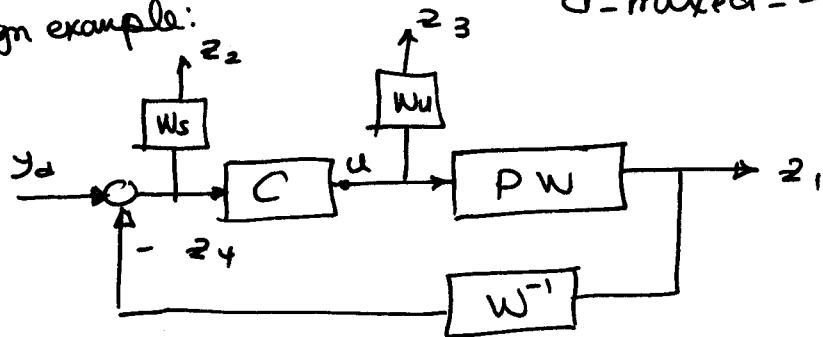
[k5,g5,normz5,info5]=h2hinfssyn(ss(P_ic),nmeas,ncon,nz2,wz, ...
'Display','off','h2max',h2max,'hinfmax',inf);
pause %strike a key to continue
clc
%
% Plot and analyze the sigma plot and objective function bode
%
echo off
[mag_g5,phase_g5]=bode(g5,omega);
sig_g5=sigma(g5(1:2,:),omega); % in this plot only
semilogx(omega,sig_g5,omega,mag_g5(1,:),'--r', ...
omega,mag_g5(2,:),'-g',omega,mag_g5(3,:),'-.m')
pause %strike a key to continue
clc

```

```
% Plot a magnitude plot of the frequency response of k1 to k5
```

```
clf  
bodemag(k1,'b',k2,'--r',k3,:g',k4,'-.m',k5,'c',omega)  
legend('hinf 1','hinf 2','hinf 3','h2','h2/hinf')
```

① Design example:



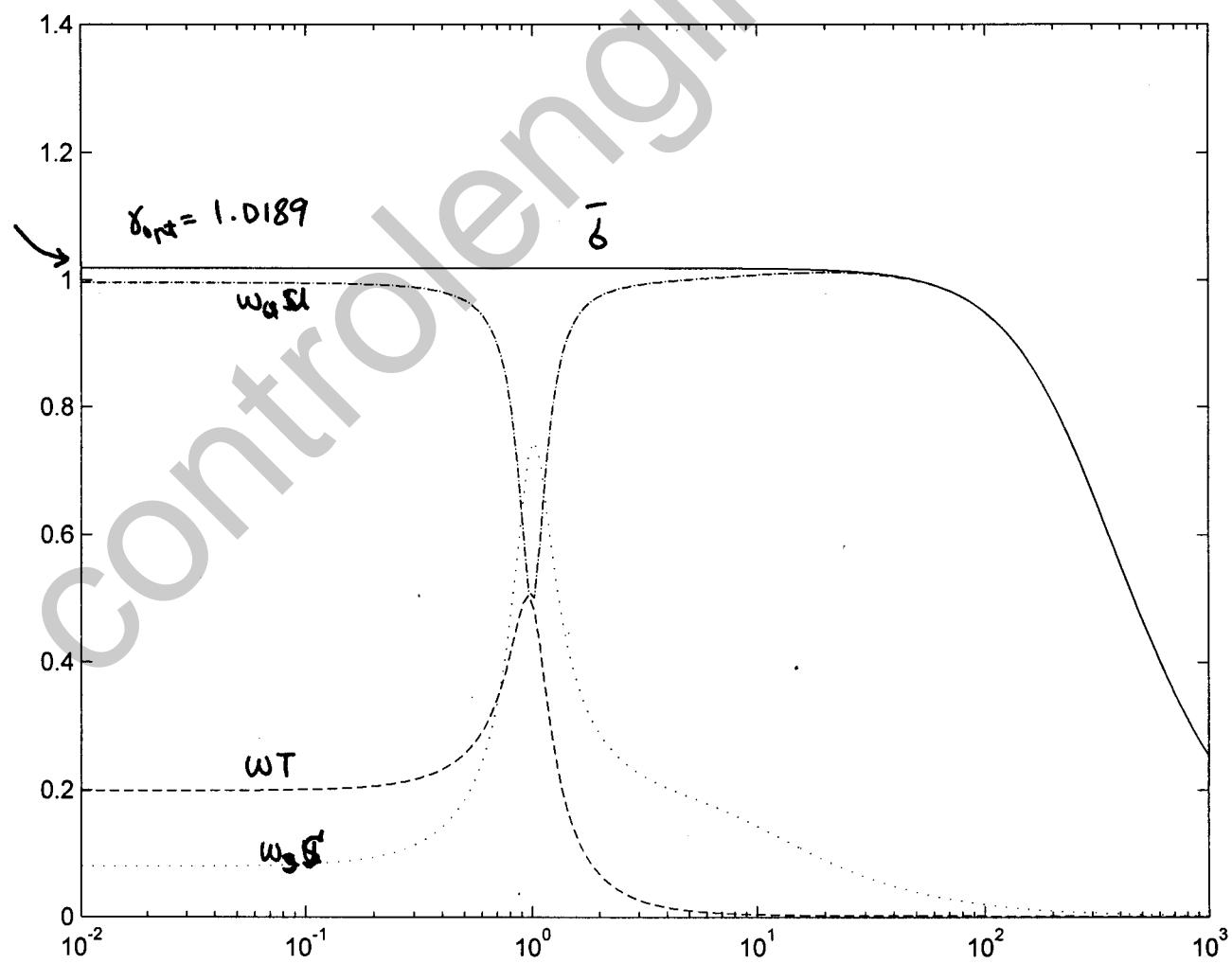
d-mixed-sensitivity.

$$\min \left\| \begin{array}{l} w_T \\ w_S S \\ w_u U \end{array} \right\|$$

$$P(s) = \frac{1}{s^2 + 0.2s + 1} ; \quad w(s) = \frac{(s+10)^2}{500} ; \quad w_s(s) = \frac{2(s+1)}{(s+0.01)(s+10)}$$

i) For now set  $w_u = 1$ .

$$Y_{opt} = \min_r \left\| \begin{array}{l} w_T \\ w_S S \\ w_u U \end{array} \right\|_\alpha = 1.0189$$



2) Design improvement: Note  $|W_{SS}| < 1$  but  $\bar{\delta} > 1$

either make  $|WT|$  smaller  $\rightarrow$  Uncertainty profile

or make  $W_u$  smaller  $\rightarrow$  this is possible (higher actuator limits)

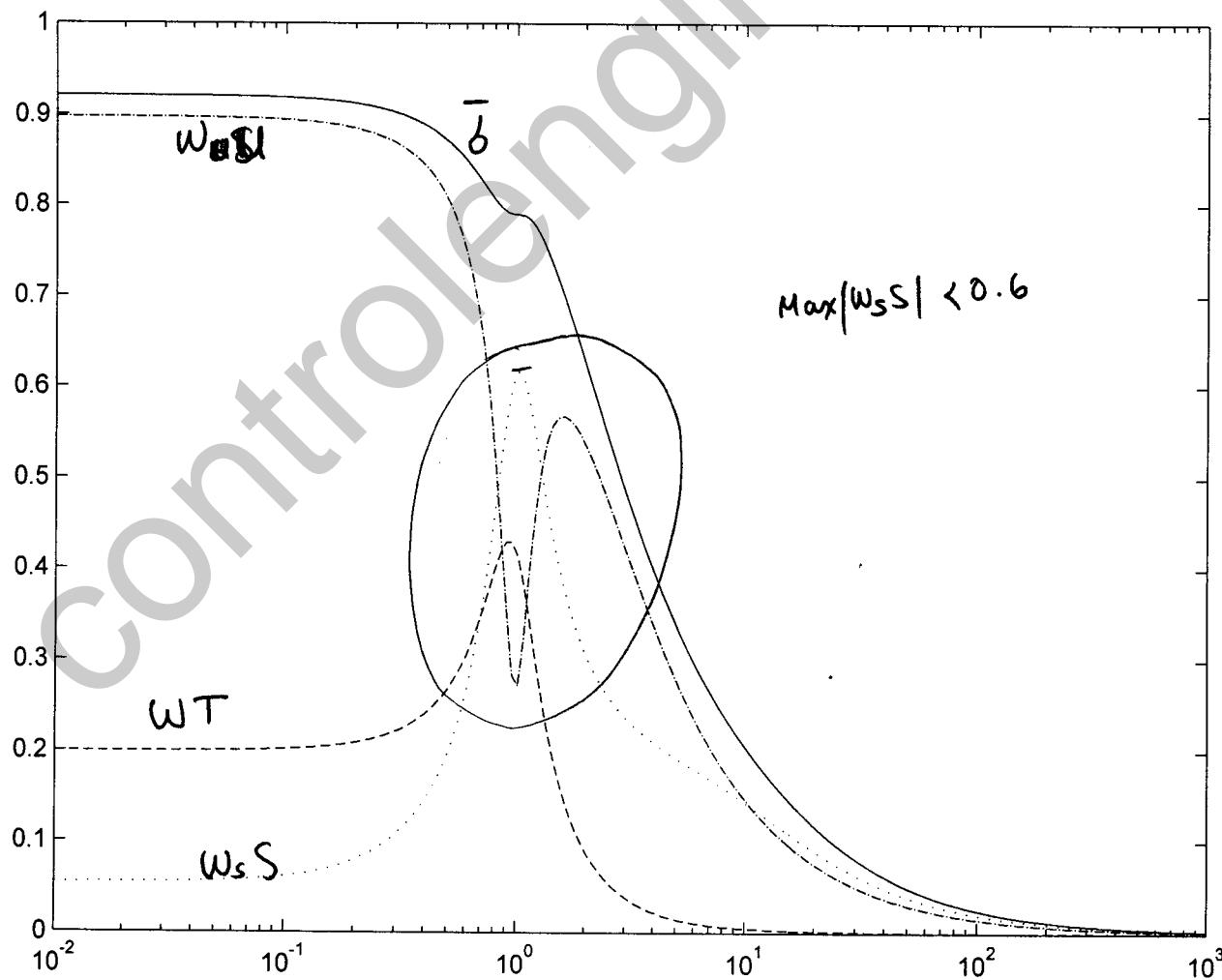
$$W_u = 0.9 \xrightarrow[\text{bared}]{\text{use frequency}} W_u = 0.9 \frac{0.9s + 1}{s + 1}$$

$$\text{Solve } \left\| \begin{matrix} WT \\ W_{SS} \\ W_u U \end{matrix} \right\|_\infty < 1 \Rightarrow \gamma_{opt} = 0.9211$$

Look at the three component contributions:

interact Sensitivity & control effort!

$WT \ll 1 \rightarrow$  good stability margin



### ③ Design improvement

$$\omega_{\text{old}} = \frac{2(s+1)}{(s+0.01)(s+10)} \longrightarrow \omega_s = \frac{2(s+5)}{(s+0.05)(s+10)}$$

keep the DC gain but increase the bandwidth from  $1 \rightarrow 5$  rad/sec

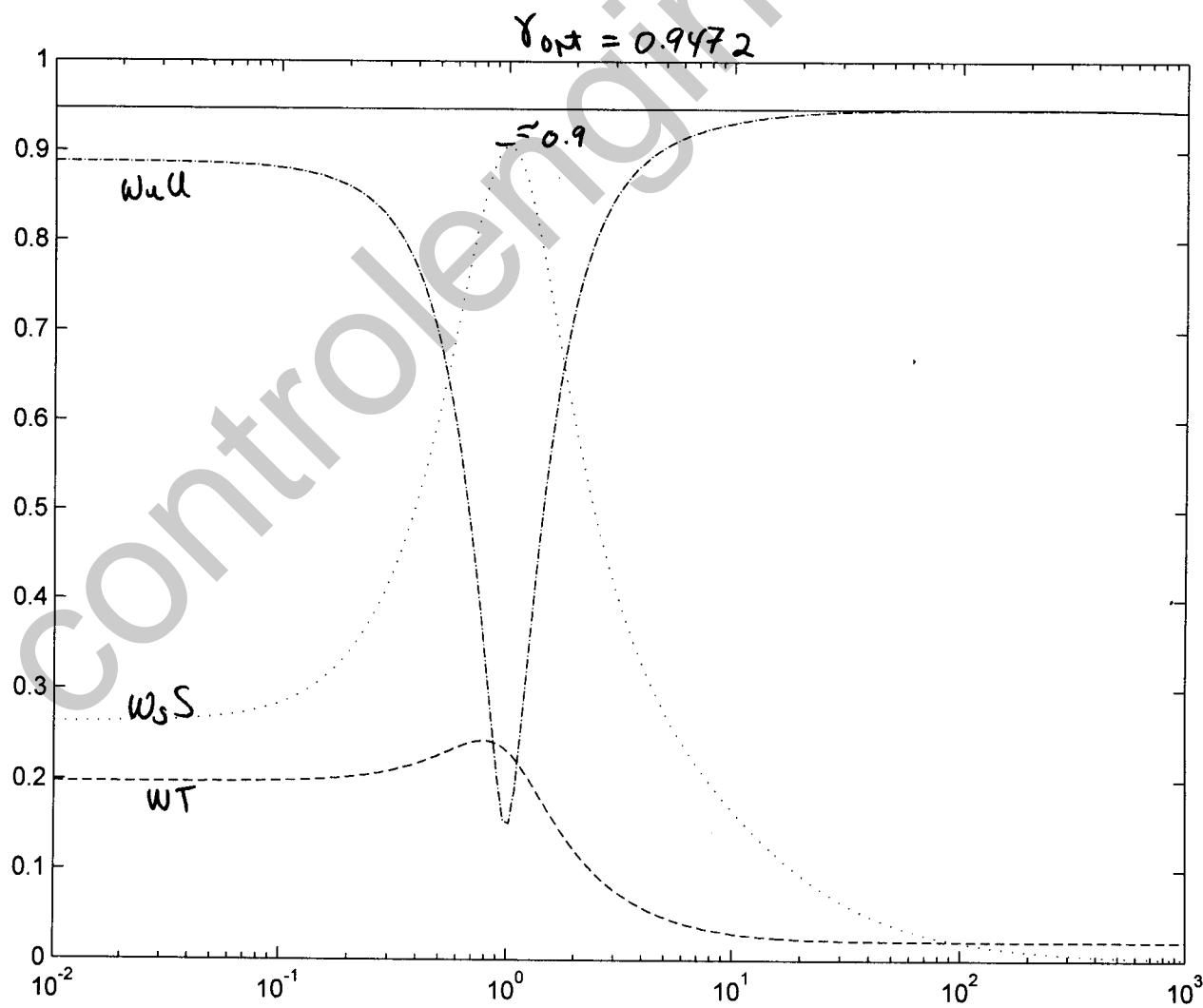
$\bar{\theta}$  is getting flat ✓

$\omega_{\text{ull}} < 1$

$\omega_s S$  is getting closer to 1

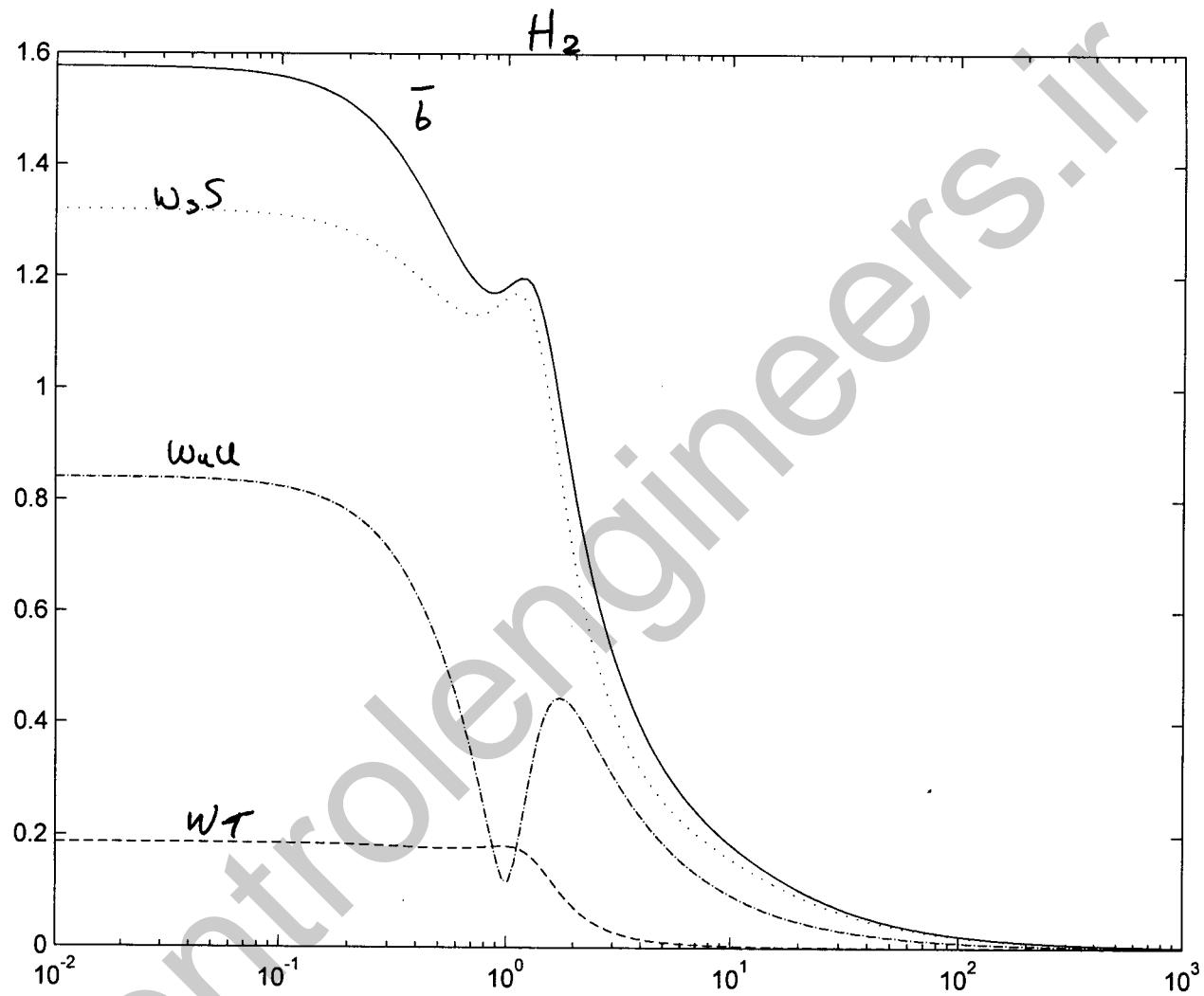
This could be named as a good compromise

$$K(s) = \frac{137903.8 (s+1)(s+9.842)(s^2 + 0.2s + 1)}{(s + 1.3 \times 10^4)(s + 10.97)(s + 10)(s + 0.2519)(s + 0.05)}$$



④ Check the  $H_2$  Solution

$$\left\| \begin{pmatrix} w_s^T \\ w_u^T \end{pmatrix} \right\|_2 < 1 \quad \rightarrow f_{2\text{opt}} = 1.1861 \quad \rightarrow \text{No claim of Robustness}$$



⑤ The nature of  $u$  is a signal its better to solve  $\| \cdot \|_\infty$

for Systems &  $\| \cdot \|_2$  for Signals

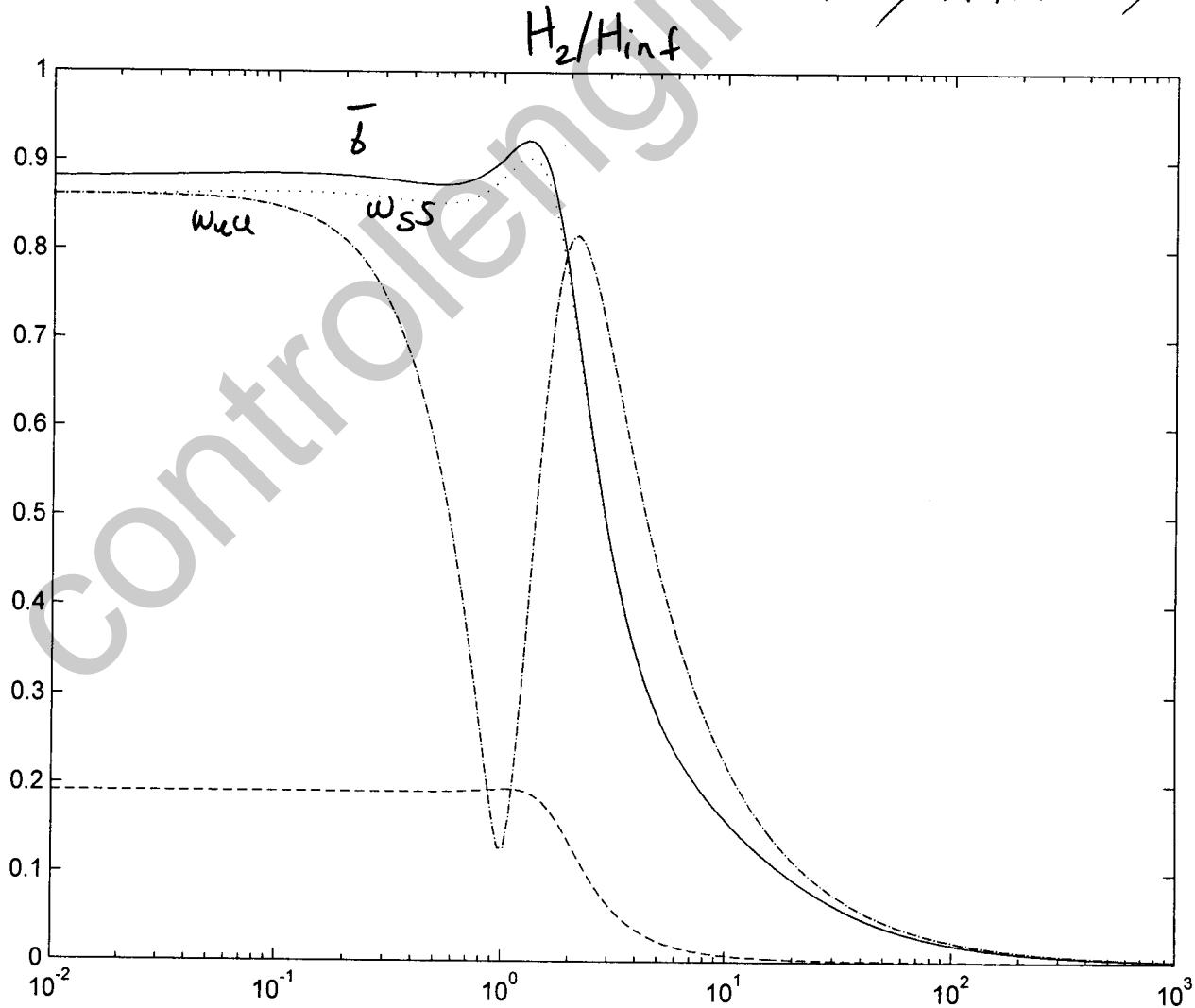
$$\alpha = 1, \beta = 0 \Rightarrow \min_{\delta_{opt}} \left\| \frac{w_s s}{w_T} \right\|_\infty < 1 \quad \text{and} \quad \| w_u u \| < \gamma = \frac{\gamma_{20\%}}{2} = 1.2$$

$$\delta_{opt} = 0.9676$$

$$\gamma_{20\%} = 1.2$$

check the closed Loop performance of the System  
using  $K_3$  &  $K_5$  to compare

$$K_5 = \frac{25 \cdot 48 (s+1)(s+9.66)(s+10)(s+0.05)(s^2+0.2s+1)^2}{(s+0.97)(s+0.05)(s+0.04961)(s^2+20s+100)(s^2+2.955s+2.3)} \times \cancel{(s^2+0.194s+1)(s^2+0.28+1)}$$



⑥

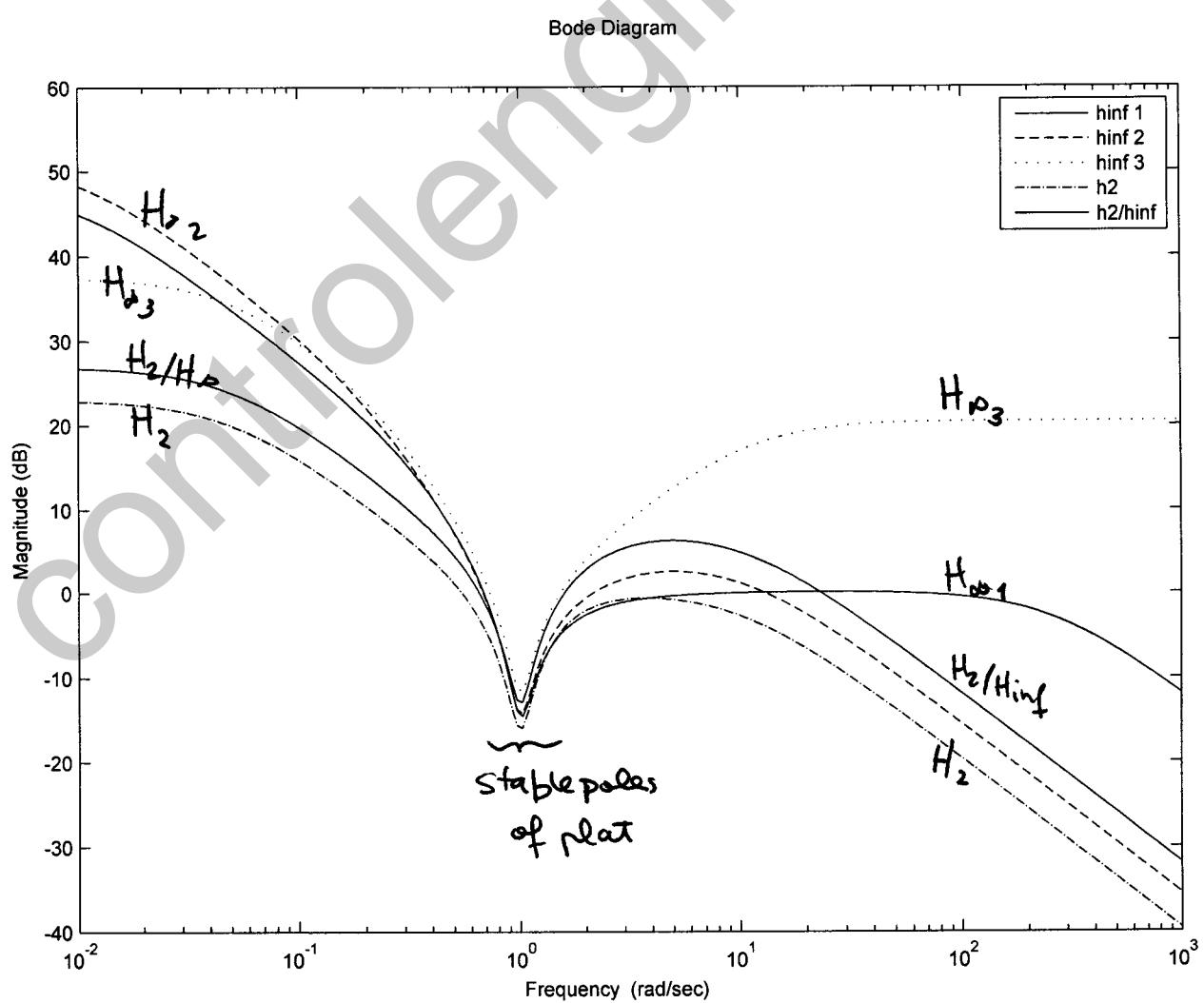
## Compare The Controllers

$H_{\infty 2}$  gives the best @ low freq but not good @ high freq.

$H_{\infty 3}$  is the best compromise in  $\infty$ -solutions

$H_2$  is the least performance @ low + high freq.

$H_2/H_{\text{inf}}$  is also good & can get's better using the optimization of gain selection in this case.



## ⑥ μ analysis and Synthesis

16-1

In this chapter we use another of structured Singular Value in order to analyse robust stability and performance, and design controllers to satisfy these requirements.

### 6-11 Complex Structured Singular Value

This section is devoted to defining the most general form of structured singular value, a matrix function  $\mu(\cdot)$ , applied on complex matrices  $M \in \mathbb{C}^{n \times n}$ . In the definition of  $\mu(M)$  is an underlying structure  $\Delta$  (a prescribed set of block diagonal matrices), which represents the forms of uncertainties, namely structured (complex or real) or unstructured. To define the structure, we must define the type of each, the total number of each block and their dimensions.

The uncertainty blocks have two types: "repeated scalar" and "Full Blocks": two integers (nonnegative)  $S$  and  $F$ , are used to represent the number of "R.S.B" and "F.B" respectively; we define  $\Delta \subset \mathbb{C}^{n \times n}$  as

$$\Delta = \left\{ \text{diag} [\delta_1 I_{r_1}, \dots, \delta_S I_{r_S}, \sigma_1, \dots, \sigma_F] : \delta_i \in \mathbb{C}, \sigma_j \in \mathbb{C}^{m_j \times m_j} \right\}$$

where the dimensions satisfy:  $\sum_{i=1}^S r_i + \sum_{j=1}^F m_j = n$

where  $r_i$  and  $\rightarrow (r_1, \dots, r_S)$  and  $m_j$  ( $m_1, \dots, m_F$ ) represent the dimension of the  $i$ th repeated scalar Block  $r_i \times r_i$  and the  $j$ th F.B. dim:  $m_j \times m_j$

Often we need norm bounds on  $\Delta$ , we introduce subsets:

$$B_\Delta = \{\Delta \in \Delta : \bar{\delta}(\Delta) \leq 1\}$$

Note: the order of S.B. and F.B. are not important also the full blocks may be not squared, but to keep notation not much complicated than that we show it in this form:

Def: Structured Singular Value:

- For  $M \in \mathbb{C}^{n \times n}$ ;  $\mu_\Delta(M)$  is defined as

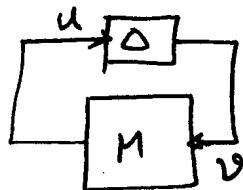
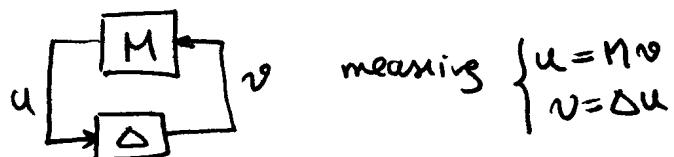
$$\mu_\Delta(M) := \frac{1}{\min \left\{ \bar{\delta}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0 \right\}}$$

unless no  $\Delta \in \Delta$  makes  $I - M\Delta$  singular, in which  $\mu_\Delta(M) := 0$

feedback representation of this def:

Let  $M \in \mathbb{C}^{n \times n}$  be given, and consider the loop

- for the sake of argument consider the  $\Delta$  block being the feedback



$$\text{meaning } \begin{cases} u = Mv \\ v = \Delta u \end{cases}$$

as long as  $(I - M\Delta)$  is nonsingular, the only solns  $u, v$  to the loop equations are  $u = v = 0$ . However, if  $I - M\Delta$  is singular, then there are infinitely many solutions to the above equati, where  $u, v \neq 0$  and their norms can get large! "Causing something like instability"

in this context,  $\mu_\Delta(M)$  is a measure of the smallest structured  $\Delta$  that causes instability

Note:  $\mu$ -funch is continuous, but is not a norm, since it doesn't satisfy triangle inequality. However

$$\mu(xM) = |x| \mu(M) \quad \forall x \in \mathbb{C}$$

it shows an indicate of the size of Magnifant of a matrix.

it can be easily shown that for two extreme cases  $\mu_\Delta(M)$  is calculated from Linear Algebra quantities:

① if  $\Delta = \{\delta I : \delta \in \mathbb{C}\}$  or  $(s=1, F=0, r_1=n)$  then

$$N_\Delta(M) = \rho(M) \quad (\text{the spectral radius of } M) \quad (\text{largest eigen value})$$

② if  $\Delta = C^{n \times n}$  ( $s=0, F=1, m_1=n$ ) then  $\mu_\Delta(M) = \bar{\delta}(M)$

We can conclude that for general  $\Delta$

$$\rho(M) \leq N_\Delta(M) \leq \bar{\delta}(M)$$

unfortunately the gap between two bonds are large and not suff.

for our purpose, there are Matrix manipulat Developed Defing matrix  $D \Rightarrow \Delta D = D \Delta$ , with specific properties (beyond the scope of this course) to reduce the size.

We use computer developed tools to find the finest bounds on  $M$ .

$Mu$  : Gives upper and lower bounds on  $\mu_\Delta(M)$

Unwind?

We need to specify  $M$  and  $\text{deltaset}$  to compute  $\mu$ .

$M$ : Matrix to calculate  $\mu$  of, a const. or varying matrix.

$\text{deltaset}$ : Block structures info about the set  $\Delta$ , the number of perturb blocks, their sizes and types as the following format

- 1) A scalar real parameter is denoted by  $[ -1 \ 1 ]$  or  $[ -1 \ 0 ]$
- 2)  $f$  times repeated real  $\sim \sim \sim \sim [ f \ 0 ]$
- 3) A  $1 \times 1$  (Scalar) unmodeled dynamics which is complex is denoted by  $[ 1 \ 1 ]$
- 4)  $\neq r \times c$  (Full Block)  $\sim \sim [ r \ c ]$

Example:

$$\text{deltaset} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ for } \Delta := \left\{ \begin{bmatrix} \delta_1 & 0 \\ \delta_2 & 0 \\ 0 & \delta_3 \delta_4 \end{bmatrix} : \delta_i \in \mathbb{R} \right\}$$

$$\sim \sim = \begin{bmatrix} 3 & 2 \\ 4 & 5 \\ 1 & 1 \end{bmatrix} \text{ for } \Delta := \left\{ \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix} : \Delta_1 \in \mathbb{R}^{3 \times 2}, \Delta_2 \in \mathbb{C}^{4 \times 5}, \Delta_3 \in \mathbb{R} \right\}$$

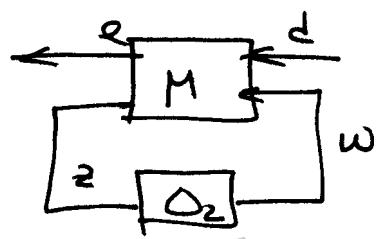
$$\sim = \begin{bmatrix} 3 & 0 \\ 2 & 2 \\ 2 & 0 \end{bmatrix} \quad \Delta := \left\{ \begin{bmatrix} \delta_1 I_3 & 0 \\ 0 & \Delta_2 \\ 0 & \delta_3 I_2 \end{bmatrix} : \delta_1, \delta_3 \in \mathbb{R}, \Delta_2 \in \mathbb{C}^{2 \times 2} \right\}$$

$$\sim = \begin{bmatrix} -20 \\ 40 \\ 33 \end{bmatrix} \quad \Delta := \left\{ \begin{bmatrix} \delta_1 I_2 & 0 \\ \delta_2 I_4 & 0 \\ 0 & \Delta_3 \end{bmatrix} : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{R}, \Delta_3 \in \mathbb{C}^{3 \times 3} \right\}$$

Note that  $\text{deltaset} = \text{abs}(\text{deltaset})$  gives the complex version of any real valued uncertainty structure.

Using Linear Fractional Transformation (LFT's) introduced before for  $\Delta$ -blocks we have in general

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \begin{cases} e = M_{11}d + M_{12}\omega \\ z = M_{21}d + M_{22}\omega \\ \omega = \Delta_2 z \end{cases}$$



$$F_L(M, \Delta_2) = M_{11} + M_{12}\Delta_2(I - M_{22}\Delta_2)^{-1}M_{21}$$

Similarly for upper loop with  $\Delta_1$  we have

$$F_U(M, \Delta_1) = M_{22} + M_{21}\Delta_1(I - M_{11}\Delta_1)^{-1}M_{12}$$

with this type of transform  $M_{11}$  has a size which is related to  $\Delta_1$

$M_{22}$  relates to  $\Delta_2$  and let's define

$$\Delta = \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \Delta_1, \Delta_2 \in \Delta_2 \right\}$$

Now  $\mu_1(M_{11})$ ,  $\mu_2(M_{22})$ ,  $\mu_\Delta(M)$  all make sense with  $\Delta_1, \Delta_2, \Delta$  respectively

Theorem: The linear Fractional transformation  $F_L(M, \Delta_2)$  is wellposed for all  $\Delta_2 \in \Delta_2$ , iff  $\mu_2(M_{22}) < 1$

$$B_2 = \left\{ \Delta_2 \in \Delta_2 : \bar{\delta}(\Delta_2) \leq 1 \right\}$$

Theorem: The following are equivalent:

$$1) \mu_\Delta(M) < 1$$

$$2) \mu_2(M_{22}) < 1 \text{ and } \max_{\Delta_2 \in B_2} \mu_1(F_L(M, \Delta_2)) < 1$$

$$3) \mu_1(M_{11}) < 1 \text{ and } \max_{\Delta_1 \in B_1} \mu_2(F_U(M, \Delta_1)) < 1$$

## 6-2 | $\mu$ -analysis

Frequency domain  $\mu$ -analysis plays an important role in robustness analysis, both in robust stability analysis, and robust performance! after completing the analysis, we give a procedure for controller design in  $\mu$ -synthesis part.

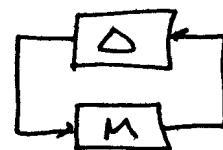
### 6-2-1| Robust Stability

- Suppose  $M(s)$  is a stable, multi-input multi-output T.F. of a linear system;  $M$  has  $n_2$  inputs and  $n_w$  outputs. Let  $\Delta$  be a block structure defined before, and assume that  $\Delta \subset \mathbb{C}^{n_2 \times n_w}$ . We want to consider feedback perturbations which are themselves dynamical systems, with the block diagonal structure of the set  $\Delta$ . Also assume the perturbations are stable. Other forms of structured uncertainty  $\Rightarrow$  coprime factor or gap metric also can be applied but the proofs are different. Let

$$S_\Delta := \{ \Delta \in \mathbb{R}H^\infty : \Delta(S_0) \in \Delta \text{ } \forall s_0 \in \bar{\mathbb{R}}_+$$

Theorem: Robust Stability

Let  $\beta > 0$ . The loop shown in Figure: is wellposed and internally stable  $\nabla \Delta \in S_\Delta$  with  $\|\Delta\|_\Delta < \frac{1}{\beta}$



iff,  $\|M\|_\Delta := \sup_{w \in \mathbb{R}} \mu_\Delta(M(jw)) \leq \beta$

Proof in: Chen + Desoer, Necessary and sufficient condition for robust stability of

and in

" Packard and Pandey ", Continuity of the real/complex S.S.V. IEEE TAC  
v38, N3, p 415-28 , 93

In Summary, the peak value on the plot of the frequency response that the perturbation sees determines the size of perturbations that the loop is robustly stable agains.

Also for admissible  $\Delta$ ,  $\|\Delta\|_\infty < 1 \Rightarrow M(\Delta) < 1$  for Robust St.

### - 6.2-2 | Robust Performance

More important than R.S. is robust performance, since no numeric solution is obtained yet. Typically there are exogenous disturbances acting on the system (winds, gusts, noises,...) which results in tracking error,  $\|W_S S\|_\infty$  is an indicator of Nominal performance.

in presence of  $\Delta$ , the worst case performance degradation must be determined

Assume  $M$  is stable,  $\in \mathbb{RH}^\infty$ , with  $n_z + n_d$

inputs and  $n_w + n_e$  outputs, partition

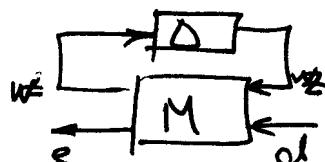
$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \Rightarrow M_{11} \text{ has } n_z \text{ inputs and } n_w \text{ outputs}$$

and so on. Let  $\delta \in \mathbb{R}^{n_w \times n_z}$  be a block structure, as in equation

defining  $\delta^*$ . Define an augmented Block structure

$$\Delta_p := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix} : \Delta \in \Delta, \Delta_F \in \mathbb{R}^{n_d \times n_e} \right\}$$

The perturbed transfer function from  $d$  to  $e$  is denoted by  $F_u(M, \Delta)$

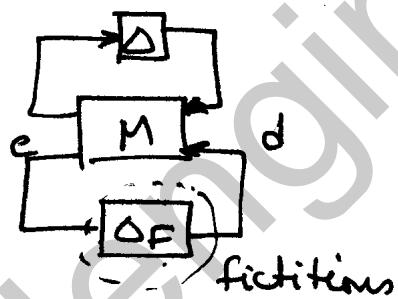


### Theorem: Robust Performance

Let  $\beta > 0$ , for  $\Delta(s) \in S_\delta$  with  $\|\Delta\|_\infty < \frac{1}{\beta}$ , the loop shown before is well-posed, internally stable, and The norm of perturbed transfer function from  $d$  to  $e$   $\|F_u(M, \Delta)\|_\infty \leq \beta$ , iff,

$$\|M\|_{\Delta_p} := \sup_{\omega \in \mathbb{R}} \mu_{\Delta_p}(M(j\omega)) \leq \beta$$

Hence for robust performance we add a fictitious uncertainty element  $\Delta_f$  between plant and controller, and check the S.S.V. of the new system.

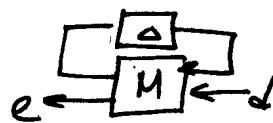


Here the summary of the procedures are as following:

#### Procedure 1: Robust Stability

- 1) Construct  $M$  and check dimensions (The controller must be already found by the)
- 2) Calculate frequency response, and form  $M_{11-g}$  by choosing the first two inputs and output channels for R.S. test.
- 3) Create delta set, depending on the form of  $\Delta$
- 4) Compute  $\mu_g(M_{11}(j\omega))$  using mu command.
- 5) Check the bounds and find its supremum  $\Rightarrow \max_w \mu_g(\Delta(j\omega)) < \frac{1}{\text{upper bound}}$
- 6) You may verify instability by violating the condition  $\textcircled{S}$

## Procedure 2: Robust Performance



- 1) recast the problem into the form  
where  $\Delta$  are generalized disturbance and  
 $\mu$  is that characterizes the performance.
- 2) Calculate freq response of  $M \rightarrow M(j\omega)$
- 3) create dataset  $\Delta$
- 4) Generate fictitious uncertainty block using dimensions of  $\Delta$
- 5) Compute  $\mu_{\Delta_p}(M(j\omega))$  on the freq. response, using the augmented  $\Delta_p$
- 6) Plot the bounds from calculation of  $\mu$ .

Suppose peak is  $\beta$  :  $\max_{\omega} \bar{\delta}(\Delta(j\omega)) < 1/\beta$ ,  $\|F_u(M, \Delta)\|_{\infty} < \rho$

the software computes a lower & upper bound for  $\beta$ , not itself exactly  $\Rightarrow$

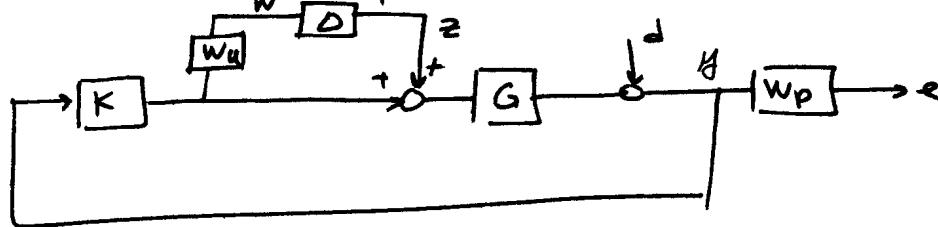
$\forall \Delta \in \Delta$  and  $\max_{\omega} \bar{\delta}(\Delta(j\omega)) < 1/\beta_u \Rightarrow$  the perturbed sys is  
stable +  $\|F_u(M, \Delta)\|_{\infty} \leq \rho_u$

$\exists \Delta \in \Delta$  satisfying  $\max_{\omega} \bar{\delta}(\Delta(j\omega)) = 1/\beta_L$  which causes

either  $\|F_u(M, \Delta)\|_{\infty} > \rho_L$  or instability.

The gap between  $\rho_L < \rho_u$  gives the region where we are unable to  
precisely state robust performance.

Example : Consider the system below



$$\text{with: } G(s) = \frac{1}{s-1} ; \quad w_u = \frac{1/4(1/2s+1)}{1/3s+1}$$

The performance  $\|W_p T_{ed}\|_\infty < 1$  or  $\|W_p S(s)\|_\infty < 1$

choose  $w_p = \frac{0.25s + 0.6}{s} \rightarrow \frac{0.25s + 0.6}{s + 0.006}$  to have no poles on jw-axis

See the program enclosed for system interconnect



Now Consider two controllers:

$$K_1 = -10 \frac{0.9s+1}{s} ; \quad K_2 = -1 \frac{2.8s+1}{s}$$

use starp(p, k1) to use LFT interconnect :

deltaset1 = uncblk = [1 1]; → multiplicative uncertainty

deltaset2 = fintblk = [1 1];

deltaset = [uncblk; fintblk];

use , vnorm (sel(Mlg, 2, 2))  $\Rightarrow \|M_{(2,2)}\|_\infty$   
 $\in (2,2)$  component of  $M_1 \Rightarrow \|T_{ed}\|_\infty$

and mu (Mlg, deltaset)  $\Rightarrow$  deltaset =  $\Delta_P = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_F \end{bmatrix} \Rightarrow$  robust performance

Conclusion: Controller  $K_2$  is robustly performing while  $K_1$  is not

In order to analyze robust Stability, two ways are plausible

1) Short Cut : "robuststab" function

2)  $\mu$ -analysis : "mussv" function

in robuststab function generation of  $\mu$ -bounds are done by calling mussv automatically and all required data for representation is generated. Old- $\mu$  user might use mussv with their step-by-step calculations.

Syntax : [stabmarg, destabunc, report, info] = robuststab(sys, opt)

sys : an uncertain System (including ureal, ucomplex, ultidyn objects)  
a uss structure, or similar ones.

stabmarg... LowerBound : Lower bound of  $\mu$  in stability analysis

. Upper Bound : Upper bound of  $\mu$

. DestabilizingFrequency : The frequency which destabilizing occur

if stabmarg is greater than 1.  $\Rightarrow$  Robust Stability is guaranteed.

destabunc : a structure of values of uncertain elements, closest to nominal  
that cause instability

use : usubs(sys, destabunc)  $\rightarrow$  to bring the uncertain Syst to  
the margin of instability

Info. Sensitivity : Sensitivity to each uncertainty (N.I.)

- Frequency :  $N \times 1$  frequency vector used for analysis

- BadUncertVal : Shows the migration (N.I.)

- MussvBnds : A  $1 \times 2$  frcl, with upper & lower bounds from mussv.

The (1,1) entry is the  $\mu$ -upper bound which corresponds to stabmarg. lower Bound, and (1,2) entry is the  $\mu$ -lower bound corresponding to stabmarg. Upper Bound

- MussvInfo : Structure of compressed data from mussv (N.I.)

Options : Options can be set using robopt function

$$\text{opt} = \text{robopt} ('Sensitivity', 'off', 'Display', 'on', \dots)$$

$$[\dots] = \text{robuststab} (\text{sys}, \text{opt})$$

To use mussv directly, the M- $\Delta$  LFT format is used

$$\text{bounds} = \text{mussv} (\text{M}, \text{BlockStructure})$$

The Block Structure is given as explained in  $\mu$  directly or is gotten from uncertain System Conversion

$$[\text{M}, \Delta] = \text{lftdata} (\text{ClosedLoop}) \quad \text{or}$$

$$[\text{M}, \Delta] = \text{aff2lft} (\text{lmi-aff-sys})$$

For Robust Performance Analysis use

[perfmargin, wcu, report, Info] = robustperf(sys, opt)

perfmargin.lowerBound = Similar to previous case

.UpperBound = . . . . .

.CriticalFrequency = the value of frequency at which the performance degradation curve crosses the  $\gamma = 1/\alpha$  curve.

Info.Sensitivity as in robuststab

- Frequency
- BadUncertainValues
- MuSSvBnd
- MuSSvInfo

in both commands robuststab & robustperf, report will give a concise report on the analysis.

```
% Copyright Dr. Hamid D. Taghirad 2005

clc
echo on

%% Robust Stability and Mu Analysis
% This demo shows how to use the Robust Control Toolbox to analyze and
% quantify the robustness of feedback control systems, including modeling
% errors and parameter variations. We'll look at how to test for robust
% stability with the |robuststab| function and gain insight into the
% connection with mu analysis and the |mussv| function.

%% System Description
% As it is described in the course notes the plant model P is uncertain,
% the signal d is disturbances, and the output e must be kept small
% for adequate disturbance rejection.
%
s=zpk('s'); % zero-pole format
plant=1/(s-1); % plant
Wu=((s/2+1)/4)/(s/32+1); % Uncertainty weighting function
Wp=(0.25*s+0.6)/(s+0.006) % Performance weighting function

%% Creating An Uncertain Plant Model
% As before we generate a full 1x1 uncertainty block

delta = ultidyn('delta',[1 1]);
P = plant*(1+Wu*delta);

%% Creating two Controllers
% For this example we use the following controllers

K1=-10*(0.9*s+1)/s;
K2=-(2.8*s+1)/s;

pause % strike a key to continue
clc

%% Creating a Closed-Loop System with SYSIC
% To build an uncertain model of the closed-loop system, we'll write the
% interconnection equations directly from the block diagram and
% use the "sysic" function to compute the interconnection:

systemnames = 'P K1 Wp';
inputvar = '[d]';
input_to_P = '[K1]';
input_to_Wp = '[d + P]';
input_to_K1 = '[d + P]';
outputvar = '[Wp]';
cleanupsysic = 'yes';
CLoop1 = sysic;
CLoop1.InputName = {'d'};
CLoop1.OutputName = {'e'};

pause % strike a key to continue
clc
%%
```

```
% Alternatively, we can use the |icsignal| and |iconnect| functions to  
% build the closed-loop model for controller K2:
```

```
IC = iconnect;  
d = icsignal(1);  
e = icsignal(1);  
y = icsignal(1);  
IC.Input = [d];  
IC.Output = [e];  
IC.Equation{1} = equate(y,d+P*K2*y);  
IC.Equation{2} = equate(e,Wp*y);  
CLoop2 = IC.System;  
ClosedLoop.InputName = {'d'};  
ClosedLoop.OutputName = {'e'};  
  
pause % strike a key to continue  
clc  
%%  
% Pick 10 random samples of the uncertain closed-loop systems  
  
clf  
Psample1=usample(CLoop1,2);  
Psample2=usample(CLoop2,2);  
  
impulse(Psample1,:b', Psample2,:r',1)  
title('Variability of closed-loop response due to model uncertainty')  
legend('K1','K2')  
  
pause % strike a key to continue  
clc  
  
%% Robust Stability Analysis  
% Does the closed-loop system remain stable for all values of |k|, |delta|  
% in the ranges specified above? We use |robuststab| to answer this basic  
% robustness question:  
  
[stabmarg1,destabunc1,report1,inf1] = robuststab(CLoop1);  
stabmarg1  
  
%%  
% The variable |stabmarg| gives upper and lower bounds on the robust  
% stability margin, a measure of how much uncertainty on |delta| the  
% feedback loop can tolerate before becoming unstable. For example, a  
% margin of 0.8 indicates that as little as 80% of the specified  
% uncertainty level can lead to instability. Here the margin is 1.06,  
% which means that the closed loop will remain stable for up to 106%  
% of the specified uncertainty. The stability margin is  
% quite narrow in this example. The |report| function summarizes  
% these results:  
  
report1  
  
pause % strike a key to continue  
clc  
% Repeat that for the second controller K2, and compare the stability  
% margins
```

```
[stabmarg2,destabunc2,report2,info2] = robuststab(CLoop2);

stabmarg2
report2

pause % strike a key to continue
clc

%% Robust Stability Analysis: Connection with Mu Analysis
% The structured singular value, or mu, is the mathematical tool used
% by |robuststab| to compute the robust stability margin. If you are
% comfortable with structured singular value analysis, you can use
% the |mussv| function directly to compute mu as a function of frequency
% and reproduce the results above. The function |mussv| is the underlying
% engine for all robustness analysis commands.
%
% To use |mussv|, we first extract the |(M,Delta)| decomposition of the
% uncertain closed-loop model |ClosedLoop|, where |Delta| is a
% block-diagonal matrix of (normalized) uncertain elements.
% The 3rd output argument of |lftdata|, |BlkStruct|, describes the block-diagonal
% structure of |Delta| and can be used directly by |mussv|

[M1,Delta1,BlkStruct1] = lftdata(CLoop1);
[M2,Delta2,BlkStruct2] = lftdata(CLoop2);

%%
% For a robust stability analysis, only the channels of |M| associated
% with the uncertainty channels are used. Based on the row/column size of
% |Delta|, select the proper columns and rows of |M|. Remember that the
% rows of |Delta| correspond to the columns of |M|, and vice versa.
% Consequently, the column dimension of |Delta| is used to specify the rows
% of |M|:

M11 = M1(1,1);
N11 = M2(1,1);

pause % strike a key to continue
clc

%%
% Mu-analysis is performed on a finite grid of frequencies. For comparison
% purposes, evaluate the frequency response of |M11| over the same
% frequency grid as used for the |robuststab| analysis.

omegal = info1.Frequency;
omega2 = info2.Frequency;
M11_g = frd(M11,omegal);
N11_g = frd(N11,omega2);

%%
% Compute |mu(M11)| at these frequencies and plot the resulting lower and
% upper bounds:

mubndsl = mussv(M11_g,BlkStruct1,'s');
```

```
mubnd2 = mussv(N11_g,BlkStruct2,'s');

semilogx(mubnd1,'b', mubnd2,'r', infol.MussvBnds,:y')

xlabel('Frequency (rad/sec)');
ylabel('Mu upper/lower bounds');
title('Mu plot of robust stability margins of two controllers');

%%%
% The robust stability margin is the reciprocal of the structured singular
% value. Therefore upper bounds from |mussv| become lower bounds
% on the stability margin.

pause % strike a key to continue
clc

%% Robust Performance Analysis
% The input/output gain of a nominally-stable uncertain system model will
% generally degrade for specific values of its uncertain elements.
% Moreover, the maximum possible degradation increases as the uncertain
% elements are allowed to further deviate from their nominal values.
%
% The simplest route to analyzing the robust performance margin of
% the closed-loop system is direct use of the |robustperf| command.

[perfmarg1,perfmargunc1,rep1,infl] = robustperf(CLoop1);
[perfmarg2,perfmargunc2,rep2,infl] = robustperf(CLoop2);
%%
% The |perfmarg| variable has upper and lower bounds on the performance
% margin.
perfmarg1

%%
% The |report| variable summarizes the robust performance analysis.
disp(rep1)

pause % strike a key to continue
clc

% Repeat the report for the second controller

perfmarg2
disp(rep2)

%%
% Finally, we plot the bounds from |mussv|, which is the
% underlying engine for the robustness analysis. The peak value is the
% reciprocal of the performance margin, and the frequency at which the
% peak occurs is the critical frequency.

semilogx(inf1.MussvBnds,'b',inf2.MussvBnds,'r')
xlabel('Frequency (rad/sec)');
ylabel('Mu upper/lower bounds');
title('Robust Performance Mu Plot');
%
```

% This concludes this example  
% For study the connection of the above robust performance analysis  
% with Mu Analysis study the Matlab mu\_demo od Robust Control Toolbox.

controlengineers.ir

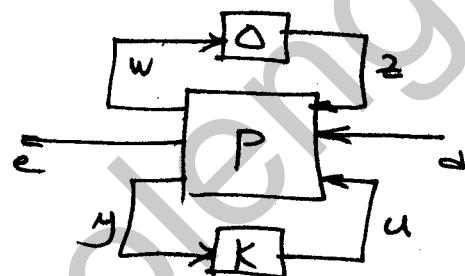
## 6-3 | $\mu$ -Synthesis

6-13

Up to now, we have enough machinery to analyse robust stability and performance, using  $\mu$ -analysis; But, to design a controller is not covered. Note that the synthesis here is based merely on the analysis section, and iteration. Despite the  $H_\infty$  method which directly derives the optimal & suboptimal solutions, here we try a  $H_\infty$  solution first check its  $\mu$ -characteristics and if it is not robustly performing, add some pre-post filters in some specific frequencies where  $\mu$  is greater or close to 1.

### 6-3-1 | Problem Setting

First recast the problem, into the LFT Setting as in figure



System  $P$  is the open loop interconnection, which contains all of the known elements, including nominal plant  $P_0$ , performance  $S$  and uncertainty weighting functions  $(W_S, W_T)$ . The  $\Delta$  block is the uncertain element from the set  $\Delta$ , which parametrizes all of the assumed model uncertainty in the problem. and the controller is  $K$ . Three set of inputs enter  $P$ , : perturbations inputs  $z$ , disturbance, and control  $u$ , The set of Plant outputs are  $w$ ,  $e$  and  $y$ . The set of systems to be controlled is described by the LFT

$$\{Fu(P, \Delta) : \Delta \in \Delta, \max_{\omega} |\Delta(j\omega)| \leq 1\}$$

The design objective is to find a stabilizing controller  $K \Rightarrow$

$\forall \Delta \in \Delta$ , the closed-loop system is stable and satisfies

$$\| F_L [F_u(P, \Delta), K] \|_\infty < 1$$

This is equivalent to  $\| T_{\text{ed}} \|_\infty < 1$  which is the same mixed sensitivity problem in presence of the uncertainty block  $\Delta$ .

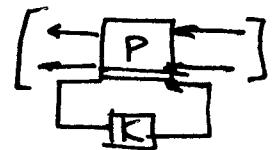
Given any  $K$  this performance can be checked by:

$$\max_w \mu_\Delta (F_L(P, K)(j\omega)) < 1. \text{ as explained in last sect.}$$

"The goal of  $\mu$ -synthesis is to minimize overall stabilizing controllers  $K$ , the peak value of  $\mu_\Delta(\cdot)$  of the  $F_L(P, K)$ ,

$$\min_K \max_w \mu_\Delta (F_L(P, K)(j\omega))$$

stabilizing



Note 1:  $F_L [F_u(P, \Delta), K] = F_u [F_L(P, K), \Delta]$

doesn't matter which LFT is computed first,  $\Rightarrow$  given  $K$  we rather use the RHS.

Note 2: As we explained briefly in  $\mu$  calculate, it is very hard to compute  $\mu$  itself, numerical solut's give their upper+lower bounds  
it is necessary to replace  $\mu$  with its upper bound

As mentioned before, we defined  $\forall D \in D_\Delta \Rightarrow D\Delta = \Delta D$ ;  $\forall \Delta \in \Delta$   
we can show

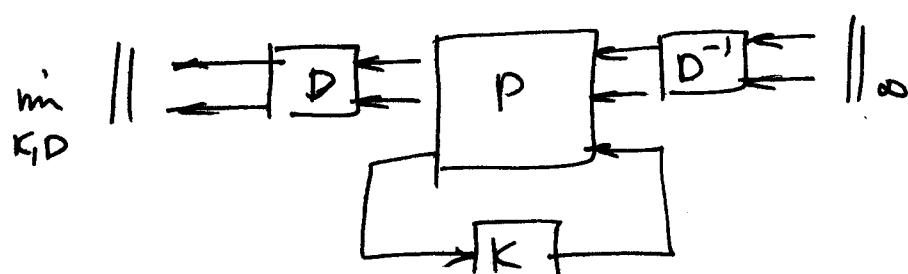
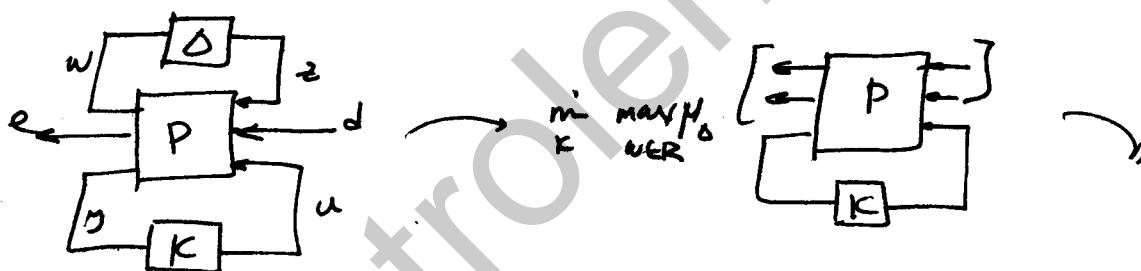
$$\mu_\Delta(M) \leq \inf_{D \in D_\Delta} \bar{\sigma}(DM^{-1})$$

Hence, the new optimization will be

$$\text{find } \min_{\substack{K \\ D \\ \text{stab.}}} \quad \min_{\substack{D(s) \in D_\Delta \\ \text{stable} \\ \text{no phase}}} \|DF_L(P, K)D^{-1}\|_\infty \quad (\text{I})$$

where both  $K + D$  which minimizes the above norm must be determined. This is solved by iteration called D-K iterat.

The problem has returned to:



"Replacing  $\mu_\Delta$  with upper Bound"

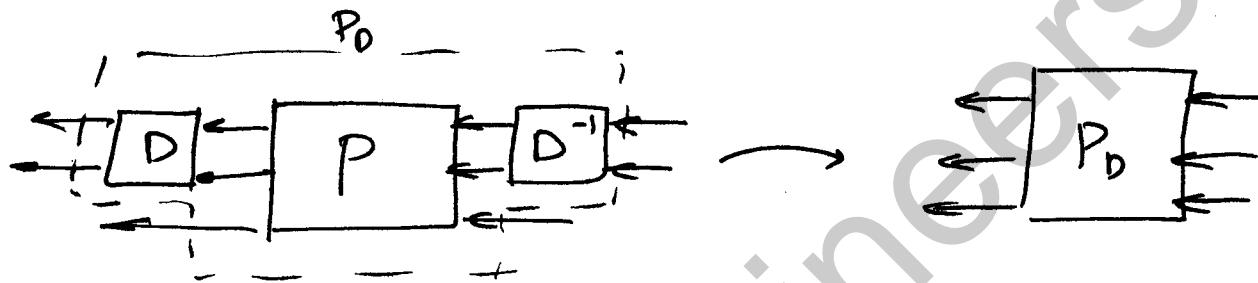
6-3-2 | D-K Iteration : Holding D Fixed → this is done by dksyn 16-16

To solve (I), first consider holding  $D(s)$  fixed at a given, stable min phase, real rational  $D(s)$ . Then, solve the optimizat

$$\min_K \left\| DF_L(P, K) D^{-1} \right\|_\infty \quad (\text{First trial } D(s)=1)$$

stab.

define  $P_D$  to be augmented Syst



$$\Rightarrow \min_K \left\| F_L(P_D, K) \right\|_\infty$$

Since  $P_D$  is known at this step, this is exactly an H<sub>∞</sub> solut which may be done easily by hinfsyn, making the effort of Solving two Riccati Eq. simultaneously, in terms of the augmented Syst  $P_D$

Note: The order of H<sub>∞</sub> controller = the order of augmented plant

If  $D(s) \neq 1$  has any order  $> 1 \Rightarrow$  the controller order increases with the same No. !

→  $K(s)$  is getting larger in dimension!

## Summary of D-K iteration procedure

The D-K iteration involves a sequence of minimizations

- 1) First over the controller variable  $K$  (Holding  $D$  variable associated with the  $\mu$  upper bound fixed),
- 2) then over  $D$  variable (holding the  $K$  variable fixed)

The D-K iteration doesn't guarantee converging to a minimum but it usually works well.

The following steps are covered in this procedure

1. (In the first iteration, this step is skipped.)

The  $\mu$  calculation (from previous step) provides a frequency-dependent scaling matrix  $D_f$ . with a fitting procedure these data fit with a rational-stable transfer

function  $\hat{D}_f$ . Plots of  $\bar{\delta} (D_f F_L(P, K) D_f^{-1})$  +  $\bar{\delta} (\hat{D}_f F_L(P, K) \hat{D}_f^{-1})$  are given

for comparison @ each iteration

The rational  $\hat{D}$  is absorbed into the open-loop interconnect as explained before  $P \rightarrow P_0$

2. A controller is designed using  $H_\infty$  synthesis, on the scaled open-loop interconnect. The controller minimizes the following  $\infty$ -norm via iterat (g-iterat)

$$\min_{K_{\text{stab}}} \|\hat{D}_f F_c(P, K) \hat{D}_f^{-1}\|_\infty$$

$$\text{OR} \quad \min_{K_{\text{stab}}} \|F_c(P_0, K)\|_\infty$$

Note the Generalized plant order is increased, and hence the Controller Order will increase.

3. The Structured Singular value ( $\mu$ ) of the closed-loop Syst is calculated and plotted.

4. An iterat summary is displayed, showing

iterat # 1

Controller Order 8

Total D-Scale order 6

Gamma achieved 2.263

Peak  $\mu$ -Value 2.014

5. choice to continue the iterat or Stop is given

Note dKinfo saves all the details of dK iterat in it.

dKinfo is a  $(1 \times n)$  array n: number of dK iterat

each array :  $\text{dKinfo}\{1, i\}$  includes the following structs.

K : the controller (an ss structure)

Bnd : Robust performance bound on the closed loop syst (double)

DL : left scaling D (ss)

DR : Right D Scaling (ss)

MussvBnds : Upper & lower bounds mpu (frd object)

Mussvinfo : Structure returned from mussv at each itra

→ dvec (frd)  
prec (frd)  
gvec (frd)  
Sens (frd)  
blk → uncertainty dataset  
bnds

To perform D-K iteration in Robust Control Toolbox use

$[k, clp, bnd, dkinfo] = \text{dksyn}(p, nmeas, ncont, prewtkinfo, opt)$

OR Simply

$[k, clp] = \text{dksyn}(p, nmeas, ncont, opt)$

where  $p$  is a USS uncertain System with  $nmeas$ : No. of measurements and  $ncont$ : No. of Controls.

$clp$  is the closed loop system :  $clp = \text{lft}(p, k)$

$bnd$  is the related  $\mu$  bounds :  $bnd = \text{robustperf}(clp)$

OPTIONS:  $opt$  can be set by  $\text{dkitopt}$

$\text{options} = \text{dkitopt}(\text{'name1'}, \text{value1}, \dots)$

whose relevant Object properties are

✓ Frequency Vector : the frequency vector for analysis

InitialController : Controller used to initiate first iteration

AutoIter : default is 'on', by 'off' you must specify Dscaling manually

✓ DisplayWhileAutoIter : Displays iteration information

StartingIterationNumber : Default is 1

✓ NumberOfAutoIterations : Number of D-K iterations (Default is 10)

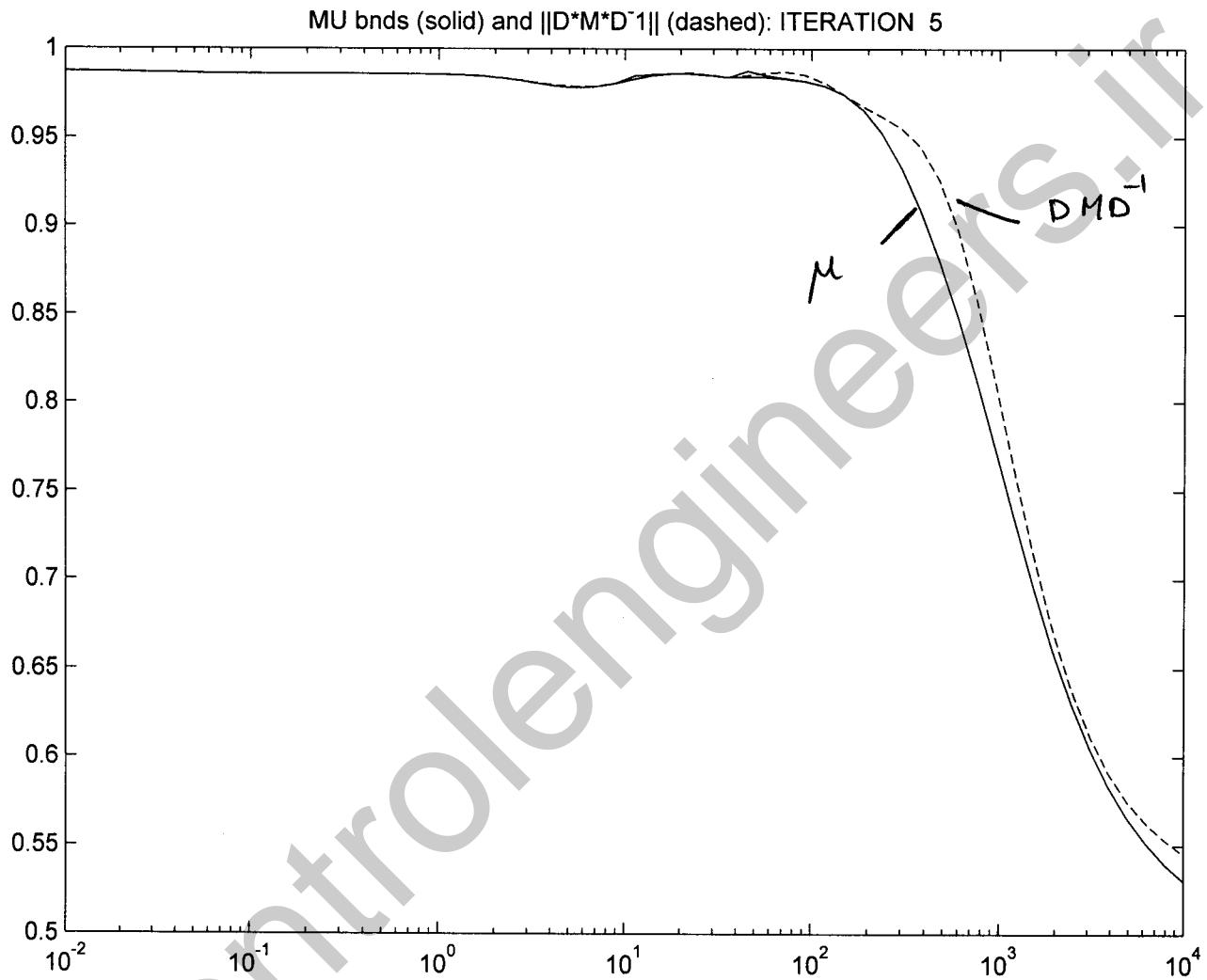
AutoScalingOrder : Maximum state order for fitting D-scaling

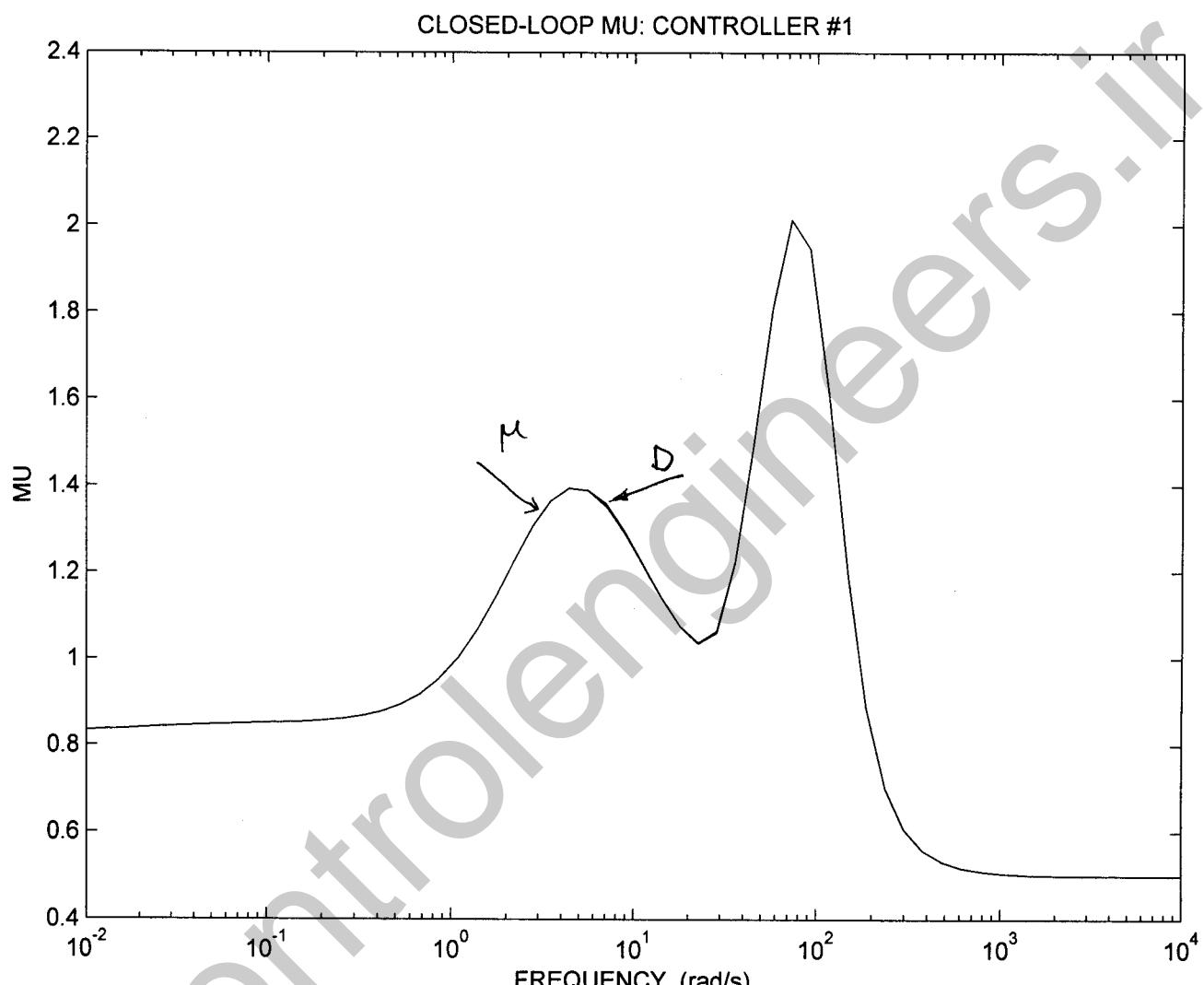
Default : the default values of the Object properties

Note 1: We have approximated  $\mu_\delta(\cdot)$  by its upper bounds, hence only when  $\mu_\delta(\cdot)$  & its upper bound are close this procedure converges & is valid, which are most of the cases.

Note 2: D-K iterat is not guaranteed to converge to a global or even a local min. This is a serious problem, which may cause the iterat unsuccessful.

look at  $\mu$ -Synthesis manual pages 7-28 - 7-30 for tutorial ex.





### 6-33) D-K Iterat : Holding K Fixed

With  $K$  held fixed, the optimization over  $D$  is carried out in a two-step procedure.

- 1) Finding the optimal frequency-dependent scaling matrix  $D$  at a large, but finite set of frequencies, where  $\mu > 1$  or close to it.  
From the upper bound calculate of  $\mu$
- 2) Fitting this optimal frequency-dependent scaling with a stable, min-phase, real-rational T.F.  $D(s) \in \mathbb{R}H^\infty$ .

Both steps are done reliably using numerical methods. The upper bound is found based on a convex optimization, which can be computed always, and quite accurately. The fitting problem is also solved by FFT or LS., which is extremely fast & effect.

This can be performed in  $\mu$ -synthesis by:

- ① GUI : dkit gui, automated, adjustable and visualizable
- ② script file dkit, automated but adjustable (The order of D to ex)
- ③ dkit command in the auto mode, to run a specified No. of iterat in an automatic mode, (No user intervention).

Example: Space Shuttle

6-4-1

(to illustrate, uncertainty LFT generation, Ho Soluti and  
 $\mu$  Robust Stability and Performance)

The System is Space Shuttle, and the lateral axis flight control  
 of system during re-entry is considered. The model is a simplified  
 version for S.Sh. from paper:

"Doyle, Lenz, Packard" in Natu ASI Series, Modelling, Robustness, and

Sensitivity Reduction in Control Systems, vol 34, Springer Verlag 1987

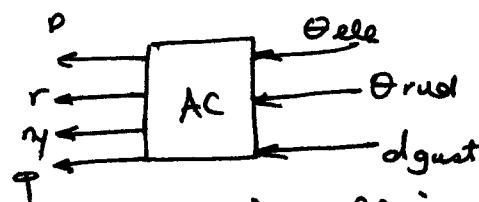
The list of program is given, with the following explanat:

1) Aircraft Model : Rigid Body

$$\dot{x} = \begin{bmatrix} \beta \\ p \\ r \\ \phi \end{bmatrix} = \begin{bmatrix} \text{sideslip angle} \\ \text{roll rate} \\ \text{yaw rate} \\ \text{bank angle} \end{bmatrix}$$

$$u = \begin{bmatrix} \theta_{ele} & (\text{rad}) \\ \theta_{rud} & (\text{rad}) \\ dgust & (\text{ft/sec}) \end{bmatrix} \begin{array}{l} \text{elevator} \\ \text{rudder} \\ \text{wind gusts} \end{array}$$

$$y = \begin{bmatrix} p \\ r \\ n_y \\ \phi \end{bmatrix} \quad n_y = \text{lateral acceleration at the pilot's locat}$$



2) uncertainty: The main uncertainty is on aerodynamic coefficients  
 which produces moving forces moments ; ( $C_{ij}$  are A.coef)

$$\begin{bmatrix} \text{side force} \\ \text{yaw moment} \\ \text{roll moment} \end{bmatrix} = \begin{bmatrix} C_{\beta\beta} & C_{\gamma\alpha} & C_{\gamma r} \\ C_{\gamma p} & C_{\gamma\alpha} & C_{\gamma r} \\ C_{\gamma p} & C_{\gamma\alpha} & C_{\gamma r} \end{bmatrix} \begin{bmatrix} \beta \\ \theta_{ele} \\ \theta_{rud} \end{bmatrix}$$

whose values are determined inaccurately and expressed as a mean value  
 a tolerance around

$$\begin{bmatrix} C_{\beta\beta} & C_{\gamma\alpha} & C_{\gamma r} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \bar{C}_{\beta\beta} & \bar{C}_{\gamma\alpha} & \bar{C}_{\gamma r} \\ - & - & - \end{bmatrix} + \begin{bmatrix} r_{ff\beta} \delta_{f\beta} & r_{ga\alpha} \delta_{g\alpha} & r_{gr} \delta_{r\alpha} \\ - & - & - \end{bmatrix}$$

where  $| \delta_{ij} | \leq 1$  and the percentage error is as following

(6-20)

$$[r_{ij}] = \begin{bmatrix} 2.19 & -1.33 & -0.57 \\ -1.52 & 1.35 & 0.87 \\ -0.72 & 0.52 & 0.24 \end{bmatrix}$$

We can put it easily into an LFT matrix

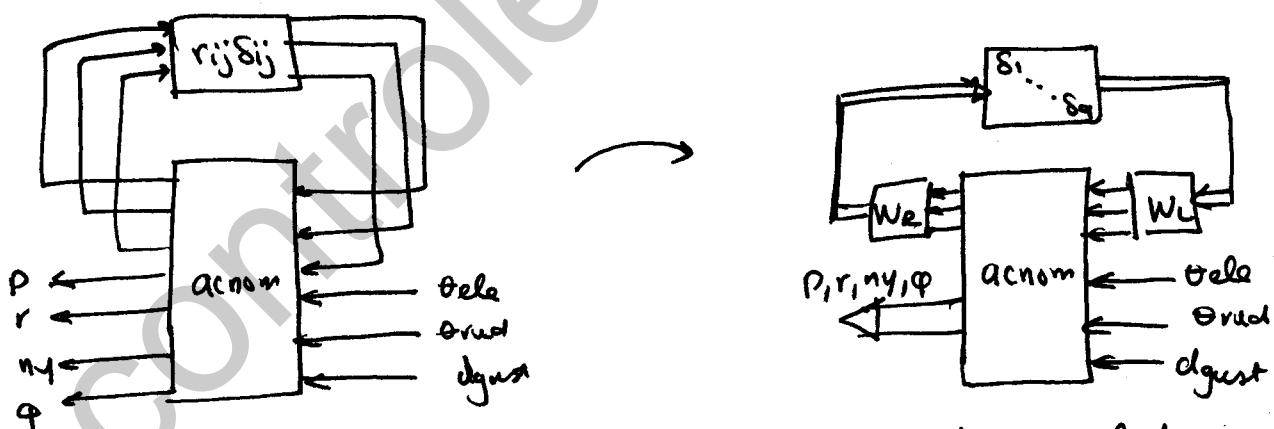
$$\begin{bmatrix} r_{ij} \delta_{ij} \end{bmatrix}_{3 \times 3} = W_L \cdot \underset{3 \times 9}{\text{diag}}(\delta_{00}, \delta_{10}, \delta_{20}, \delta_{30}, \dots, \delta_{L0}) \cdot W_R \underset{9 \times 9}{\delta_{11} \ \delta_{21} \ \delta_{31} \ \dots \ \delta_{55}} \underset{9 \times 3}{\text{ }} \text{ }$$

By simply using

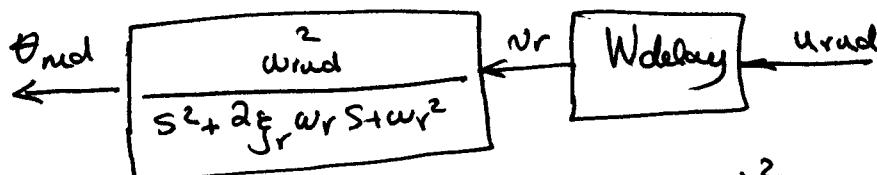
$$W_L = \begin{bmatrix} 2.19 & 0 & 0 & -1.33 & 0 & 0 & -0.57 & 0 & 0 \\ 0 & -1.52 & 0 & 0 & 1.35 & 0 & 0 & 0.87 & 0 \\ 0 & 0 & -0.72 & 0 & 0 & 0.52 & 0 & 0 & 0.24 \end{bmatrix}_{3 \times 9}$$

$$W_R^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Hence the overall system comes to



3) Actuator Models: actuators for elevator+rudder is modeled as a second order system + delay (Modeled by Padé 2nd order approx.)

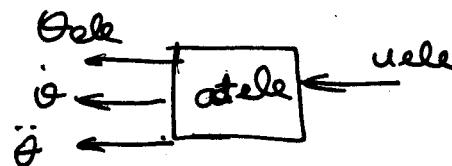
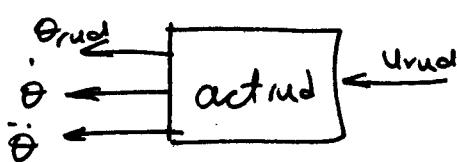


$$W_{\text{delay}} = \frac{1 - 2g_{\text{del}}(s/w_{\text{del}}) + (s/w_{\text{del}})^2}{1 + 2g_{\text{del}}(s/w_{\text{del}}) + (s/w_{\text{del}})^2}$$

Similarly for θele

	$\zeta$	$\omega$
ele	0.72	14
rudd	0.35	21

hence both actuators are modeled as



#### ④ Disturbances and noises:

we assume all of those are frequency content base signals:

$$d_{gust} \in \{W_{gust} + \eta_{gust} : W_{gust} = 30 \frac{1+s/2}{1+s}; \|\eta_{gust}\|_2 \leq 1\}$$

Sensor noise

$$\begin{aligned} P_{meas} &= P + W_p \eta_p & \eta_{meas} &= \eta_p + W_p \eta_p \\ r_{meas} &= r + W_r \eta_r, \dots & \eta_{meas} &= \eta_r + W_r \eta_r \end{aligned}$$

$$W_p = 3 \times 10^{-4} \frac{1+s/0.01}{1+s/0.5}, \quad W_\varphi = 7 \times 10^{-4} \frac{1+s/0.01}{1+s/10}$$

$$W_r = 0.25 \frac{1+s/0.05}{1+s/10}, \quad W_\varphi_{normal} := 0.5 \frac{1+s/2}{1+s/0.5}$$

$$W_{noise} = \text{diag} \{W_p, W_r, W_\varphi, W_\varphi\}$$

#### ⑤ Performance: tracking errors, ...

→ Actuator level (control effort limits)

the angles, rates & acc of rudder-elevator must be kept small

$$e_{act} = W_{act} \begin{bmatrix} \theta_e \\ \dot{\theta}_e \\ \ddot{\theta}_e \\ \theta_r \\ \dot{\theta}_r \\ \ddot{\theta}_r \end{bmatrix} \quad W_{act} = \text{diag}(4, 1, 0.005, 2, 0.2, 0.009)$$

→ ideal  $\varphi_{com}$  response model

$$P_{ideal} := \frac{1}{1+2\xi(s/\omega) + (s/\omega)^2} \varphi_{com} \quad \omega_{mod} = 1.2 \quad \xi = 0.7$$

Performance weighting function:

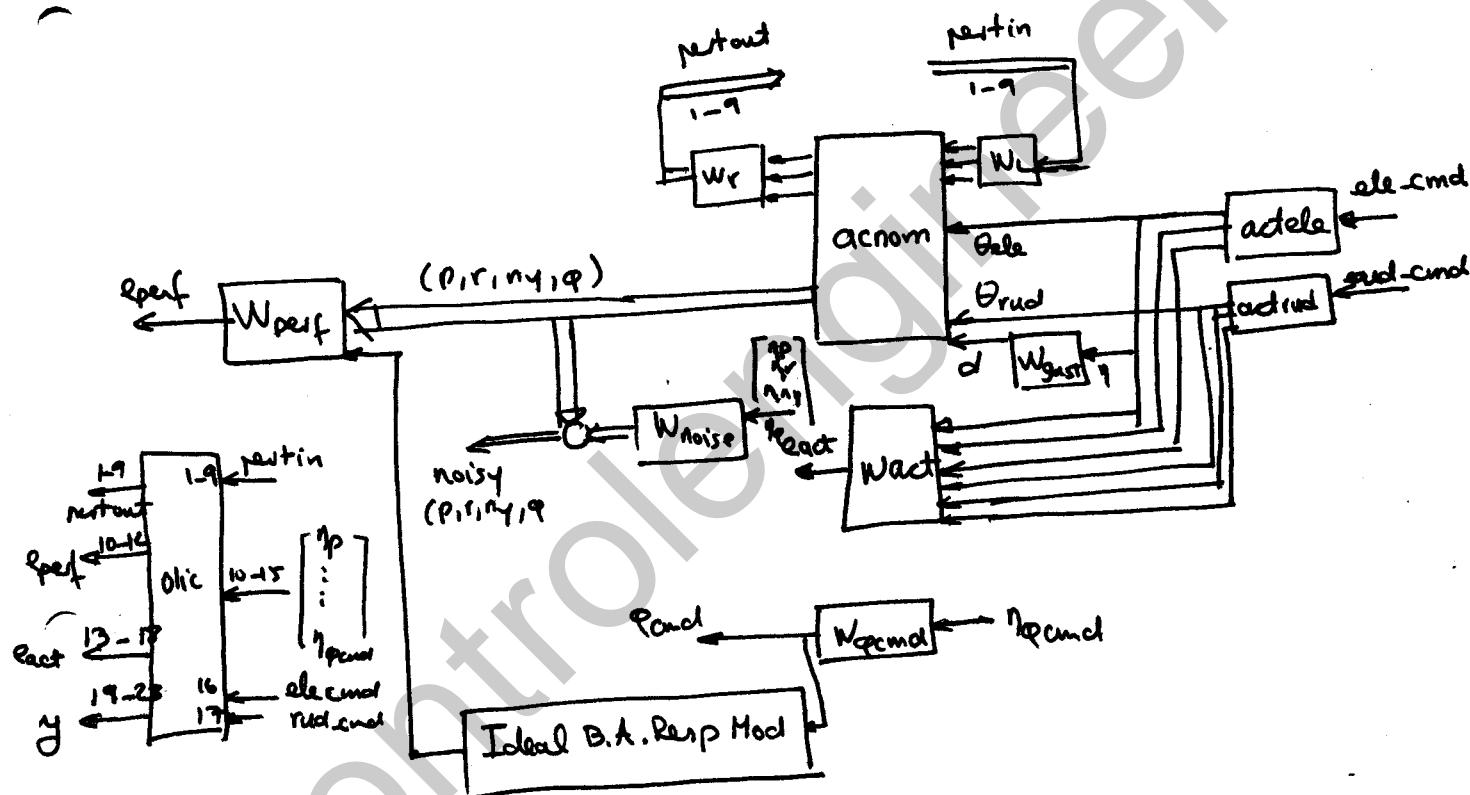
6-22

$$W_{perf} := \begin{bmatrix} 0.8 & \frac{1+s}{1+s/0.1} & 0 & 0 & ny \\ 0 & 500 & \frac{1+s}{1+s/0.01} & 0 & r-0.037q \\ 0 & 0 & 0 & 250 & P - P_{ideal} \end{bmatrix}$$

for example the  $\varphi$  tracking weighting is  $500 \frac{1+s}{1+s/0.01}$

bandwidth 1 rad/sec, st. st error =  $\frac{1}{500} = 0.2\%$ .

Finally The whole syst is as follows:

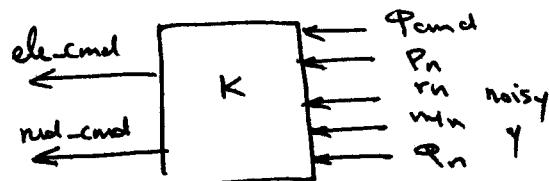


Controllers:

We overview only the H<sub>∞</sub> controller here, the controllers we are

Comparing

- 1)  $k-h$ : H<sub>∞</sub> to minimize  $\|T_d\|_{H\infty} \leq 1$   
no model uncertainty
- 2)  $k-mu$ : D-K iteration using  $\mu$ -synthesis
- 3)  $k-v$ : a tradeoff between  $k-h$  &  $k-mu$



H<sub>oo</sub> controller is designed by neglecting pert-in & pert-out

$$\text{olic\_h} = \text{sel}(\text{olic}, [10:23], [10:17]);$$

$$k\_h = \text{hinf}\text{syn}(\text{olic\_h}, 5, 2, 0, 2, 1e-3);$$

this finds suboptimal Solution to  $\|T_{dell}\|$  where

$$d = \text{exogenous disturbances} = \begin{bmatrix} \eta_p \\ \eta_r \\ \eta_{ny} \\ \eta_p \\ \eta_{gust} \\ \eta_{wind} \end{bmatrix} \quad e = \begin{bmatrix} e_{pervf} \\ e_{act} \end{bmatrix} = \begin{bmatrix} \text{weighted } \gamma_y \\ \gamma_r \\ \gamma_{acc} \\ \gamma_{vel} \\ \gamma_{pos} \\ \gamma_{rad acc} \\ \gamma_{rel pos} \\ \gamma_{rel pos} \end{bmatrix}$$

The closed loop system obtains from

$$\text{clp\_h} = \text{starp}(\text{olic}, k\_h, 5, 2)$$

The two other controllers are designed separately and loaded here  
load shutcont;

The rest gives 1) Robust Nominal performance

$$\text{Select the Closed loop syst } \frac{e}{[10:18]} \rightarrow [10:15] \quad \|T_{dell}\|_\infty$$

$$r_{hg} = \text{sel}(\text{clp\_hg}, [10:18], [10:15])$$

use vnorm(np-hg)

Figure 1,

2) Robust Stability

$$\Delta = \{\text{diag}[\delta_1, \dots, \delta_9]; \delta_i \in \mathbb{C}\}$$

$$\text{delsetrs} = \text{ones}(9, 2); \text{clp\_hgtrs} = \text{sel}(\text{clp\_hg}, 1:9, 1:9)$$

$$[\text{bnd}_{\text{h}}]_{\text{clp\_h}} = \text{mu}(\text{clp\_hgtrs}, \text{delsetrs});$$

Figure 2.

3) Robust Performance

$$\Delta := \{\text{diag}[\delta_1, \dots, \delta_9, \Delta_{10}]; \delta_i \in \mathbb{R}, \Delta_{10} \in \mathbb{C}^{6 \times 9}\}$$

$$\text{delsetrp} = [-\text{ones}(9, 1) \ \text{zeros}(9, 1); 6 \ 9];$$

$$[\text{bnd}_{\text{h}, \text{ph}}]_{\text{clp\_hg}} = \text{mu}(\text{clp\_hg}, \text{delsetrp})$$

vplot(bnd\_h)

Figure 3.

## Conclusions:

- 1) LFT representation of parametric uncertainty
- 2)  $H_\infty$  design based on a complete Model
- 3) Nominal Performance: Figure 1

Nominal performance, is well achieved by  $k_h$ , since it is designed for that. relatively  $k_{mu}$  is not as good as  $k_h$ , but acceptable

### 3) Robust Stability

only  $k_{mu}$  has  $\mu < 1$  and performs robust stability  
the size of perturbation for  $k_h$  &  $k_x$  for R-S are much smaller.

### 4) Robust Performance

$k_{mu}$  has the best performance  $\mu = 1.22$

$k_x$  is the second best  $\mu = 1.56$

$k_h$  gets  $\mu \approx 14$  at low frequencies, and R.P. is really poor.

A better illustration can be made by time responses. It is easy to use Simulink to import the plant, controller, perturbations and the type of input and plot the outputs;

In perturbation block you may use

$pertx = \text{diag}([1 \ 1 \ 1 \ -1 \ -1 \ -1 \ 1 \ 1 \ -1])$  or any combination to see the perturbed responses.

```
% Copyright Dr. Hamid D. Taghirad 2005

clear all
clc
echo on
% This constructs the aircraft model for the SSLAFCS
% example. the SYSTEM matrix produced has
%
% ACNOM: 4 states, 7 outputs, 6 inputs
%
% OUTPUTS: INPUTS:
% 1) pert1o 1) pert1i
% 2) pert2o 2) pert2i
% 3) pert3o 3) pert3i
% 4) p (roll rate (rad/s)) 4) theta_el (elevon angular def (rad))
% 5) r (yaw rate(rad/s)) 5) theta_rud (rudder angular def (rad))
% 6) ny (normal accel (ft/s^2)) 6) gust disturbance (ft/s)
% 7) phi (bank angle (rad))
%

echo off
clear all

a = [-0.0946 0.141 -0.99 0.0364;
      -3.59 -0.428 0.281 0;
      0.395 -0.0126 -0.0814 0;
      0 1 -0.141 0 ];
%
b = [-0.0124 0.0102 -0.000000109 ;
      6.57 1.26 -0.00000413 ;
      0.378 -0.256 0.000000453;
      0 0 0];
%
c = [ 0 1 0 0; 0 0 1 0; -68 -1.74 -4.06 -3.72e-5; 0 0 0 1];
%
bp = [ 0.0128 0 0;
       0 -0.0311 -3.12 ;
       0 -0.19 -0.0644;
       0 0 0];
%
cp = [ 1 0 0 0 ; 0 0 0 0 ; 0 0 0 0];
d12 = [ 0 0 0.00000115 ; 1 0 0 ; 0 1 0];
dp = zeros(3,3);
d21 = [zeros(2,3) ; 11.1 -11.1 -11.1 ; 0 0 0 ];
d22 = [zeros(2,3) ; 26.7 -2.95 -0.0000781; 0 0 0];

acnom = ss(a,[bp b],[cp ; c],[dp d12;d21 d22]);
clear cp d12 d21 d22 bp c b a

echo on
%----- mk_act -----%
%
% This makes up the 2 actuator models as specified in
% the handout. The actuator models each have 1 input
% and 3 outputs. They are listed below:
```

```
%  
%  
% ACTRUD: 4 states, 3 outputs, 1 input  
% OUTPUTS: INPUTS:  
% 1) position 1) rudder_cmd  
% 2) rate  
% 3) acceleration  
%  
% ACTELE: 4 states, 3 outputs, 1 input  
% OUTPUTS: INPUTS:  
% 1) position 1) elevon_cmd  
% 2) rate  
% 3) acceleration  
  
echo off  
s=zpk('s'); % Laplace variable s  
wrud = 21.0;  
zetarud = 0.75;  
wele = 14.0;  
zetaele = 0.72;  
  
wdel = 173.0;  
zetadel = 0.866;  
delaytf =(s^2-2*zetadel*wdel*s+wdel^2)/(s^2+2*zetadel*wdel*s+wdel^2);  
  
int1 = 1/s;  
int2 = 1/s;  
c1 = wrud^2;  
c2 = wrud^2;  
c3 = 2*zetarud*wrud;  
systemnames = 'c1 c2 c3 int1 int2';  
inputvar = '[u]';  
outputvar = '[int2;int1;c1-c2-c3]';  
input_to_c1 = '[u]';  
input_to_c2 = '[int2]';  
input_to_c3 = '[int1]';  
input_to_int1 = '[c1-c2-c3]';  
input_to_int2 = '[int1]';  
sysoutname = 'rudder';  
cleanupsysic = 'yes';  
sysic  
actrud = rudder*delaytf;  
  
c1 = wele^2;  
c2 = wele^2;  
c3 = 2*zetaele*wele;  
systemnames = 'c1 c2 c3 int1 int2';  
inputvar = '[u]';  
outputvar = '[int2;int1;c1-c2-c3]';  
input_to_c1 = '[u]';  
input_to_c2 = '[int2]';  
input_to_c3 = '[int1]';  
input_to_int1 = '[c1-c2-c3]';  
input_to_int2 = '[int1]';  
sysoutname = 'elevon';  
cleanupsysic = 'yes';
```

```
sysic
actele = elevon*delaytf;

echo on
%
% This makes up the weighting functions

echo off

% EXOGENOUS SIGNALS
%
% WIND
wgust = 30*(.5*s+1)/(s+1);

% SENSOR NOISE
wr = 3e-4*(100*s+1)/(2*s+1); % rads/sec
wp = wr; % rads/sec
wphi = 7e-4*(100*s+1)/(0.5*s+1); % rads
wny = 0.25*(20*s+1)/(0.1*s+1); % ft/sec/sec
wnoise = append(wp,wr,wny,wphi);

% PILOT COMMAND
wphicmd = 0.5*(0.5*s+1)/(2*s+1);

% ERROR WEIGHTING
% ACTUATOR WEIGHTINGS
wact = diag([4 1 .005 2 .2 .009]);

% IDEAL PHI_COMMAND RESPONSE MODEL
wmod = 1.2;
zmod = .7;
idmod = wmod^2/(s^2+2*zmod*wmod*s+wmod^2);

% PERFORMANCE VARIABLES
nyerr = 0.8*(s+1)/(10*s+1);
cterr = 500*(s+1)/(100*s+1);
baerr = 250*(s+1)/(100*s+1);
fix = [0 0 1 0 0;0 1 0 -0.037 0;0 0 0 1 -1];
wperf = append(nyerr,cterr,baerr);
wperf=wperf*fix;

% PERTURBATION WEIGHTS
wr = [1 0 0;1 0 0;1 0 0;0 1 0;0 1 0;0 1 0;0 0 0 1;0 0 1;0 0 1];
wll = diag([2.194 -1.517 -.718]);
wlm = diag([-1.327 1.347 .5185]);
wlr = diag([-0.3656 .8667 .2347]);
wl = [wll wlm wlr];

echo on
%
% SYSIC program to form interconnection

echo off
systemnames = 'acnom wr wl wnoise wact wgust wphicmd wperf idmod';
systemnames = [systemnames ' actele actrud'];
```

```
inputvar = '[pertin{9} ; noise{4} ; gust ; comd ; elec ; rfdc]';  
outputvar = '[wr; wperf; wact ; wphicmd; acnom(4:7) + wnoise]';  
input_to_acnom = '[wl ; actele(1) ; actrud(1) ; wgust ]';  
input_to_wl = '[pertin]';  
input_to_wr = '[acnom(1:3)]';  
input_to_wnoise = '[noise]';  
input_to_wact = '[actele ; actrud]';  
input_to_wgust = '[gust]';  
input_to_wphicmd = '[comd]';  
input_to_wperf = '[acnom(4:7) ; idmod]';  
input_to_idmod = '[wphicmd]';  
input_to_actele = '[elec]';  
input_to_actrud = '[rfdc]';  
cleanupsysic = 'yes';  
sysoutname = 'olic';  
sysic  
  
clear c1 c2 c3 actrud rudder delaytf int1 int2 zetadel wdel zetaele wele  
clear wgust wr wp wphi wny wnoise wphicmd wact wmod zmod idmod nyerr  
clear cterr baerr fix wpert wr wll wlm wlr wl zetarud wrud acnom actele  
  
echo on  
pause % Strike a key to continue  
clc  
  
%  
% Design of H-infinity controller:  
% mixed sensitivity of(Nominal Performance & Actuator Effort)  
%  
echo off  
  
olic_h =olic(10:23,10:17);  
k_h = hinfsyn(olic_h,5,2,'Display','on');  
  
echo on  
pause % Strike a key to continue  
clc  
  
%  
% Design H2/Hinf controller:  
%  
% [K,CL,NORMZ,INFO] = H2HINFSYN(P,NMEAS,NCON,NZ2,WZ,key1,value1,...)  
%  
% Here the actuator effort vector has 6 components nmeas=5 & ncont=2  
% We are optimizing the infinity norm while keeping h2 norm bounded  
  
nmeas=5;  
ncon =2;  
nz2 =6;  
h2max=10;  
alpha=1;  
beta =0;  
wz=[alpha beta]; % alpha = 1; beta = 0 cost:= alpha* inf_norm + beta* 2-norm
```

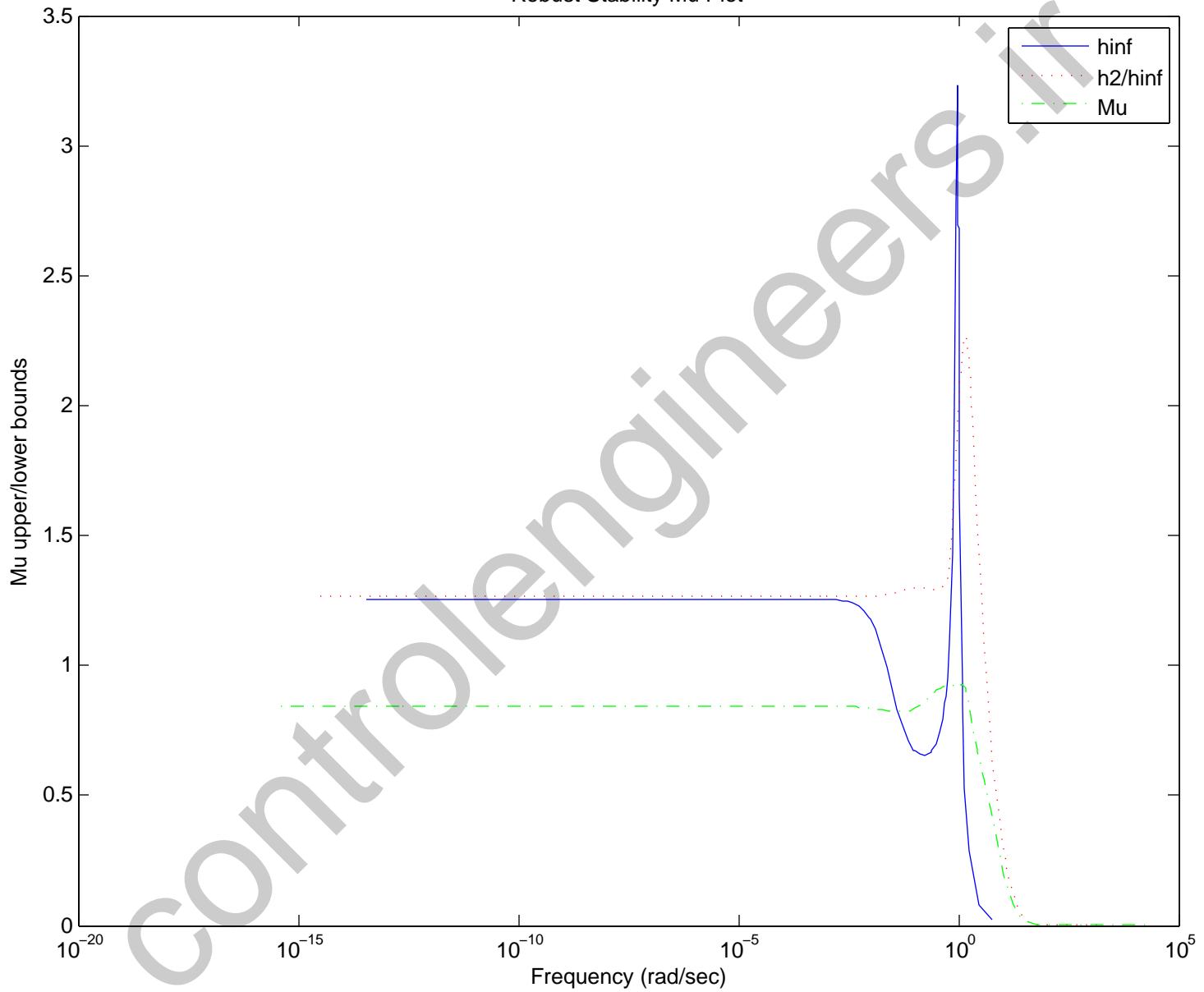
```
[k_mix,g_x,normz_x,info_x]=h2hinfsyn(ss(olic_h),nmeas,ncon,nz2,wz,...  
    'Display','off','h2max',h2max,'hinfmax',inf);  
  
pause % Strike a key to continue  
clc  
  
%  
% load the mu controller and another trade off controller  
%  
  
load shutcont;  
k_mu=mat2lti(k_mu);  
k_x=ss(k_mix);  
  
echo on  
%  
% Plotting Sample Sigma plots for closed loop systems  
%  
  
echo off  
  
omega = logspace(-2,3,50);  
  
clp_h = lft(olic,k_h,2,5);  
clp_x = lft(olic,k_x,2,5);  
clp_mu = lft(olic,k_mu,2,5);  
  
np_hg = clp_h(10:18,10:15);  
np_xg = clp_x(10:18,10:15);  
np_mug = clp_mu(10:18,10:15);  
  
sigma(np_hg(1,1),'-', np_xg(1,1),'r--', np_mug(1,1),'g-.');  
grid; title('Nominal Performancs: All three controllers')  
legend('hinf','h2/hinf','Mu')  
  
echo on  
% Strike a key to continue  
  
pause  
%  
% Analysing the robust stability  
%  
echo off  
  
% First generate uncertain models  
  
del1=ucomplex('del1',0);  
del2=ucomplex('del2',0);  
del3=ucomplex('del3',0);  
del4=ucomplex('del4',0);  
del5=ucomplex('del5',0);  
del6=ucomplex('del6',0);  
del7=ucomplex('del7',0);  
del8=ucomplex('del8',0);  
del9=ucomplex('del9',0);
```

```
delta=blkdiag(dell1,del2,del3,del4,del5,del6,del7,del8,del9);
clear dell1 del2 del3 del4 del5 del6 del7 del8 del9

echo on
%
% Use lft to combine uncertainty to the closed loop model
%
echo off
pclp_h = lft(delta,clp_h);
pclp_x = lft(delta,clp_x);
pclp_mu = lft(delta,clp_mu);
echo on
%
% Use robuststab to analyse robust stability
%

[stm1,stmul1,rep1,inf1] = robuststab(pclp_h);
%
% The robust stability analysis report of hinfinity controller
%
rep1
[stm2,stmu2,rep2,inf2] = robuststab(pclp_x);
%
% The robust stability analysis report of h2/hinf controller
%
rep2
%
% Warning: This task will take a few minutes to accomplish ...
%
[stm3,stmu3,rep3,inf3] = robuststab(pclp_mu);
%
% The robust stability analysis report of mu-synthesis controller
%
rep3
echo off
semilogx(inf1.MussvBnds(:,1),'b',inf2.MussvBnds(:,1),':r',...
    inf3.MussvBnds(:,1),'-.g')
xlabel('Frequency (rad/sec)');
ylabel('Mu upper/lower bounds');
title('Robust Stability Mu Plot');
legend('hinf','h2/hinf','Mu')
echo on
%
% This completes this example
%
```

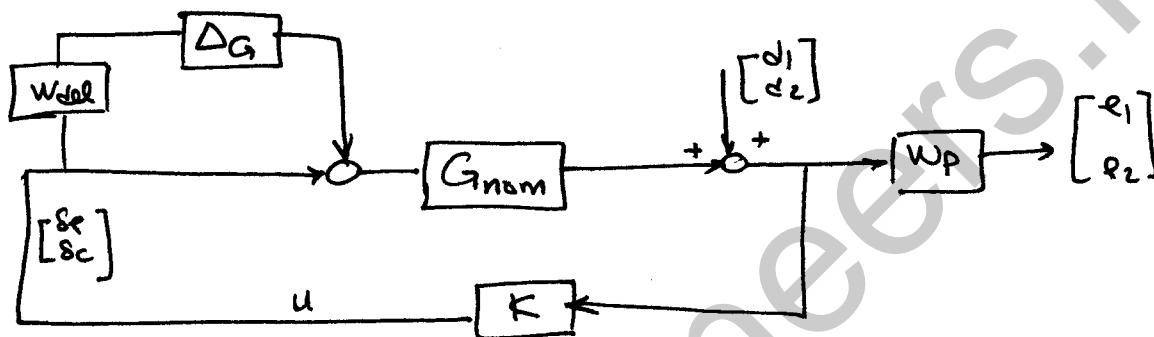
Robust Stability Mu Plot



Design of Robust Performance Controller, for a MIMO System with multiplicative uncertainty. using D-K iteration.

### ① System Model and performance requirements

The block diagram of system is given



$$\dot{x} = \begin{bmatrix} \dot{v}_v \\ \alpha \\ \dot{\theta} \\ \theta \end{bmatrix} \quad \begin{array}{l} \text{perturbation of velocity} \\ \text{angle of attack} \\ \text{rate of attitude angle} \\ \text{attitude angle} \end{array}$$

$$\text{Control inputs} = \begin{bmatrix} \delta_e \\ \delta_c \end{bmatrix} \quad \begin{array}{l} \text{angle of elevon} \\ \text{o r control} \end{array}$$

$$\text{Output measurements } y = \begin{bmatrix} \alpha \\ \theta \end{bmatrix}$$

Performance:

- ① Control the vertical velocity at a constant  $\dot{v}_v$  (keep altitude, while velo. vector  $\overset{\text{toto}}{\rightarrow}$ )
- ② Control the flight path angle at constant angle of attack

We put it into a disturbance reject routine  $\|T_d\|_\infty < 1$ ,  $\forall \Delta G \in \Delta$   
and  $\|\Delta G\|_\infty < 1$ , hence having good tracking for  $\alpha, \theta$  despite external  
disturbances & uncertainty

### ② System data:

The Nom. plant is given in the program, uncertainty weight is a  $2 \times 2$   
t.f., with diag [wdel, wdel]

$$w_{del} = \frac{40(s+100)}{s+10^4} \quad \text{meaning } 40\% \text{ uncertainty at } \omega=0$$

100% " at  $\omega = 17.3$

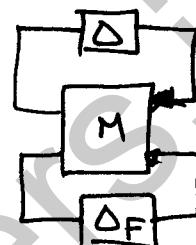
much more at higher frequency

Sensitivity weighting fact is also diagonal one  $W_p(s) = w_p \cdot I_2$  6-26

where  $\alpha_p(s) = \frac{0.5(s+3)}{s+0.03}$  : b.w. Brad/S  
st. st. error = 2%.

Similar to previous example, syst interconnect and H<sub>∞</sub>-controller design is made  $[k, dp] = hinfsyn(himat_ic, 2, 2, 0.8, 6.0, 0.01, 2)$ ;

Bode plots of 4 transfer fact is produced.  
assuring robust performance.



$\Delta := \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}; \Delta_1 \in \mathbb{C}^{2 \times 2}, \Delta_2 \in \mathbb{C}^{2 \times 2} \right\}$

deltaset = [2 2; 2 2]

[bncls1, pvl] = mu([clp-g1, deltaset]);

use vnom to plot

$\mu_{\Delta}(M) = 1.69 > 1$  go for D-K iterat

### ③ D-K Iteration

we use dkit script for that. To use this script we need to assign ref.

NOMINAL\_DK = himat\_ic;

NMEAS\_DK = 2;

NCONT\_DK = 2

BIK\_DK = [2 2; 4 2]

OMEGA\_DK = logspace(-3, 3, 60);

DK\_DEF\_NAME = 'himat\_dk'

dkit

1<sup>st</sup> iteration  $\rightarrow \begin{cases} \gamma = 2.1516, \text{ we may change the frequency range now} \\ \mu_{\text{real}} = 2.075 \end{cases}$

2<sup>nd</sup> iteration  $\rightarrow \begin{cases} \gamma = 1.0730 \\ \mu_{\text{real}} = 1.073 \end{cases}$

3<sup>rd</sup> iteration  $\rightarrow \begin{cases} \gamma = 0.970 \\ \mu_{\text{real}} = 0.979 \end{cases} \rightarrow \text{close the D-K iteration in 3<sup>rd</sup> step.}$

the final controller is in  $k$ -dB variable,

which is of order ~~30~~ ! 28

The main drawback of D-K iterat, which increases the Model order and hence the controller order, we usually need to reduce the controller order by different Methods.

### Model Reduction Methods

Given a controller system matrix  $[A \ B; C \ D]$  the simplest

- method is to truncate a part of Syst. (which has smaller singular value with respect to others)
- function "strunc" performs this function.  
in order to reduce the effect of high poles truncation into low ~~order~~ frequency, we may use residualize the truncated modes and compensate for the zero frequency contributions of each truncated mode with an additional D matrix term in the resulting reduced order system matrix. "sresid" does this function.

A balanced realization on the input system matrix is performed which entails balancing the observability and Controllability Grammians, in its simplest form, this command will remove all unobservable and uncontrollable modes.

- "sysbal" returns a vector of the Hankel singular values of the System and performs balanced realization

16-28  
"hankmr" performs Hankel norm approximation of a system matrix. sysbal & hankmr are restricted to be used on a LTI continuous and stable system.

We usually use a balanced realization + another method:

as in example hinat

$$[K_{dK3bal}, h_{sv}] = \text{sysbal}(k_{dK3})$$

$$[K_{dK3red}] = \text{strunc}(k_{dK3bal}, 12)$$

We may close the loop and check the  $M_\Delta(\cdot)$  here is obtained

$$M_\Delta(\cdot) = 0.9991$$

We may use singular or other forms to perform robust performance tests on time responses for the system.

```
% Copyright Dr. Hamid D. Taghirad 2005

clear all
clc
echo on
%
% This program is generated to illustrate h-infinity controller
% design and d-k iteration used for highly maneuverable aircraft HiMAT
%
%% Model of NASA's HiMAT aircraft
% For illustration purposes, let's use a four-state model of the
% longitudinal dynamics of the HiMAT aircraft.
%
% This model has two control inputs:
%
% * Elevon deflection
% * Canard deflection
%
% It also has two measured outputs:
%
% * Angle of attack alpha
% * Pitch angle theta
%

echo off
ag =[ -0.0226 -36.6000 -18.9000 -32.1000
      0 -1.9000 0.9830 0
      0.0123 -11.7000 -2.6300 0
      0 0 1.0000 0];
bg = [ 0 0
      -0.4140 0
      -77.8000 22.4000
      0 0];
cg = [ 0 57.3000 0 0
      0 0 0 57.3000];
dg = [ 0 0;
      0 0];
G=ss(ag,bg,cg,dg);
G.InputName = {'elevon','canard'};
G.OutputName = {'alpha','theta'};

clf
step(G), title('Open-loop step response of HiMAT aircraft')

echo on
pause % press a key to continue
clc

%
% Creating an interconnection for plant model and
% specifying performance weighting function.

s=zpk('s'); % Laplace variable s
WDELL1 = 50*(s+100)/(s+1e4);
WP1    = 0.5*(s+3)/(s+0.03);
```

```
WDEL = append(WDELL1,WDELL1);
WP    = [WP1 0;0 WP1];

echo off
systemnames = 'G WDEL WP';
inputvar = '[pert(2); dist(2); control(2)]';
outputvar = '[ WDEL ; WP; G + dist ]';
input_to_G = '[ control + pert ]';
input_to_WDEL = '[ control ]';
input_to_WP = '[ G + dist ]';
sysoutname = 'himat_ic';
cleanupsysic = 'yes';
sysic

echo on
pause % press a key to continue
clc

%
% The next step is to design an H-infinity control law for PLANT
%
% In this example, the system interconnection structure is
% himat_ic, with 2 measurements, 2 controls.

[k1,clp1,gamma1,info1]=hinfsyn(himat_ic,2,2,'Display','on');

echo on
pause % press a key to continue
clc
%
% The closed loop system is neither robust stable nor robustly
% performed. This can be visualized using robuststab and robustperf
% functions as in previous example. We like to use mu-synthesis to design a
% controller which is robust in performance.
%
% First generate the uncertain system structure.

delta = ultidyn('delta',[2 2]);
himat_pert=lft(delta,himat_ic);
clpm1=lft(himat_pert,k1);

% Next, we perform a mu-synthesis to see if the specs can be met
% robustly when taking into account the modeling errors (uncertainty
% "delta"). We use the command "dksyn" to perform the synthesis and set
% the frequency grid used for mu-analysis and the number of D-K
% iterations with "dkitopts".

fmu = logspace(-2,4,60);
opt = dkitopt('FrequencyVector',fmu,'NumberofAutoIterations',5, ...
    'DisplayWhileAutoIter','on');
[kmu,clpmu,bnd,dkinfo] = dksyn(himat_pert,2,2,opt);

%
% The order of controller is high, use a balance realization
% and a truncation to make it small.
```

```
[kmu13,info_kmu13] = reduce(kmu,13);
clpmu13=lft(himat_pert,kmu13);

%
% Analyse the robust performance
%

[perfmargin1,perfmarginunc1,rep1,inf1] = robustperf(clpm1);
[perfmargin2,perfmarginunc2,rep2,inf2] = robustperf(clpmu);
[perfmargin3,perfmarginunc3,rep3,inf3] = robustperf(clpmu13);

semilogx(inf1.MussvBnds(:,1),'b',inf2.MussvBnds(:,1),':r',...
    inf3.MussvBnds(:,1),'--g')
xlabel('Frequency (rad/sec)');
ylabel('Mu upper/lower bounds');
title('Robust Performance Mu Plot');
legend('hinf', 'mu' , 'mu reduced')

pause % press a key to continue
clc
%
% For the final case compare the time response of three controllers
% with model uncertainty. Only disturbance rejection in channel 2 is
% displayed in this simulations.

P1 = usample(clpm1,5);      % 5 random samples of the plant
P2 = usample(clpmu,5);      % 5 random samples of the plant
P3 = usample(clpmu13,5);    % 5 random samples of the plant

subplot(311)
impulse(P1(2,2),0.1)
xlabel('')
legend('hinf')
subplot(312)
impulse(P2(2,2),':r',0.1)
xlabel('')
legend('mu synthesis')
subplot(313)
impulse(P3(2,2),'--g',0.1)
legend('mu reduced')

%
% This concludes this example
%
```

